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ON A PROPER CLASS AND RELATED MATTERS

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This note establishes that a subdirect product of certain semigroups (abelian semigroups, monoids, abelian monoids, groups, abelian groups, rings and logical theories [1]) does not exist and *a fortiori* that certain classes of logico-algebraic structures are *proper* classes (i.e., classes which are *not* sets) in the sense of K. Gödel's set theory [2].

Consider an aggregate $\{S_i\}$ of *all* pair-wise inequivalent semigroups S_i. If $\{S_i\}$ is a set then its ordinary direct product P (*alias*, subdirect product; *alias* complete direct product) exists *à la* category theory. If P exists then, trivially, P is a semigroup. In such a case P is not inequivalent to every S_i for the following reason. If P is inequivalent to every S_i then, surely, $P \notin \{S_i\}$ for, otherwise, $P \cong P$; a contradiction of inequivalence. But, then, $\{S_i\}$ does not contain all pair-wise inequivalent semigroups, *viz.*, it does not contain P; but this contradicts the definition of $\{S_i\}$ as the class of all pair-wise inequivalent semigroups. Hence $P \cong S_j$ for some j. However if $P \cong S_j$ and $P \cong S_k$, $j \neq k$, then $S_j \cong S_k$; a contradiction of pair-wise inequivalence. Hence $P \cong S_j$ for *precisely one* j. Let S_j be respesented as $(S_j{}', o)$ and let P be represented as $(P', *)$ where 'o' and '$*$' denote associative composition laws. Since $S_j{}'$ is a set one can form its power set Π, [2]. But there is a semigroup model for every infinite cardinal. Hence there is an S_k of the form $(\Pi, \oplus)$ for some k. However since $P \cong S_j$, card (P') = card $(S_j{}')$ and, therefore, card $(\Pi) >$ card $(S_j{}')$ = card (P'). But this contradicts the theorem which asserts that the cardinality of a cartesian product of nonempty sets is greater than or equal to the cardinality of any one of its factors. Thus we arrive at the *Metatheorem*: P does not exist and $\{S_i\}$ is *not* a set in Gödel's set theory but rather it is a *proper* class [2]. *Metacorollary:* The class of all semigroups is not a set, for otherwise, upon forming its power set it would follow that $\{S_i\}$ is a set; a contradiction to the metatheorem.

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