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Infinitesimal Comparisons: Homomorphisms between Giordano's Ring and the Hyperreal Field

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Abstract The primary purpose of this paper is to analyze the relationship between the familiar non-Archimedean field of hyperreals from Abraham Robinson's nonstandard analysis and Paolo Giordano's ring extension of the real numbers containing nilpotents. There is an interesting nontrivial homomorphism from the limited hyperreals into the Giordano ring, whereas the only nontrivial homomorphism from the Giordano ring to the hyperreals is the standard part function, namely, the function that maps a value to its real part. We interpret this asymmetry to mean that the nilpotent infinitesimal values of Giordano's ring are "smaller" than the hyperreal infinitesimals. By viewing things from the "point of view" of the hyperreals, all nilpotents are zero, whereas by viewing things from the "point of view" of Giordano's ring, nonnilpotent, nonzero infinitesimals register as nonzero infinitesimals. This suggests that Giordano's infinitesimals are more fine-grained.

1 Introduction

The primary purpose of this paper is to analyze the relationship between the familiar non-Archimedean field of hyperreals from Abraham Robinson's nonstandard analysis and Paolo Giordano's ring extension of the real numbers containing nilpotents (i.e., nonzero infinitesimals ϵ such that $\epsilon^k = 0$ for some integer k). Giordano refers to these as the Fermat reals.[1]

One of our goals is to show that nilpotents are smaller than their nonnilpotent infinitesimal counterparts. Below, we will demonstrate this in an indirect fashion. To demonstrate this directly, one may pursue two courses. First, one may work inside Giordano's ring of Fermat reals by applying the ultrapower construction directly to the ring to produce invertible infinitesimals for direct comparison.[2] Alternatively, one may work within the logically subtle models of smooth infinitesimal analysis

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(SIA). Specifically, Moerdijk and Reyes [5] developed two models containing both invertible infinitesimals and nilpotents. However, in neither model may one use classical logic. This is required by the topos-theoretic setting in which these models were developed. This weakening of the logic enables the local system of real numbers to satisfy the field axioms while not requiring that $\epsilon = 0$ follows from $\epsilon^2 = 0$ (but neither may we assert that $\epsilon \neq 0$).[3] Whether in the augmented Fermat reals or the models of SIA, there are nontrivial nilsquares ϵ and invertible infinitesimals δ such that $\epsilon < \delta$.

In contradistinction, our proof provides an indirect means of comparing infinitesimals while permitting the full use of classical logic. We do this by proving some model-theoretic results. It turns out that there is an interesting nontrivial homomorphism from the limited hyperreals into the Fermat reals, whereas the only nontrivial homomorphism from the Fermat reals to the hyperreals is the standard part function, namely, the function that maps a value to its real part. We interpret this asymmetry to mean that the nilpotent infinitesimal values of the Fermat reals are "smaller" than the hyperreal infinitesimals. By viewing things from the "point of view" of the hyperreals, all nilpotents are zero, whereas by viewing things from the "point of view" of the Fermat reals, nonnilpotent, nonzero infinitesimals register as nonzero infinitesimals. This suggests that nilpotent infinitesimals are more fine-grained.

1.1 Hyperreals The hyperreals are more familiar, so we will not spend as much time detailing them here. The hyperreals can be defined to be members of $\mathbb{R}^{\mathbb{N}}$ modulo some nonprinciple ultrafilter. We will be operating at a high enough level of generality that any model of hyperreals will do. Now, we must clarify some basic hyperreal concepts.

- For any $c \in {}^*\mathbb{R}$, c is *limited* if there is some $r \in \mathbb{R}$ such that $|c| < r$.
- For any $c \in {}^*\mathbb{R}$, c is *unlimited* if c is not limited.
- For any $c \in {}^*\mathbb{R}$, c is *infinitesimal* if, for all standard $r \in \mathbb{R}$, $r > 0$, $|c| < r$.
- For any $x, y \in {}^*\mathbb{R}$, x and y are *infinitely close* (written $x \simeq y$) if $|x - y| = \epsilon$ for some infinitesimal $\epsilon \geq 0$.

We will not go into any further detail on the relations or operations on the hyperreals, given their familiarity and the level of abstraction at which the below constructions take place.

1.2 Giordano's ring of Fermat reals The initial motivation for Giordano's work is SIA. As we mentioned in the Introduction, SIA requires intuitionistic logic, whereas the Fermat reals provide a classical venue for nilpotent infinitesimals. Furthermore, the ring is constructed from more familiar mathematical objects rather than emerging from category-theoretic investigations.

The Fermat reals are constructed from so-called little-o polynomials, written $\mathbb{R}_o[t]$. To be precise, a little-o polynomial x_t is a function in t from the nonnegative reals ($\mathbb{R}_{\geq 0}$) into the reals, defined as follows:

$$x_t = \sum_{k=0}^{n} \alpha_k t^{a_k} + o(t),$$

$$n \in \mathbb{Z}^+, \alpha_k \in \mathbb{R}, a_k \in \mathbb{R}^+, \text{where } o(t) : \mathbb{R}_{\geq 0} \to \mathbb{R} \text{ and } \lim_{t \to 0^+} \frac{o(t)}{t} = 0.$$

In some circumstances, one might place greater requirements on $o(t)$. For our purposes, $o(t)$ can be any function from the nonnegative reals into the reals satisfying the limit requirement. Using the little-o is primarily a notational feature that permits a natural transition between $\mathbb{R}_o[t]$ and ${}^\bullet\mathbb{R}$. Indeed, ${}^\bullet\mathbb{R}$ is simply $\mathbb{R}_o[t]$ modulo the equivalence relation on $\mathbb{R}_o[t]$ defined as follows. If $x_t, y_t \in \mathbb{R}_o[t]$, then $x \sim y$ when

$$\lim_{t \to 0^+} \frac{x_t - y_t}{t} = 0.$$

Explicitly, ${}^\bullet\mathbb{R} = \mathbb{R}_o[t]/\sim$. Giordano [1] proved that there is a unique representation of each member of ${}^\bullet\mathbb{R}$ provided that the exponent values are restricted to $(0, 1]$. We will assume this below.

Addition and multiplication on members of $\mathbb{R}_o[t]$ are just like normal polynomials. That is, let $x, y \in \mathbb{R}_o[t]$, so that

$$\begin{gathered} x = \sum_{k=0}^{n_1} \alpha_k t^{a_k} + o_1(t), \qquad y = \sum_{k=0}^{n_2} \beta_k t^{b_k} + o_2(t), \\ x +_o y = \alpha_0 + \beta_0 + \sum_{k=1}^{n_1} \alpha_k t^{a_k} + \sum_{k=1}^{n_2} \beta_k t^{b_k} + o_3(t), \end{gathered} \tag{1}$$

and

$$\begin{aligned} (x \times_o y) = \alpha_0\beta_0 &+ \sum_{k=1}^{n_1} \beta_0 \alpha_k t^{a_k} + \sum_{k=1}^{n_2} \alpha_0 \beta_k t^{b_k} \\ &+ \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \alpha_i \beta_j t^{a_i + b_j} + o_4(t). \end{aligned} \tag{2}$$

Addition and multiplication on ${}^\bullet\mathbb{R}$ are the same only where $o_j(t)$ is absent (and where any exponent $a_i + b_j > 1$ vanishes under $\sim$). For addition and multiplication, respectively, we write $x \,{}^\bullet\!+\, y$ and $x \,{}^\bullet\!\times\, y$ for $x, y \in {}^\bullet\mathbb{R}$.

What is the order relation on ${}^\bullet\mathbb{R}$? If $x, y \in {}^\bullet\mathbb{R}$, then $x \,{}^\bullet\!\leq\, y$ if and only if there is some real $\epsilon > 0$ such that, for all nonnegative $t < \epsilon$, $x_t \leq y_t$. Giordano [1] proved that ${}^\bullet\!\leq$ linearly orders ${}^\bullet\mathbb{R}$. Maintaining conventions, the reader will find that $x \,{}^\bullet\!< y$ is simply defined to be $x \,{}^\bullet\!\leq\, y$ and $x \neq y$.

Finally, in the constructions below, we will need the function called the *order of* x, $\omega(x) : {}^\bullet\mathbb{R} \to \mathbb{R}$. It is defined to be the largest infinitesimal value in x. In particular, if $x = \sum_{k=0}^{n} \alpha_k t^{a_k}$ with $a_k \in (0, 1]$, $\alpha_k \in \mathbb{R}$, then $\omega(x)$ is the least member of $\{a_k : 1 \leq k \leq n\}$. When viewed as a polynomial expression, the order of $x(t)$ is like the opposite of the degree of $x(t)$. The largest infinitesimal will have the lowest exponent value. As such, ω plays a very significant role in determining order relations between values of ${}^\bullet\mathbb{R}$, given how ${}^\bullet\!\leq$ is defined.

1.3 Notational matters There is an isomorphic copy of $\mathbb{R}$ in both ${}^*\mathbb{R}$ and ${}^\bullet\mathbb{R}$. For this reason, we will ignore distinctions of what form $\mathbb{R}$ takes in either structure and speak univocally about members of $\mathbb{R}$. In other words, if $c \in \mathbb{R}$, we will simply refer to $c \in {}^\bullet\mathbb{R}$ and $c \in {}^*\mathbb{R}$. Relatedly, one convenient feature of the definition of ${}^\bullet\mathbb{R}$ is that, for some $x \in {}^\bullet\mathbb{R}$, the standard part of x, written ${}^\circ x$, is simply x evaluated at zero; that is, $x = x(t) \in \mathbb{R}_o[t]$ and ${}^\circ x = x(0)$. The standard part function for $x \in {}^*\mathbb{R}$ will be written $\mathrm{sh}(x)$.[4]

For the following, let ${}^{\bullet}\mathfrak{R}$ be the Fermat reals with relational structure $\langle {}^{\bullet}\mathbb{R}, {}^{\bullet}\leq, {}^{\bullet}+, {}^{\bullet}\times \rangle$, and let ${}^{*}\mathfrak{R}$ be $\langle {}^{*}\mathbb{R}, {}^{*}\leq, {}^{*}+, {}^{*}\times \rangle$, where ${}^{*}\mathbb{R}$ is some fixed model of hyperreals. Furthermore, let ${}^{*}\mathbb{R}_L$ be the limited members of ${}^{*}\mathbb{R}$. In other words,

$$ {}^{*}\mathbb{R}_L = \left\{ x \in {}^{*}\mathbb{R} : (\exists r \in \mathbb{R})\ x < r \right\}. $$

Finally, let ${}^{*}\mathfrak{R}_L$ be the relational structure $\langle {}^{*}\mathbb{R}_L, {}^{*}\leq, {}^{*}+, {}^{*}\times \rangle$, where all operations and relations on ${}^{*}\mathbb{R}$ are appropriately restricted. There will be no setting below where an ambiguity will exist between operations and relations on ${}^{*}\mathbb{R}$ and operations and relations on ${}^{*}\mathbb{R}_L$. Distinct notation is unnecessary.

2 The View from the Hyperreals

The results in this section are fairly routine. We rehearse them for exhaustiveness.

Lemma 2.1 *The standard part function is a homomorphism from ${}^{\bullet}\mathfrak{R}$ into ${}^{*}\mathfrak{R}$.*

Proof In this section, we will mainly work in $\mathbb{R}_o[t]$ due to greater ease of maneuverability. This will make no significant difference. Now, let

$$ x_t = \sum_{k=0}^{n_1} \alpha_k t^{a_k} + o_1(t) \qquad \text{and} \qquad y_t = \sum_{k=0}^{n_2} \beta_k t^{b_k} + o_2(t). $$

Using the notation where the standard part of x is written as ${}^{\circ}x$, note then that

$$ \begin{aligned} {}^{\circ}(x \,{}^{\bullet}\!+ y) &= {}^{\circ}\!\left[\alpha_0 + \beta_0 + \sum_{k=1}^{n_1} \alpha_k t^{a_k} + \sum_{k=1}^{n_2} \beta_k t^{b_k} + o_3(t) \right] \\ &= \alpha_0 + \beta_0 + \sum_{k=1}^{n_1} \alpha_k (0)^{a_k} + \sum_{k=1}^{n_2} \beta_k (0)^{b_k} + o_3(0) \\ &= \alpha_0 \,{}^{\bullet}\!+ \beta_0 = \underbrace{{}^{\circ}x \,{}^{\bullet}\!+ {}^{\circ}y}_{\in \mathbb{R}} = {}^{\circ}x \,{}^{*}\!+ {}^{\circ}y. \end{aligned} $$

Similarly,

$$ \begin{aligned} {}^{\circ}(x \,{}^{\bullet}\!\times y) &= {}^{\circ}\!\Bigg[\alpha_0\beta_0 + \sum_{k=1}^{n_1} \beta_0 \alpha_k t^{a_k} + \sum_{k=1}^{n_2} \alpha_0 \beta_k t^{b_k} \\ &\qquad + \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \alpha_i \beta_j t^{a_i + b_j} + o_4(t) \Bigg] \\ &= \alpha_0\beta_0 + \sum_{k=1}^{n_1} \beta_0 \alpha_k (0)^{a_k} + \sum_{k=1}^{n_2} \alpha_0 \beta_k (0)^{b_k} \\ &\qquad + \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \alpha_i \beta_j (0)^{a_i + b_j} + o_4(0) \\ &= \alpha_0 \,{}^{\bullet}\!\times \beta_0 = \underbrace{{}^{\circ}x \,{}^{\bullet}\!\times {}^{\circ}y}_{\in \mathbb{R}} = {}^{\circ}x \,{}^{*}\!\times {}^{\circ}y. \end{aligned} $$

Finally, note that, for $x, y \in {}^{\bullet}\mathbb{R}$, $x \,{}^{\bullet}\!\leq y \implies \underbrace{{}^{\circ}x \,{}^{\bullet}\!\leq {}^{\circ}y}_{\in \mathbb{R}} \iff {}^{\circ}x \,{}^{*}\!\leq {}^{\circ}y$.[5] □

Lemma 2.2 *If φ is a nontrivial homomorphism from ${}^{\bullet}\mathfrak{R}$ into ${}^{*}\mathfrak{R}$, then $\varphi|_{\mathbb{R}}$ is the identity function on ${}^{\bullet}\mathbb{R}$.*

Proof Let φ be a nontrivial homomorphism from ${}^{\bullet}\mathfrak{R}$ into ${}^{*}\mathfrak{R}$. It follows from the uniqueness of the additive and multiplicative identities that $\varphi({}^{\bullet}0) = {}^{*}0$ and $\varphi({}^{\bullet}1) = {}^{*}1$; indeed, $\varphi({}^{\bullet}{-1}) = {}^{*}{-1}$. Hence, $\mathbb{Z}$ is fixed by φ. Furthermore, since $\mathbb{Q} \subseteq {}^{\bullet}\mathbb{R}$ and each member of $\mathbb{Q}$ has a *unique* multiplicative inverse, φ fixes $\mathbb{Q}$. Lastly, let $r \in \mathbb{R} \setminus \mathbb{Q}$, and let r_n be some monotonically increasing sequence in $\mathbb{Q}$ such that $r_n \to r$. Fix some $k \in \mathbb{N}$. So, there is some $N \in \mathbb{N}$ such that, for all $m \geq N$, $r - {r_m}^{\bullet} \leq \frac{1}{k}$. Since φ is a homomorphism and fixed on $\mathbb{Q}$, it follows that $\varphi(r) - {r_m}^{*} \leq \frac{1}{k}$. Hence, $r_n \to \varphi(r)$. Therefore, $r = \varphi(r)$. Hence, φ fixes $\mathbb{R}$—it follows that $\varphi|_{\mathbb{R}}$ is the identity. □

Theorem 2.3 *The only nontrivial homomorphism from ${}^{\bullet}\mathfrak{R}$ into ${}^{*}\mathfrak{R}$ is the standard part function in ${}^{\bullet}\mathbb{R}$.*

Proof Suppose that φ is a nontrivial homomorphism. Since $\mathbb{R}$ is fixed by φ, it suffices to consider ${}^{\bullet}\mathbb{R} \setminus \mathbb{R}$. So, let $r \notin \mathbb{R}$. Now put $\varepsilon = (r - {}^{\circ}r)$. So, ε is a (possibly trivial) infinitesimal. Hence, there is some positive integer k such that $\varepsilon^k = 0$. Since φ is a homomorphism, $\varphi(\varepsilon)^k = 0$. Since ${}^{*}\mathbb{R}$ is a field, $\varphi(\varepsilon) = 0$. So, $\varphi(r - {}^{\circ}r) = 0$. Again, since φ is a homomorphism, it follows that $\varphi(r) - \varphi({}^{\circ}r) = 0$. In that case, $\varphi(r) - {}^{\circ}r = 0$, since ${}^{\circ}r \in \mathbb{R}$. So, $\varphi(r) = {}^{\circ}r$. □

3 The View from Giordano's Ring of Fermat Reals

Now, in the opposite direction, one must take care. In order to get any interesting results, one must restrict ${}^{*}\mathbb{R}$ to the limited values. Otherwise, the following holds.

Proposition 3.1 *The only homomorphism from ${}^{*}\mathfrak{R}$ into ${}^{\bullet}\mathfrak{R}$ is trivial.*

Proof Suppose for contradiction that there is some nontrivial homomorphism from ${}^{*}\mathfrak{R}$ into ${}^{\bullet}\mathfrak{R}$. For identical reasons as above, if φ is nontrivial, then $\mathbb{R}$ is fixed. Now, there is some $s \in {}^{*}\mathbb{R}$ such that, for all $r \in \mathbb{R}$, $r^{*} \leq s$. Since φ is a homomorphism, it should follow that, for all $r \in \mathbb{R}$, $r^{\bullet} \leq \varphi(s)$. In that case, there is some $x \in {}^{\bullet}\mathbb{R}$ such that, for all $r \in \mathbb{R}$, $r^{\bullet} \leq x$—a contradiction. There are no unlimited values in ${}^{\bullet}\mathbb{R}$. □

Theorem 3.2 *There is a nontrivial homomorphism from ${}^{*}\mathfrak{R}_L$ into ${}^{\bullet}\mathfrak{R}$ other than the standard part function.*

Proof As before, a homomorphism will fix $\mathbb{R}$. Now, we will define a homomorphism Ψ such that $\Psi(x) \neq \mathrm{sh}(x)$ for some $x \in {}^{*}\mathbb{R}$.

First, let $I \subseteq^{*} \mathbb{R}_L$ be the set of infinitesimals; that is, $I = \{x \in {}^{*}\mathbb{R} : (\forall r \in \mathbb{R})\ 0 \leq x < r\}$. Define the set of nilsquares, $D_2 = \{x \in {}^{\bullet}\mathbb{R} : x^2 = 0\}$. We are interested in a special subclass of D_2, namely, those where the exponential values of the Fermat polynomial are in $(\frac{1}{2}, 1)$. That is, we are interested in

$$\Big\{y \in D_2 : y = \Big[\sum_{i=1}^{n} \alpha_i t^{a_i}\Big]_{\sim},\ a_i \in \Big(\frac{1}{2}, 1\Big)\Big\}. \tag{3}$$

Denote by ${}^{\bullet}D$ the set in (3).

For the moment, consider ${}^{\bullet}D$ and I as vector spaces with scalars in $\mathbb{R}$. Clearly, $\{t^{\alpha} : \alpha \in (\frac{1}{2}, 1)\}$ is an uncountable basis for ${}^{\bullet}D$. Call this $\mathfrak{B}_{{}^{\bullet}D}$. Since I forms a vector space, it has a basis, $\mathfrak{B}_I$. Without loss of generality, suppose that $\mathfrak{B}_I$ contains only positive members of I. Since infinitesimals corresponding to the sequences $\{\frac{1}{n^k}\}_n$ for any positive integer k are clearly linearly independent, $\mathfrak{B}_I$ is at least denumerably infinite. Now, by the axiom of choice, the members of these bases can be well-ordered.[6] So, we will write for simplicity $\mathfrak{B}_I = \{i_0, i_1, \ldots\}$ and $\mathfrak{B}_{{}^\bullet D} = \{d_0, d_1, \ldots\}$. Note that the orders of the well-orderings have nothing to do with either ${}^*{<}$ or ${}^\bullet{<}$. Furthermore, define $\mathfrak{B}_I|_\kappa = \{i_\alpha : \alpha < \kappa\}$. In the following, we will define Ψ in stages to be a homomorphism.

Stage 0. Define $\Psi(x) = x$ whenever $x \in \mathbb{R}$. Since homomorphisms must fix $\mathbb{R}$, we build this in from the beginning.

Stage I. In this stage, we will define Ψ by transfinite induction on $\mathfrak{B}_I$, mapping all of $\mathfrak{B}_I$ into $\mathfrak{B}_{{}^\bullet D} \cup \{0\}$. To ensure that Ψ is well defined, if necessary, reorder $\mathfrak{B}_I$ so that $i_0 = i_\alpha$ where α is the least ordinal such that i_α is not the product of any two members of $\mathfrak{B}_I$. So, on the newest ordering, define $\Psi(i_0) = d_0$ and $\Psi(i_\alpha) = 0$ if $i_\alpha = i_\beta \,{}^*{\times}\, i_\gamma$ for any α, β, γ. Now, we must break the problem into some cases.[7]

Case A. For all $\beta < \alpha$, $i_\alpha \,{}^*{>}\, i_\beta$. In this case, put $\Psi(i_\alpha) = d_\eta$ such that η is the least ordinal such that $d_\eta \in \mathfrak{B}_{{}^\bullet D} \setminus \Psi[\mathfrak{B}_I|_\alpha]$ and $d_\eta \,{}^\bullet{>}\, \Psi(i_\beta)$, for all $\beta < \alpha$. There will always be such a value, because for any $d \in \mathfrak{B}_{{}^\bullet D}$, there is some d' such that $d' \,{}^\bullet{>}\, d$.

Case B. For all $\beta < \alpha$, $i_\alpha \,{}^*{<}\, i_\beta$. In this case, put $\Psi(i_\alpha) = d_\eta$ such that η is the least ordinal such that $d_\eta \in \mathfrak{B}_{{}^\bullet D} \setminus \Psi[\mathfrak{B}_I|_\alpha]$ and $d_\eta \,{}^\bullet{<}\, \Psi(i_\beta)$ (if it exists); otherwise, put $\Psi(i_\alpha) = 0$.

Case C. For some $\beta, \gamma < \alpha$, $i_\gamma {}^*{<} i_\alpha {}^*{<} i_\beta$. This divides into two subcases.

1. There is some pair $\kappa, \lambda < \alpha$ such that $i_\kappa {}^*{<} i_\alpha {}^*{<} i_\lambda$ and $(i_\lambda - i_\kappa)$ is the least among pairs i_β, i_γ satisfying $i_\gamma {}^*{<} i_\alpha {}^*{<} i_\beta$ (and nonzero by definition). Now put $\Psi(i_\alpha) = d_\eta$ such that $\eta \geq \alpha$ is the least ordinal such that $d_\eta \in \mathfrak{B}_{{}^\bullet D} \setminus \Psi[\mathfrak{B}_I|_\alpha]$ and $\Psi(i_\kappa) \,{}^\bullet{<}\, d_\eta \,{}^\bullet{<}\, \Psi(i_\lambda)$. There will always be such a value since $\mathfrak{B}_{{}^\bullet D}$ is dense.
2. No pair κ, λ satisfies the description in subcase 1. This implies that i_α is an accumulation point. Pick some strictly monotonic subsequence $\{i_n\}$ in $\mathfrak{B}_I|_\alpha$ such that $i_n \to i_\alpha$. To isolate our desired value, we must analyze the sequence $\{\omega(\Psi(i_n))\}$.

Claim 3.3 *There is some $c \in (\frac{1}{2}, 1)$ such that $\omega(\Psi(i_n)) \to c$.*

Proof Since, for any n, $\omega(\Psi(i_n)) \in (\frac{1}{2}, 1)$, $\{\omega(\Psi(i_n))\}$ is bounded. Since $\{i_n\}$ is strictly monotonic, by construction $\{\Psi(i_n)\}$ is strictly monotonic. Since $\mathfrak{B}_{{}^\bullet D} = \{t^\alpha : \alpha \in (\frac{1}{2}, 1)\}$ and $t^\alpha \,{}^\bullet{<}\, t^\beta \iff \beta < \alpha$, it follows that $\{\omega(\Psi(i_n))\}$ is strictly monotonic with the opposite ordering of $\{\Psi(i_n)\}$. But now since $\{\omega(\Psi(i_n))\}$ is bounded and strictly monotonic, we know it converges to some $c \in (\frac{1}{2}, 1)$. So, put $\Psi(i_\alpha) = t^c$. □

Stage II. Now, we extend Ψ to all of I by requiring that $\Psi(ri_\alpha) = r\Psi(i_\alpha)$ and that $\Psi(i_\alpha + i_\beta) = \Psi(i_\alpha) + \Psi(i_\beta)$ for any $i_\alpha, i_\beta \in \mathfrak{B}_I$ and $r \in \mathbb{R}$. This is well defined since the bases are (by definition) linearly independent. As before, put $\Psi(i) = 0$ if $i = i_\beta \,{}^*{\times}\, i_\gamma$ for $i_\beta, i_\gamma \in \mathfrak{B}_I$.

Stage III. Now we extend Ψ to all of ${}^*\mathbb{R}_L$ in the obvious way. For $s \in {}^*\mathbb{R}_L$, let $s = r + i$, the unique decomposition where $r \in \mathbb{R}$ and $i \in I$. Therefore, put $\Psi(s) = r + \Psi(i)$. This completes the construction of Ψ. □

Now that the construction is complete, we will show that Ψ is a well-defined homomorphism.

Lemma 3.4 *The function Ψ is well defined.*

Proof Let $s, t \in {}^*\mathbb{R}_L$ and $s = t$. Let $s = r_1 + i_1$ and $t = r_2 + i_2$ for $r_j \in \mathbb{R}$ and $i_j \in I$. Hence, $r_1 + i_1 = r_2 + i_2$. Note that $(r_1 - r_2) + i_1 = i_2$; that is, $(r_1 - r_2) + i_1 \in I$. So, $r_1 - r_2 \in I$. So, $r_1 - r_2 = 0$, since $r_1 - r_2 \in \mathbb{R}$. Now, because $\Psi(r_1 + i_1) = r_1 + \Psi(i_1)$ and $\Psi(r_2 + i_2) = r_2 + \Psi(i_2)$, it follows that

$$\Psi(s) = \Psi(t) \iff r_1 + \Psi(i_1) = r_2 + \Psi(i_2) \iff \Psi(i_1) = \Psi(i_2).$$

Hence, it suffices to show that $\Psi(i_1) = \Psi(i_2)$ for $i_1 = i_2$ to establish that Ψ is well defined. But the truth of this latter fact was noted to hold in Stage II of the construction. □

Lemma 3.5 *The function Ψ is a ring homomorphism.*

Proof For the reader's ease, we will not distinguish addition or multiplication when moving from ${}^*\mathbb{R}_L$ into ${}^\bullet\mathbb{R}$. First, Ψ fixes $\mathbb{R}$, so $\Psi(0) = 0$ and $\Psi(1) = 1$ as necessary. Since Ψ fixes $\mathbb{R}$, note that if $s = r_1 + i_1$ and $t = r_2 + i_2$ (as above), then

$$\Psi(s + t) = \Psi(r_1 + i_1 + r_2 + i_2) = r_1 + r_2 + \Psi(i_1 + i_2).$$

Now, since $\Psi(i_1 + i_2) = \Psi(i_1) + \Psi(i_2)$ is satisfied by construction, $\Psi(s) + \Psi(t)$ follows easily. Similarly,

$$\begin{aligned}\Psi(st) &= \Psi(r_1r_2 + r_1i_2 + r_2i_1 + i_1i_2)\\ &= r_1r_2 + r_1\Psi(i_2) + r_2\Psi(i_1) + \Psi(i_1i_2).\end{aligned} \tag{4}$$

Before proceeding, we must establish that, for any $i_1, i_2 \in I$, $\Psi(i_1i_2) = 0$. Let $i_1, i_2 \in I$, so that $i_1 = s_1j_1 + \cdots + s_mj_m$ and $i_2 = t_1k_1 + \cdots + t_nk_n$ for $m, n \geq 1$ and $j_p, k_q \in \mathfrak{B}_I$. So,

$$\begin{aligned}\Psi(i_1i_2) &= \Psi\big((s_1j_1 + \cdots + s_mj_m)(t_1k_1 + \cdots + t_nk_n)\big)\\ &= \sum_{\substack{1\leq q\leq n\\ 1\leq p\leq m}} s_pt_q \underbrace{\Psi(j_p)\Psi(k_q)}_{=0} = 0.\end{aligned}$$

Now, since $\Psi(i_1i_2) = 0 = \Psi(i_1)\Psi(i_2)$, then in (4), $\Psi(st) = \Psi(s)\Psi(t)$. □

Finally, suppose that $s \mathrel{{}^*{\leq}} t$ where s, t are defined as in the proofs of Lemmas 3.4 and 3.5 above. Either $r_1 < r_2$ or $r_1 = r_2$. If $r_1 < r_2$, then it follows naturally that $\Psi(s) \mathrel{{}^\bullet{\leq}} \Psi(t)$. Otherwise, the result will hold provided that $\Psi(i_1) \mathrel{{}^\bullet{\leq}} \Psi(i_2)$ holds. In other words, it suffices to show that $i_1 \mathrel{{}^*{\leq}} i_2 \implies \Psi(i_1) \mathrel{{}^\bullet{\leq}} \Psi(i_2)$. We establish this final claim in the form of a series of lemmas.

Lemma 3.6 *For any $j, k \in \mathfrak{B}_I$,*

$$j \mathrel{{}^*{<}} k \implies \frac{j}{k} \simeq 0.$$

Proof Let $j, k \in \mathfrak{B}_I$, and suppose that $j \mathbin{{}^*{<}} k$. It follows that $j/k < 1$. Now, $j/k = c$ for some limited $c > 0$. So, $c = r_c + i_c$ where $r_c \in \mathbb{R}$ and $i_c \in I$. So, $j = (r_c + i_c)k$; that is, $j = r_c k + i_c k$. Now, $i_c k \in I$, from which it follows that $i_c k = s_1 i_1 + \cdots + s_n i_n$ for $s_\ell \in \mathbb{R}$ and $i_\ell \in \mathfrak{B}_I$ where $i_\ell \neq k$. Therefore, $j = r_c k + s_1 i_1 + \cdots + s_n i_n$; that is, $0 = -j + r_c k + s_1 i_1 + \cdots + s_n i_n$. But, $j \in \mathfrak{B}_I$, so $r_c = 0$, $s_\ell i_\ell = j$ for some positive integer $\ell \leq n$, and for all $m \neq \ell$, $s_m = 0$. So, $j/k = c = i_c \simeq 0$. □

Lemma 3.7 *Let $n \in \mathbb{N}$. Suppose that, for integers $k \in [1, n]$, $s_k \in \mathbb{R}$ and $i_k \in \mathfrak{B}_I$. Also, suppose that $i_n \mathbin{{}^*{<}} \cdots \mathbin{{}^*{<}} i_1$ and that each $s_k \neq 0$. Then,*

$$0 < s_1 \iff 0 \mathbin{{}^*{<}} \sum_{k=1}^{n} s_k i_k.$$

Proof Fix some $n \in \mathbb{N}$, and suppose that $i_n \mathbin{{}^*{<}} \cdots \mathbin{{}^*{<}} i_1$ and that $i_m \in \mathfrak{B}_I$ for all relevant m. Given that $s_1 \in \mathbb{R}$, $s_1 > 0$ is equivalent to

$$0 \mathbin{{}^*{<}} s_1 + s_2 \underbrace{\left(\frac{i_2}{i_1} \right)}_{\simeq 0} + \cdots + s_n \underbrace{\left(\frac{i_n}{i_1} \right)}_{\simeq 0}. \tag{5}$$

Since $i_1 > 0$, (5) is true if and only if

$$0 \cdot i_1 \mathbin{{}^*{<}} \left[s_1 + s_2 \left(\frac{i_2}{i_1} \right) + \cdots + s_n \left(\frac{i_n}{i_1} \right) \right] \cdot i_1,$$

which is equivalent to $0 \mathbin{{}^*{<}} \sum_{k=1}^{n} s_k i_k$. □

Now we give our final lemma to establish the theorem.

Lemma 3.8 *If $i_1, i_2 \in I$, then*

$$i_1 \mathbin{{}^*{\leq}} i_2 \implies \Psi(i_1) \mathbin{{}^\bullet{\leq}} \Psi(i_2).$$

Proof Suppose that $i_1 \mathbin{{}^*{<}} i_2$. The stronger hypothesis will do, since we already established that it is well defined. Let $i_1 = s_1 j_1 + \cdots + s_m j_m$ and $i_2 = t_1 k_1 + \cdots + t_n k_n$ for $m, n \geq 1$ and $j_p, k_q \in \mathfrak{B}_I$. Also without loss of generality, suppose that each of $j_m \mathbin{{}^*{<}} \cdots \mathbin{{}^*{<}} j_1$ and $k_n \mathbin{{}^*{<}} \cdots \mathbin{{}^*{<}} k_1$. This leaves a few distinct possibilities, based upon what Lemma 3.7 demonstrates about how ${}^*{<}$ works:

1. $m < n$, $t_i = s_i$ and $k_i = j_i$ for all $i \leq m$, and $t_{m+1} > 0$;
2. $s_i < t_i$ where $j_i = k_i$ and $i \leq \min(n, m)$ is the least positive integer such that $s_i j_i \neq t_i k_i$;
3. $s_i, t_i \lesseqgtr 0$, $j_i \mathbin{{}^*{<}} k_i$ where i is as described above;
4. $s_i < 0 < t_i$ where i is described as above.

Fortunately, ${}^\bullet{<}$ works exactly the same. Let us establish a few general things. By construction, $j_m \mathbin{{}^*{<}} \cdots \mathbin{{}^*{<}} j_1$ and $k_n \mathbin{{}^*{<}} \cdots \mathbin{{}^*{<}} k_1$, respectively, imply $\Psi(j_m) \mathbin{{}^\bullet{<}} \cdots \mathbin{{}^\bullet{<}} \Psi(j_1)$ and $\Psi(k_n) \mathbin{{}^\bullet{<}} \cdots \mathbin{{}^\bullet{<}} \Psi(k_1)$. Also by construction, we have $\Psi(i_1) = s_1 \Psi(j_1) + \cdots + s_m \Psi(j_m)$ and $\Psi(i_2) = t_1 \Psi(k_1) + \cdots + t_n \Psi(k_n)$. Let us go case by case.

1. We must establish here that $0 \mathbin{{}^\bullet{<}} t_{m+1} \Psi(k_{m+1}) + \cdots + t_n \Psi(k_n)$. Since $\Psi(k_q) \mathbin{{}^\bullet{<}} \Psi(k_{m+1})$ for integers $q \in (m+1, n]$ and $t_{m+1} > 0$, the result follows.

2. After reducing this case, we are ultimately determining whether
$$s_i\Psi(j_i)+\cdots+s_m\Psi(j_m) \mathrel{\bullet\!\!<} t_i \underbrace{\Psi(j_i)}_{j_i=k_i}+\cdots+t_n\Psi(k_n).$$
Since $j_i = k_i$, $\omega(\Psi(i_1)) = \omega(\Psi(i_2))$, and since $s_i < t_i$, the result follows immediately.
3. Here, since s_i, t_i have the same sign, our result follows given that $\omega(\Psi(j_i)) > \omega(\Psi(k_i))$.
4. In this case, $\Psi(i_1) \mathrel{\bullet\!\!<} 0 \mathrel{\bullet\!\!<} \Psi(i_2)$ from Lemma 3.7. □

4 Conclusion

The purpose of this paper is to demonstrate that Giordano's nilpotents are more fine-grained infinitesimals than their nonnilpotent counterparts within the hyperreals. A hyperreal "sieve" allows all of the nilpotents of Fermat reals to fall through, catching only the real numbers; a Fermat real "sieve" does not allow as much through, catching real numbers and infinitesimals alike.

Notes

1. See Giordano's [1] and [2].

2. Thanks to Professor Giordano for personal correspondence (3 June 2014) and discussion on these matters.

3. Thanks to an anonymous referee for drawing these details to our attention. Extensive discussion is found in [5, Chapter VI].

4. Our presentation of the hyperreals roughly follows the lucid one in Goldblatt [3].

5. Thanks to an anonymous referee for finding an error in this proof.

6. The proof of the existence of the hyperreals requires a principle that is independent of ZF but weaker than ZFC. We do not take it as much of a stretch to assume the axiom of choice given that it is not significantly stronger than the Boolean prime ideal principle. For a careful discussion, see Tech [4].

7. Due to the fact that we will be performing something more like a complete induction, we will not distinguish between successor and limit stages. We will be sensitive to the difference in various subcases, but will not distinguish them in the definition of Ψ.

References

[1] Giordano, P., "Infinitesimals without logic," *Russian Journal of Mathematical Physics*, vol. 17 (2010), pp. 159–91. Zbl 1264.26035. MR 2660917. DOI 10.1134/S1061920810020032. 207, 213

[2] Giordano, P., "The ring of Fermat reals," *Advances in Mathematics*, vol. 225 (2010), pp. 2050–75. Zbl 1205.26051. MR 2680199. DOI 10.1016/j.aim.2010.04.010. 213

[3] Goldblatt, R., *Lectures on the Hyperreals: An Introduction to Nonstandard Analysis*, vol. 188 of *Graduate Texts in Mathematics*, Springer, New York, 1998. Zbl 0911.03032. MR 1643950. DOI 10.1007/978-1-4612-0615-6. 213

[4] Jech, T. J., *The Axiom of Choice*, vol. 75 in *Studies in Logic and the Foundations of Mathematics*, North-Holland, Amsterdam, 1973. Zbl 0259.02051. MR 0396271. 213

[5] Moerdijk, I., and G. E. Reyes, *Models for Smooth Infinitesimal Analysis*, Springer, New York, 1991. MR 1083355. DOI 10.1007/978-1-4757-4143-8. 206, 213

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