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Ramsey Algebras and Formal Orderly Terms

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Abstract Hindman's theorem says that every finite coloring of the natural numbers has a monochromatic set of finite sums. A Ramsey algebra is a structure that satisfies an analogue of Hindman's theorem. In this paper, we present the basic notions of Ramsey algebras by using terminology from mathematical logic. We also present some results regarding classification of Ramsey algebras.

1 Introduction

The set of natural numbers $\{0, 1, 2, \dots\}$ is denoted by ω. Suppose that $\langle x_i \rangle_{i\in\omega}$ is a sequence of natural numbers. Let $\mathrm{FS}(\langle x_i \rangle_{i\in\omega})$ denote $\{\sum_{i\in F} x_i \mid F \in \mathcal{P}_f(\omega) \setminus \{\emptyset\}\}$, where $\mathcal{P}_f(\omega)$ is the set of all finite subsets of ω. Hindman's theorem [8, Theorem 3.1] says that, for every finite partition of the set of positive natural numbers $\mathbb{N} = X_0 \cup X_1 \cup \cdots \cup X_N$, there exists a sequence $\langle x_i \rangle_{i\in\omega}$ of positive natural numbers such that, for some $0 \leq j \leq N$, we have $\mathrm{FS}(\langle x_i \rangle_{i\in\omega}) \subseteq X_j$. In fact, such a sequence $\langle x_i \rangle_{i\in\omega}$ can be chosen to be a sum subsystem of any given sequence $\langle y_i \rangle_{i\in\omega}$ of natural numbers.

To us an algebra is a structure that consists of a set together with a collection of operations on the set. A Ramsey algebra is a structure which possesses the property analogous to that possessed by the semigroup $(\mathbb{N}, +)$ as in Hindman's theorem. The definition of Ramsey algebra was suggested by Carlson and introduced in [14] by using standard set-theoretic notation. In this paper, we will define Ramsey algebras by using the notation and terminology commonly employed in mathematical logic. We give a characterization of Ramsey algebras by using this alternative method.

In the remainder of this section, we give a historical account and motivation for Ramsey algebras. In addition, we point out the connection between Ramsey algebras and idempotent ultrafilters.

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In 1988 Carlson [1] presented an abstract version of Ellentuck's theorem [4, Theorem 9]. He called structures that have properties analogous to those of Ellentuck's space *Ramsey spaces*. The main objects of study there were certain spaces of infinite sequences of multivariable words. In particular, the fact that the spaces of infinite sequences of variable words—single variable—are Ramsey ([1, Theorem 2]), has as corollaries many earlier Ramsey-theoretic results including Hindman's theorem, Ellentuck's theorem, the dual Ellentuck theorem [2, Theorem 4.1], the Galvin–Prikry theorem [6, Theorem 2], and the Hales–Jewett theorem [7, Theorem 1]. Since then, there has been active study of Ramsey spaces (see [15]).

Carlson's interesting spaces of infinite sequences of variable words can be associated to some algebras of variable words. Conversely, every algebra induces a space of infinite sequences under the analogous Ellentuck topology. His abstract Ellentuck's theorem reduces the topological question of whether such a space is Ramsey to a more combinatorial question. The notion of Ramsey algebras is formulated precisely to capture this combinatorial property, and hence, such a space is Ramsey if and only if the associated algebra is Ramsey. This relation between Ramsey algebras and Ramsey spaces is addressed in [14, Section 6].

Therefore, there are interesting Ramsey algebras of variable words which are *not* semigroups. The collection of operations in each of these algebras is finite but can be arbitrarily large depending on the size of the underlying finite alphabet. Because no infinite integral domain—involving two associative binary operations—is a Ramsey algebra [14], there appears to be a nice interplay among the operations in a Ramsey algebra of variable words. This fine interplay allows the construction of certain idempotent ultrafilters, a key feature in Carlson's proof. These ultrafilters in turn allow the construction of sequences with certain homogeneity properties, showing that the corresponding algebra of variable words is Ramsey. This approach generalizes the Galvin–Glazer proof (see [3] or [10]) of Hindman's theorem.

Hindman [9] showed that no ultrafilter on $\mathbb{N}$ is idempotent for addition and multiplication simultaneously. On the other hand, the ultrafilter constructed in Carlson's proof is idempotent for every operation in the corresponding Ramsey algebra of variable words. Furthermore, this author [13] has shown that assuming Martin's axiom every nondegenerate Ramsey algebra has a nonprincipal strongly reductible ultrafilter, analogous to the existence of strongly summable ultrafilters under Martin's axiom (see [11]). Strongly reductible ultrafilters are necessarily idempotent. Hence, a positive answer to the following open problem, which is due to Carlson, is a generalization of the existence of idempotent ultrafilters for a semigroup.

Question Can the existence of idempotent ultrafilters for a Ramsey algebra be proven in ZFC (Zermelo–Fraenkel set theory with the axiom of choice)?

2 Preliminaries

Suppose that A is any set. The set of infinite sequences in A is denoted by ${}^{\omega}A$. The collection of subsets of A of size ω is denoted by $[A]^{\omega}$. For $n \in \omega$, let $[A]^n$ denote the collection of subsets of A of size n.

An *algebra* is a pair $(A, \mathcal{F})$, where A is a nonempty set and $\mathcal{F}$ is a collection of operations on A, none of which is nullary. If f is a binary operation on A, then $(A, \{f\})$ is commonly known as a *groupoid*, and we will write it as (A, f).

We do not assume the reader is familiar with the syntax and semantics of first-order logic. Hence, we will introduce the logical notions that we will use in this paper. As we do not need relation symbols or constant symbols in our languages, our definitions will differ from the standard ones for our purposes. For interested readers, a standard introductory textbook for logic is that of Enderton [5].

We assume that a fixed list of distinct symbols, called the *syntactic variables*, $v_0, v_1, v_2, \ldots$, is given. The *index* of v_i is i. Henceforth, we will simply refer to the v_i's as variables when it is understood.

To us a *language* $\mathcal{L}$ is a set L (disjoint from the set of syntactic variables) along with a function θ from L into the set of positive natural numbers. The elements of the set L are referred to as the *function symbols* of the language $\mathcal{L}$. The image of a function symbol f of $\mathcal{L}$ under θ is referred to as the *arity* of f. The set of *terms* of a language $\mathcal{L}$ is the set of expressions that can be built up by finitely many applications of the following rules:

1. v_i is a term for every $i \in \omega$;
2. if f is an n-ary function symbol of $\mathcal{L}$, then the expression $f t_1 t_2 \cdots t_n$ is a term whenever $t_1, t_2, \ldots, t_n$ are terms.

An $\mathcal{L}$-*algebra* $\mathfrak{A}$ is a pair $(|\mathfrak{A}|, \{f^{\mathfrak{A}}\}_{f\in L})$ such that $|\mathfrak{A}|$ is a set, called the *universe of* $\mathfrak{A}$, and such that, for each function symbol f of $\mathcal{L}$, if f is n-ary, then $f^{\mathfrak{A}}$ is an n-ary operation on $|\mathfrak{A}|$. Suppose that $\mathfrak{A}$ is an $\mathcal{L}$-algebra. An *assignment* is (identified with) an infinite sequence in $|\mathfrak{A}|$. The *interpretation* of a term t under $\mathfrak{A}$ via the assignment $\vec{a}$, denoted by $t^{\mathfrak{A}}[\vec{a}]$, is defined inductively as follows:

1. ${v_i}^{\mathfrak{A}}[\vec{a}] = \vec{a}(i)$ for each variable v_i;
2. if f is an n-ary function symbol of $\mathcal{L}$ and $t_1, \ldots, t_n$ are terms, then

$$(f t_1 \cdots t_n)^{\mathfrak{A}}[\vec{a}] = f^{\mathfrak{A}}\big(t_1^{\mathfrak{A}}[\vec{a}], \ldots, t_n^{\mathfrak{A}}[\vec{a}]\big).$$

Note that $t^{\mathfrak{A}}[\vec{a}]$ depends only on the values of $\vec{a}(i)$ for i such that v_i appears in t. Suppose that the variables appearing in t are exactly $v_{i_1}, \ldots, v_{i_p}$. We will write $t^{\mathfrak{A}}[v_{i_1}, \ldots, v_{i_p} \mid x_1, \ldots, x_p]$ to mean $t^{\mathfrak{A}}[\vec{a}]$ for some (or any) $\vec{a}$ such that $\vec{a}(i_k) = x_k$ for $1 \leq k \leq p$.

Suppose that $\mathfrak{A}$ is an algebra $(A, \mathcal{F})$. Note that $\mathfrak{A}$ can be regarded as an $\mathcal{L}$-algebra for some language $\mathcal{L}$. More explicitly, for each $f \in \mathcal{F}$ let $\underline{f}$ denote $\{f\}$. Note that $\underline{f}$ and $\underline{g}$ are distinct whenever f and g are distinct. Let $\mathcal{L}_{\mathcal{F}}$ denote the language with $\{\, \underline{f} \mid f \in \mathcal{F}\,\}$ being the collection of function symbols such that $\underline{f}$ has the same arity as f whenever $f \in \mathcal{F}$. We will identify $\mathfrak{A}$ with an $\mathcal{L}_{\mathcal{F}}$-algebra $\mathfrak{B}$ with universe A such that $\underline{f}^{\mathfrak{B}} = f$ for every function symbol $\underline{f}$ of $\mathcal{L}_{\mathcal{F}}$.

3 Orderly Terms

In this section, we define orderly terms and Ramsey algebras.

Definition 3.1 Suppose that $(A, \mathcal{F})$ is an algebra. An operation f on A is an *orderly composition* of $\mathcal{F}$ if and only if there exist $g, h_1, \ldots, h_n \in \mathcal{F}$ such that $f(\bar{x}_1, \ldots, \bar{x}_n) = g(h_1(\bar{x}_1), \ldots, h_n(\bar{x}_n))$.[1] We say that $\mathcal{F}$ is *closed under orderly composition* if and only if $f \in \mathcal{F}$ whenever f is an orderly composition of $\mathcal{F}$. The collection of *orderly terms* over $\mathcal{F}$ is the smallest collection of operations on A that contains $\mathcal{F}$ and the identity function on A and that is closed under orderly composition.

Equivalently, the collection of orderly terms over $\mathscr{F}$ is the collection of operations on A that can be generated by finitely many applications of the following rules:

1. the identity function on A is an orderly term;
2. every operation in $\mathscr{F}$ is an orderly term;
3. if f is an operation on A given by $f(\bar{x}_1, \ldots, \bar{x}_n) = g(h_1(\bar{x}_1), \ldots, h_n(\bar{x}_n))$ for some $g \in \mathscr{F}$ and some orderly terms $h_1, \ldots, h_n$, then f is an orderly term.

Remark 3.2 The definition of orderly composition is due to Carlson [1]. In fact, Carlson's definition is more general because it is defined for any heterogeneous algebra, that is, an indexed collection of distinct sets with a collection of operations on it.

Before introducing the syntactical version of orderly terms, we try to shed some light on orderly composition. Fix a finite alphabet/set L and a distinct variable v not contained in L. A *variable word* of L is a finite sequence w of elements of $L \cup \{v\}$ such that the variable v occurs at least once in w. Denote the set of variable words of L by W. Assume that $w \in W$ and $a \in L \cup \{v\}$. Then $w(a)$ is the result of replacing every occurrence of v in w by a. In particular, $w(v) = w$. A particular Ramsey algebra in [1] has the form $(W, \mathscr{F})$ for some $\mathscr{F}$, where an n-ary operation f on W is an orderly term over $\mathscr{F}$ if and only if there exist $a_1, \ldots, a_n \in L \cup \{v\}$ (with v among the list of a_i's) such that

$$f(s_1, \ldots, s_n) = s_1(a_1) * \cdots * s_n(a_n).$$

Notice that f is composed from the concatenation operation $*$ and the unary functions $w \to w(a)$ for $a \in L$ in some orderly fashion.

The general notion of idempotentness of ultrafilters is not needed in this paper. Nevertheless, it is worth pointing out that idempotentness of ultrafilters is preserved under orderly composition. In other words, if an ultrafilter is idempotent for a collection of operations $\mathscr{F}$, then it is idempotent for the collection of orderly terms over $\mathscr{F}$ (see [1, Lemma 3.7]).

Definition 3.3 Suppose that $\mathscr{L}$ is a language. An *orderly term* of $\mathscr{L}$ is a term of $\mathscr{L}$ such that the indices of the variables appearing in it from left to right are strictly increasing. The set of orderly terms of $\mathscr{L}$ is denoted by $\mathrm{OT}(\mathscr{L})$.

The next proposition justifies that the two usages of orderly terms are compatible. Before that, we need a definition.

Definition 3.4 Suppose that $\mathfrak{A}$ is an $\mathscr{L}$-algebra and that t is an orderly term of $\mathscr{L}$ such that the variables appearing in t from left to right are exactly $v_{i_1}, \ldots, v_{i_p}$. Let φ_t denote the p-ary operation on $|\mathfrak{A}|$ given by $\varphi_t(x_1, \ldots, x_p) = t^{\mathfrak{A}}[v_{i_1}, \ldots, v_{i_p} \mid x_1, \ldots, x_p]$ for all $x_1, \ldots, x_p \in |\mathfrak{A}|$.

Proposition 3.5 *Suppose that $\mathfrak{A}$ is an algebra $(A, \mathscr{F})$, and suppose that f is an operation on A. The following are equivalent.*

1. *f is a p-ary orderly term over $\mathscr{F}$.*
2. *$f = \varphi_t$ for some orderly term t of $\mathscr{L}_{\mathscr{F}}$ such that p many variables appear in t.*
3. *For every list of p distinct variables $v_{i_1}, v_{i_2}, \ldots, v_{i_p}$, there exists an orderly term t of $\mathscr{L}_{\mathscr{F}}$ such that the variables appearing in t are exactly $v_{i_1}, v_{i_2}, \ldots, v_{i_p}$ and $f = \varphi_t$.*

Proof (3 ⇒ 2) This is immediate.

(2 ⇒ 1) It suffices to show that φ_t is an orderly term over $\mathcal{F}$ whenever $t \in \mathrm{OT}(\mathcal{L}_{\mathcal{F}})$. We prove this by induction on the complexity of the orderly term t. Suppose that t is a variable. Then φ_t is the identity function on A and, hence, trivially is an orderly term over $\mathcal{F}$. Suppose that $t = \underline{g}t_1 \cdots t_n$ for some n-ary (possibly unary) function symbol $\underline{g}$ of $\mathcal{L}_{\mathcal{F}}$ and some orderly terms $t_1, \ldots, t_n$ of $\mathcal{L}_{\mathcal{F}}$ such that $t_1 < \cdots < t_n$. By the induction hypothesis, φ_{t_i} is an orderly term over $\mathcal{F}$ for each $1 \le i \le n$. Suppose that, for each $1 \le i \le n$, we have a list $\bar{v}_i$ of the variables appearing in t_i from left to right. Clearly, $\bar{v}_1, \ldots, \bar{v}_n$ is a list of the variables appearing in t from left to right. Thus, we have

$$\begin{aligned}\varphi_t(\bar{x}_1, \ldots, \bar{x}_n) &= t^{\mathfrak{A}}[\bar{v}_1, \ldots, \bar{v}_n \mid \bar{x}_1, \ldots, \bar{x}_n] \\ &= \underline{g}^{\mathfrak{A}}\big(t_1^{\mathfrak{A}}[\bar{v}_1 \mid \bar{x}_1], \ldots, t_n^{\mathfrak{A}}[\bar{v}_n \mid \bar{x}_n]\big) \\ &= g\big(\varphi_{t_1}(\bar{x}_1), \ldots, \varphi_{t_n}(\bar{x}_n)\big).\end{aligned}$$

Therefore, φ_t is an orderly term over $\mathcal{F}$, since $g \in \mathcal{F}$ and the collection of orderly terms is closed under orderly composition.

(1 ⇒ 3) We prove this by induction on the generation of orderly terms over $\mathcal{F}$. If f is the identity function on A, then take t to be the variable v_{i_1}. Likewise, if f is a p-ary operation in $\mathcal{F}$, then we can take t to be $\underline{f} v_{i_1} \cdots v_{i_p}$. Now, suppose that $f(\bar{x}_1, \ldots, \bar{x}_n) = g(h_1(\bar{x}_1), \ldots, h_n(\bar{x}_n))$ for some $g \in \mathcal{F}$ and some orderly terms $h_1, \ldots, h_n$ over $\mathcal{F}$. Suppose that $\bar{v}_1, \ldots, \bar{v}_n$ is a list of variables with increasing indices such that for each $1 \le i \le n$ the length of $\bar{v}_i$ equals the arity of h_i. By the induction hypothesis, for each $1 \le i \le n$ we can choose $t_i \in \mathrm{OT}(\mathcal{L}_{\mathcal{F}})$ such that $h_i(\bar{x}_i) = \varphi_{t_i}(\bar{x}_i) = t_i{}^{\mathfrak{A}}[\bar{v}_i \mid \bar{x}_i]$, and $\bar{v}_i$ is exactly the list of variables appearing in t_i from left to right. Take t to be $\underline{g}t_1 \cdots t_n$. It is clear that t is an orderly term of $\mathcal{L}_{\mathcal{F}}$ as $t_1 < \cdots < t_n$. Furthermore, $f(\bar{x}_1, \ldots, \bar{x}_n) = \underline{g}^{\mathfrak{A}}(t_1^{\mathfrak{A}}[\bar{v}_1 \mid \bar{x}_1], \ldots, t_n^{\mathfrak{A}}[\bar{v}_n \mid \bar{x}_n]) = t^{\mathfrak{A}}[\bar{v}_1, \ldots, \bar{v}_n \mid \bar{x}_1, \ldots, \bar{x}_n]$. Therefore, $f = \varphi_t$. □

Definition 3.6 Suppose that t, t' are orderly terms of $\mathcal{L}$. Define $t < t'$ to mean that the index of the last variable in t is less than the index of the first variable in t'.

Definition 3.7 Suppose that $\mathfrak{A}$ is an algebra $(A, \mathcal{F})$, and suppose that $\vec{a}$, $\vec{b}$ are infinite sequences in A. We say that $\vec{a}$ is a *reduction* of $\vec{b}$ (with respect to $\mathcal{F}$), and write $\vec{a} \le_{\mathcal{F}} \vec{b}$ if and only if there exists an infinite sequence $\vec{t}$ in $\mathrm{OT}(\mathcal{L}_{\mathcal{F}})$ that is $<$-increasing such that $\vec{a}(k) = \vec{t}(k)^{\mathfrak{A}}[\vec{b}]$ for all $k \in \omega$.

Remark 3.8 Definition 3.7 is a rephrase, using the syntactical orderly terms, of the definition of a reduction that appears in [14, Definition 3.2], which itself is a special case of [1, Definition 4.11].

It is easy to check that $\le_{\mathcal{F}}$ is a reflexive and transitive relation on ${}^{\omega}A$.

Our definition of $\le_{\mathcal{F}}$ is equivalent to a special case of the one given in [1, Definition 4.11], where the collection of operations contains all projections.

Definition 3.9 Suppose that $\mathfrak{A}$ is an algebra $(A, \mathcal{F})$, and suppose that $\vec{a}$ is an infinite sequence in A. The set $\mathrm{FR}_{\mathcal{F}}(\vec{a})$ of *finite reductions* of $\vec{a}$ (with respect to $\mathcal{F}$) is defined by

$$\mathrm{FR}_{\mathcal{F}}(\vec{a}) = \big\{ t^{\mathfrak{A}}[\vec{a}] \bigm| t \in \mathrm{OT}(\mathcal{L}_{\mathcal{F}}) \big\}.$$

Definition 3.10 Suppose that $(A, \mathcal{F})$ is an algebra. We say that $(A, \mathcal{F})$ is *Ramsey* if and only if, for every $\vec{a} \in {}^{\omega}A$ and every $X \subseteq A$, there exists $\vec{b} \leq_{\mathcal{F}} \vec{a}$ such that $\vec{b}$ is *homogeneous* for X (with respect to $\mathcal{F}$); that is, $\mathrm{FR}_{\mathcal{F}}(\vec{b})$ is either contained in or disjoint from X.

When $\mathcal{F} = \{f\}$, we will write $\leq_f$ for $\leq_{\{f\}}$, $\mathcal{L}_f$ for $\mathcal{L}_{\{f\}}$, and $\mathrm{FR}_f(\vec{a})$ for $\mathrm{FR}_{\{f\}}(\vec{a})$.

The following (see [12, Part V, Section 2]) is a consequence of Hindman's theorem.

Theorem 3.11 *Every semigroup is a Ramsey algebra. Hence, every group is a Ramsey algebra.*

4 Main Results

Definition 4.1 Suppose that $\mathfrak{A}$ is a groupoid (A, f), and suppose that $\vec{a} \in {}^{\omega}A$. We say that f is *orderly associative* on $\vec{a}$ if and only if, for every $t_1, t_2, t_3 \in \mathrm{OT}(\mathcal{L}_f)$ such that $t_1 < t_2 < t_3$, we have

$$(\underline{f}\,\underline{f}t_1t_2t_3)^{\mathfrak{A}}[\vec{a}] = (\underline{f}t_1\underline{f}t_2t_3)^{\mathfrak{A}}[\vec{a}].$$

Lemma 4.2 *Suppose that $\mathfrak{A}$ is a groupoid (A, f), that $\vec{a} \in {}^{\omega}A$, and that f is orderly associative on $\vec{a}$. Suppose that $s, t \in \mathrm{OT}(\mathcal{L}_f)$. If the same variables appear in both terms, then $s^{\mathfrak{A}}[\vec{a}] = t^{\mathfrak{A}}[\vec{a}]$.*

Proof We proceed by induction on the number of variables appearing in t. The result is trivial if a single variable appears in t because then both s and t are equal to the variable itself. Suppose that the variables appearing in t are exactly $v_{i_0}, v_{i_1}, \ldots, v_{i_n}$, where $i_0 < i_1 < \cdots < i_n$. It suffices to show that $t^{\mathfrak{A}}[\vec{a}] = (\underline{f}v_{i_0}t')^{\mathfrak{A}}[\vec{a}]$ for some orderly term t' (over $\mathcal{L}_f$)[2] such that the variables appearing in t' are exactly $v_{i_1}, v_{i_2}, \ldots, v_{i_n}$. This is sufficient because $(\underline{f}v_{i_0}t')^{\mathfrak{A}}[\vec{a}] = f(\vec{a}(i_0), t'^{\mathfrak{A}}[\vec{a}])$ and, by the induction hypothesis, $t'^{\mathfrak{A}}[\vec{a}]$ is independent of t' provided that the variables appearing in t' are exactly $v_{i_1}, v_{i_2}, \ldots, v_{i_n}$. Now, t must be equal to $\underline{f}t_1t_2$ for some orderly terms t_1 and t_2. If t_1 is v_{i_0}, then we can take t' to be t_2. Otherwise, by the induction hypothesis, $t_1^{\mathfrak{A}}[\vec{a}] = (\underline{f}v_{i_0}t_1')^{\mathfrak{A}}[\vec{a}]$ for every orderly term t_1' such that the variables appearing in t_1' are exactly those appearing in t_1, excluding v_{i_0}. Choose one such t_1'. Since f is orderly associative on $\vec{a}$, we have $t^{\mathfrak{A}}[\vec{a}] = f(t_1^{\mathfrak{A}}[\vec{a}], t_2^{\mathfrak{A}}[\vec{a}]) = f((\underline{f}v_{i_0}t_1')^{\mathfrak{A}}[\vec{a}], t_2^{\mathfrak{A}}[\vec{a}]) = (\underline{f}\,\underline{f}v_{i_0}t_1't_2)^{\mathfrak{A}}[\vec{a}] = (\underline{f}\,v_{i_0}\underline{f}t_1't_2)^{\mathfrak{A}}[\vec{a}]$. We can take t' to be $\underline{f}t_1't_2$. □

In other words, say, the groupoid operation is denoted by multiplication, orderly associativity on $\langle a_0, a_1, a_2, \ldots \rangle$ implies that the product $a_{i_0}a_{i_1} \cdots a_{i_n}$ is independent of the bracketing whenever $i_0 < i_1 < \cdots < i_n$.

The property that f being orderly associative on a sequence $\langle a_0, a_1, a_2, \ldots \rangle$ is not equivalent to the property that $f(f(a_i, a_j), a_k) = f(a_i, f(a_j, a_k))$ whenever $i < j < k$. Here is a simple example. Suppose that $f(1,1) = 2$ and $f(1,2) = f(2,1) = f(2,2) = 1$. Then $f(f(1,1),1) = f(1, f(1,1))$. Nevertheless, f is not orderly associative on $\langle 1, 1, 1, \ldots \rangle$ because $f(f(f(1,1),1),1) \neq f(f(1,1), f(1,1))$.

Theorem 4.3 *Suppose that $\mathfrak{A}$ is a groupoid (A, f), that $\vec{a} \in {}^{\omega}A$, and that f is orderly associative on $\vec{a}$. Then for every $X \subseteq A$, there exists $\vec{b} \leq_f \vec{a}$ such that $\vec{b}$ is homogeneous for X.*

Proof Suppose that $\vec{a} = \langle a_i \rangle_{i \in \omega}$ is as stated. Let $T \colon \mathcal{P}_f(\omega) \backslash \{\emptyset\} \to A$ be the function defined by $T(\{i_0, i_1, \ldots, i_n\}) = f(a_{i_0}, f(a_{i_1}, \cdots f(a_{i_{n-1}}, a_{i_n}) \cdots))$, where $i_0 < i_1 < \cdots < i_n$, with the understanding that $T(\{i_0\}) = a_{i_0}$. Fix $X \subseteq A$. Since $(\mathcal{P}_f(\omega) \backslash \{\emptyset\}, \cup)$ is a semigroup, it is a Ramsey algebra by Theorem 3.11. Hence, we can choose a reduction $\langle S_i \rangle_{i \in \omega}$ of $\langle \{0\}, \{1\}, \{2\}, \ldots \rangle$ with respect to $\cup$ that is homogeneous for $T^{-1}[X]$. Note that $\max S_i < \min S_j$ whenever $i < j$. Take $\vec{b}(i) = T(S_i)$ for all $i \in \omega$. It is easy to see that $\vec{b} \leq_f \vec{a}$. The homogeneity of $\vec{b}$ for X follows immediately from the homogeneity of $\langle S_i \rangle_{i \in \omega}$ for $T^{-1}[X]$ and the following claim, since it implies that $\mathrm{FR}_f(\vec{b}) \subseteq T[\mathrm{FR}_{\cup}(\langle S_i \rangle_{i \in \omega})]$.

Claim *Suppose that t is an orderly term of the language $\mathcal{L}_f$. Then $t^{\mathfrak{A}}[\vec{b}] = T(S)$, where $S = \bigcup\{ S_i \mid v_i$ appears in $t \}$.*

We will prove the claim inductively on the complexity of the orderly term t. The conclusion holds if t is a variable, say v_i, as ${v_i}^{\mathfrak{A}}[\vec{b}] = \vec{b}(i) = T(S_i)$. Suppose that t is equal to $\underline{f} t_1 t_2$ for some orderly terms t_1 and t_2 such that $t_1 < t_2$. By the induction hypothesis, $t_1^{\mathfrak{A}}[\vec{b}] = T(S')$ and $t_2^{\mathfrak{A}}[\vec{b}] = T(S'')$, where $S' = \bigcup\{ S_i \mid v_i \text{ appears in } t_1 \}$ and $S'' = \bigcup\{ S_i \mid v_i \text{ appears in } t_2 \}$. Since $t_1 < t_2$, by the choice of $\langle S_i \rangle_{i \in \omega}$, we have $\max S' < \min S''$. Now, $t^{\mathfrak{A}}[\vec{b}] = (\underline{f} t_1 t_2)^{\mathfrak{A}}[\vec{b}] = f(t_1^{\mathfrak{A}}[\vec{b}], t_2^{\mathfrak{A}}[\vec{b}]) = f(T(S'), T(S''))$. Since f is orderly associative on $\vec{a}$ and $\max S' < \min S''$, by Lemma 4.2, we have $f(T(S'), T(S'')) = T(S' \cup S'')$. It remains to see that $S' \cup S'' = \bigcup\{ S_i \mid v_i \text{ appears in } t \}$. □

Corollary 4.4 *Suppose that (A, f) is a groupoid. Suppose that, for every $\vec{a} \in {}^{\omega}A$, there exists $\vec{b} \leq_f \vec{a}$ such that f is orderly associative on $\vec{b}$. Then (A, f) is a Ramsey algebra.*

Proof This follows from Theorem 4.3 and the transitivity of $\leq_f$. □

Example 4.5 Suppose that $C \subseteq \omega^2$. Let

$$f(x, y) = \begin{cases} x, & (x, y) \in C, \\ y, & (x, y) \in \omega^2 \backslash C. \end{cases}$$

The groupoid (ω, f) is trivially Ramsey. To see this, suppose that $X \subseteq \omega$ and $\vec{a} \in {}^{\omega}\omega$. By the pigeonhole principle, choose a subsequence $\vec{b}$ of $\vec{a}$ that is either a sequence in X or a sequence in $\omega \backslash X$. It follows that $\vec{b}$ is a reduction of $\vec{a}$ homogeneous for X because $\mathrm{FR}_f(\vec{b}) = \{ \vec{b}(i) \mid i \in \omega \}$.

Alternatively, (ω, f) is Ramsey due to Corollary 4.4. Suppose that $\vec{a} = \langle a_i \rangle_{i \in \omega} \in {}^{\omega}\omega$. We will find $\vec{b} \leq_f \vec{a}$ such that f is orderly associative on $\vec{b}$. Consider the coloring $c \colon [\omega]^2 \to \{0, 1\}$ defined by

$$c(\{i, j\}) = \begin{cases} 0, & (a_i, a_j) \in C, \\ 1, & (a_i, a_j) \in \omega^2 \backslash C, \end{cases}$$

where $i < j$. By the Ramsey theorem for pairs, we can choose $H \in [\omega]^{\omega}$ such that $[H]^2$ is monochromatic. Suppose that $i_0 < i_1 < i_2 < \cdots$ is an increasing enumeration of the elements of H. Let $\vec{b}(k) = a_{i_k}$ for all $k \in \omega$. Then $\vec{b}$, being a subsequence of $\vec{a}$, is a reduction of $\vec{a}$. It remains to show that f is orderly associative on $\vec{b}$. We may assume $[H]^2$ is colored 0; that is, $c(\{i, j\}) = 0$ whenever $i, j \in H$ such that $i < j$. Then it is not hard to prove by induction on the complexity of the orderly term that if v_k is the first variable appearing in an orderly term t, then $t^{\mathfrak{A}}[\vec{b}] = \vec{b}(k)$. Thus, f is orderly associative on $\vec{b}$.

Corollary 4.4 implies and Example 4.5 illustrates that a nonassociative groupoid can be a Ramsey algebra due to orderly associativity on sequences. Hence, the class of Ramsey algebras would not be more appealing than the class of semigroups if the converse of Corollary 4.4 holds. This is not the case, as our next example shows.

Example 4.6 Let $f\colon \omega^2 \times \omega^2 \to \omega^2$ be defined by $f((x_1, y_1), (x_2, y_2)) = (y_2, x_1)$. First, we show that (ω^2, f) is a Ramsey algebra. Fix a sequence $\langle (a_i, b_i)\rangle_{i\in\omega}$ in ω^2 and a subset X of ω^2. Consider the coloring $c\colon [\omega]^2 \to \{0, 1\}$ defined by

$$c(\{i, j\}) = \begin{cases} 0, & (a_j, a_i) \in X, \\ 1, & (a_j, a_i) \in \omega^2 \setminus X, \end{cases}$$

where $i < j$. By the Ramsey theorem for pairs, choose $H \in [\omega]^\omega$ such that $[H]^2$ is monochromatic. Suppose that $i_0 < i_1 < i_2 < \cdots$ is an increasing enumeration of the elements of H. We may assume that $i_{k+1} - i_k \geq 2$. We claim that $\langle (a_{i_{2k+1}}, a_{i_{2k}})\rangle_{k\in\omega}$ is a reduction of $\langle (a_i, b_i)\rangle_{i\in\omega}$ that is homogeneous for X.

To show homogeneity, we may assume that $[H]^2$ is colored 0. It is easy to verify that every finite reduction of $\langle (a_{i_{2k+1}}, a_{i_{2k}})\rangle_{k\in\omega}$ is of the form $(a_{i_j}, a_{i_{j'}})$, where $j > j'$. Therefore, by the color of $[H]^2$, we have $\mathrm{FR}_f(\langle (a_{i_{2k+1}}, a_{i_{2k}})\rangle_{k\in\omega}) \subseteq X$. Meanwhile, since $f((x_1, y_1), f((x_2, y_2), (x_3, y_3))) = (x_2, x_1)$, it is not hard to show that $\langle (a_{i_{2k+1}}, a_{i_{2k}})\rangle_{k\in\omega}$ is a reduction of $\langle (a_i, b_i)\rangle_{i\in\omega}$, provided the set of i_k's is sparse enough. This is achieved by our convenient assumption that $i_{k+1} - i_k \geq 2$.

Suppose that $\langle (a_i, b_i)\rangle_{i\in\omega}$ is a sequence in ω^2 such that $b_0 < a_0 < b_1 < a_1 < b_2 < a_2 < \cdots$. We will show that f is not orderly associative on any reduction $\langle (c_i, d_i)\rangle_{i\in\omega}$ of $\langle (a_i, b_i)\rangle_{i\in\omega}$. Suppose that $\langle (c_i, d_i)\rangle_{i\in\omega}$ is such a reduction. Then it is not hard to see that $d_0 < c_0 < d_1 < c_1 < d_2 < c_2 < \cdots$. Therefore, $(d_3, d_2) = f(f((c_1, d_1), (c_2, d_2)), (c_3, d_3)) \neq f((c_1, d_1), f((c_2, d_2), (c_3, d_3))) = (c_2, c_1)$.

Definition 4.7 Suppose that $\mathfrak{A}$ is an algebra $(A, \mathcal{F})$, that $\vec{a} \in {}^{\omega}A$, and that $X \subseteq A$. We say that $\vec{a}$ is *prehomogeneous* for X (with respect to $\mathcal{F}$) if and only if, for every $t_1, t_2 \in \mathrm{OT}(\mathcal{L}_{\mathcal{F}})$ such that the same variables appear in both orderly terms, we have

$$t_1^{\mathfrak{A}}[\vec{a}] \in X \qquad \text{if and only if} \qquad t_2^{\mathfrak{A}}[\vec{a}] \in X.$$

If f is orderly associative on $\vec{a}$, then for each $X \subseteq A$ the sequence $\vec{a}$ is clearly prehomogeneous for X. As far as finding homogeneous sequences is concerned, the following theorem and its corollary suggest that prehomogeneity is an optimal generalization of orderly associativity.

Theorem 4.8 *Suppose that $\mathfrak{A}$ is an algebra $(A, \mathcal{F})$ such that $\mathcal{F}$ contains a binary operation on A. Suppose that $\vec{a} \in {}^{\omega}A$ and $X \subseteq A$ such that $\vec{a}$ is prehomogeneous for X. Then there exists $\vec{b} \leq_{\mathcal{F}} \vec{a}$ such that $\vec{b}$ is homogeneous for X.*

Proof The proof is similar to that of Theorem 4.3. Fix $\vec{a} \in {}^{\omega}A$ and $X \subseteq A$. Consider a subset Y of $\mathcal{P}_f(\omega)\backslash\{\emptyset\}$ defined as follows: for $i_0 < i_1 < \cdots < i_m$, the set $\{i_0, i_1, \ldots, i_m\}$ is in Y if and only if $t^{\mathfrak{A}}[\vec{a}] \in X$ for every orderly term t such that the variables appearing in t are exactly $v_{i_0}, v_{i_1}, \ldots, v_{i_m}$. Since $(\mathcal{P}_f(\omega)\backslash\{\emptyset\}, \cup)$ is a semigroup, it is a Ramsey algebra. Hence, we can choose a reduction $\langle S_i \rangle_{i \in \omega}$ of $\langle \{0\}, \{1\}, \{2\}, \ldots \rangle$ with respect to $\cup$ that is homogeneous for Y. For each $i \in \omega$, choose an orderly term s_i such that the set of indices of the variables appearing in s_i is exactly S_i. Such s_i exists since there is a binary function symbol in $\mathcal{L}_{\mathcal{F}}$. Take $\vec{b}(i) = s_i^{\mathfrak{A}}[\vec{a}]$ for all $i \in \omega$. It is clear that $\vec{b} \leq_f \vec{a}$ since $s_0 < s_1 < s_2 < \cdots$.

Claim *Suppose that t is an orderly term of $\mathcal{L}_{\mathcal{F}}$. Then $t^{\mathfrak{A}}[\vec{b}] = t'^{\mathfrak{A}}[\vec{a}]$, for some orderly term t' such that the set of indices of the variables appearing in t' is $\bigcup\{ S_i \mid v_i$ appears in $t \}$.*

We will prove the claim inductively on the complexity of the orderly term t. The conclusion holds if t is a variable, say v_i, as $v_i{}^{\mathfrak{A}}[\vec{b}] = \vec{b}(i) = s_i^{\mathfrak{A}}[\vec{a}]$. Suppose that $t = \underline{f} t_1 \cdots t_n$ for some n-ary function symbol $\underline{f}$ of $\mathcal{L}_{\mathcal{F}}$ and some orderly terms $t_1, \ldots, t_n$ of $\mathcal{L}_{\mathcal{F}}$ such that $t_1 < \cdots < t_n$. By the induction hypothesis, for each $1 \leq j \leq n$, we have $t_j^{\mathfrak{A}}[\vec{b}] = t_j'^{\mathfrak{A}}[\vec{a}]$ for some orderly term t_j' such that the set of indices of the variables appearing in t_j' is $S^{(j)} = \bigcup\{ S_i \mid v_i$ appears in $t_j \}$. Since $t_1 < \cdots < t_n$, by the choice of $\langle S_i \rangle_{i \in \omega}$, we have $\max S^{(j_1)} < \min S^{(j_2)}$ whenever $1 \leq j_1 < j_2 \leq n$ and so $t_1' < \cdots < t_n'$. Now, $t^{\mathfrak{A}}[\vec{b}] = (\underline{f} t_1 \cdots t_n)^{\mathfrak{A}}[\vec{b}] = f(t_1^{\mathfrak{A}}[\vec{b}], \ldots, t_n^{\mathfrak{A}}[\vec{b}]) = f(t_1'^{\mathfrak{A}}[\vec{a}], \ldots, t_n'^{\mathfrak{A}}[\vec{a}]) = (\underline{f} t_1' \cdots t_n')^{\mathfrak{A}}[\vec{a}]$. We can take t' to be $\underline{f} t_1' \cdots t_n'$. It remains to observe that the set of indices of the variables appearing in t' is $S^{(1)} \cup \cdots \cup S^{(n)}$, which is equal to $\bigcup\{ S_i \mid v_i$ appears in $t \}$. The claim is proved.

Finally, to see that $\vec{b}$ is homogeneous for X, we may assume $\mathrm{FR}_{\cup}(\langle S_i \rangle_{i \in \omega})$ is disjoint from Y. (The other case is similar and easier.) We will show that $\mathrm{FR}_{\mathcal{F}}(\vec{b})$ is disjoint from X. Suppose that t is an orderly term. By the claim, $t^{\mathfrak{A}}[\vec{b}] = t'^{\mathfrak{A}}[\vec{a}]$ for some orderly term t' such that the set of indices of the variables appearing in t' is $S = \bigcup\{ S_i \mid v_i$ appears in $t \}$. Clearly, S is in $\mathrm{FR}_{\cup}(\langle S_i \rangle_{i \in \omega})$ and hence is not in Y. Therefore, $t''^{\mathfrak{A}}[\vec{a}] \notin X$ for some orderly term t'' such that the set of indices of the variables appearing in t'' is exactly S. Since $\vec{a}$ is prehomogeneous for X, we know $t'^{\mathfrak{A}}[\vec{a}] \in X$ if and only if $t''^{\mathfrak{A}}[\vec{a}] \in X$. It follows that $t^{\mathfrak{A}}[\vec{b}] \notin X$ as required. □

Of course, Theorem 4.3 is a corollary of Theorem 4.8.

Corollary 4.9 *Suppose that $(A, \mathcal{F})$ is an algebra such that $\mathcal{F}$ contains a binary operation on A. Then $(A, \mathcal{F})$ is Ramsey if and only if, for every $\vec{a} \in {}^{\omega}A$ and every $X \subseteq A$, there exists $\vec{b} \leq_{\mathcal{F}} \vec{a}$ such that $\vec{b}$ is prehomogeneous for X.*

Proof The forward direction follows from the definition of a Ramsey algebra and the simple fact that a homogeneous sequence is prehomogeneous. The backward direction is obtained by Theorem 4.8 and the transitivity of $\leq_{\mathcal{F}}$. □

Remark 4.10 The assumption that $\mathcal{F}$ contains a binary operation on A in Corollary 4.9 is necessary. The algebra $(\omega, +_3)$, where $+_3(x, y, z) = x + y + z$, is not Ramsey, although for every $X \subseteq \omega$, every infinite sequence $\vec{a}$ of natural numbers is clearly prehomogeneous for X.

5 Conclusion

Certainly, one would aim for a complete and natural classification of Ramsey algebras. Our characterization of Ramsey algebras in this paper is a step forward toward that goal. A further direction for research is to look into the construction of new Ramsey algebras from the known ones. In particular, we hope to address whether the Cartesian product of two Ramsey groupoids is Ramsey.

Notes

1. For notational convenience, we will use a symbol with a bar over it to indicate a list.

2. Henceforth, to increase readability, the underlying language will be omitted when it is understood from the context.

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