Notre Dame Journal of Formal Logic
Volume 57, Number 4, 2016

Modal Consequence Relations Extending S4.3: An Application of Projective Unification

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Abstract We characterize all finitary consequence relations over **S4.3**, both syntactically, by exhibiting so-called (admissible) passive rules that extend the given logic, and semantically, by providing suitable strongly adequate classes of algebras. This is achieved by applying an earlier result stating that a modal logic L extending **S4** has *projective unification* if and only if L contains **S4.3**. In particular, we show that these consequence relations enjoy the *strong finite model property*, and are *finitely based*. In this way, we extend the known results by Bull and Fine, from logics, to consequence relations. We also show that the lattice of consequence relations over **S4.3** (the lattice of quasivarieties of **S4.3**-algebras) is countable and *distributive* and it forms a Heyting algebra.

1 Introduction

Modal logics extending **S4.3** form a subset of the lattice of all modal logics in which some theorems on particular logics were generalized to the whole area. Recall that **S4.3** is an extension of **S4** with the axiom $\Box(\Box\alpha \to \Box\beta) \vee \Box(\Box\beta \to \Box\alpha)$.

First, Bull [4] proved that every such logic has the finite model property, FMP. Next, Fine [11] showed that all modal logics extending **S4.3** have the finite frame property, which is equivalent to the FMP, and that they all are finitely axiomatizable. Fine also characterized all such logics by means of finite quasilinear frames (chains of clusters), or lists. Hence he provided a semantic description of all modal logics extending **S4.3**.[1]

Following this line of investigations, we lift these results from the level of theoremhood into the level of derivability. In other words, we extend these results from

Received November 30, 2012; accepted November 11, 2013
First published online July 19, 2016
2010 Mathematics Subject Classification: Primary 03B45, 08C15, 68T15; Secondary 06E25, 03B35
Keywords: admissible rules, consequence relations, projective unification, **S4.3**, structural completeness, quasivarieties
 10.1215/00294527-3636512

logics to global consequence relations extending **S4.3**, that is, to all deductive systems extending **S4.3**. Hence we consider not only theorems of a modal logic L but also rules of inference in L, especially admissible rules in L. Note that we consider only finitary rules and finitary consequence relations.

Our considerations are essentially based on the result of [10, Corollary 3.19], stating that every modal logic L extending **S4.3**, in short, $L \in$ NExt**S4.3**, enjoys projective unification. Here we give another proof of that, which is shorter but less constructive. The proof is based on Ghilardi's characterization of projective formulas in modal logics extending **K4** (see [13]).

Recall that a rule $r : \alpha/\beta$ is admissible in L if adding r to L does not change (the theorems of) L; that is, if $\tau(\alpha) \in L$, then $\tau(\beta) \in L$, for every substitution τ. A rule $r : \alpha/\beta$ is derivable in L if $\alpha \vdash_L \beta$. A logic L is (almost) structurally complete if every rule which is admissible in L (and has unifiable premises) is derivable in L; a formula α is unifiable in L if $\vdash_L \tau(\alpha)$, for some substitution τ.

According to [10], each logic L in NExt**S4.3** is almost structurally complete, and such L is structurally complete if and only if McKinsey's formula $M : \Box \Diamond x \to \Diamond\Box x$ is in L. Note that **S4.3.1** is another name for **S4.3M**. Since there is a particular splitting of the lattice NExt**S4.3**, the complement of the filter generated by **S4.3.1** is the ideal generated by the logic of 2-element cluster $\mathfrak{2}$; this ideal contains all almost structurally complete logics which are not structurally complete, and hence can be extended by adding admissible rules. In fact, the only rules that are admissible but not derivable in logics from NExt**S4.3** are so-called *passive rules*, that is, rules $r : \alpha/\beta$ such that α is not unifiable (or $\nvdash_L \tau(\alpha)$, for every substitution τ).

We give a detailed description of all passive rules in logics from NExt**S4.3**. By doing a step-by-step transformation, we reduce any passive rule into a particularly simple form (see Theorem 5.19): $\Diamond \mathrm{Ker}(h)/\Box\delta$, where $\mathrm{Ker}(h)$ is a formula representing the kernel of a suitable homomorphism h (see Theorems 5.12–5.18), and $\mathrm{Ker}(h)$ and δ have no variables in common.

Next we show that any consequence relation $\vdash$ extending **S4.3** is finitely based; that is, $\vdash$ can be given by adding to $\vdash_L$, for some $L \in$ NExt**S4.3**, a finite number of passive rules. This can be done effectively. Hence, we provide a syntactic characterization of all consequence relations extending **S4.3**, by exhibiting the appropriate passive rules (see Theorem 6.2(i)), and moreover, we show that every such $\vdash$ is decidable. In algebraic terms, we exhibit the form of quasiequations defining quasivarieties of S4.3-algebras and show that all quasivarieties of S4.3-algebras are finitely based and decidable.

We provide a semantic characterization of any consequence relation $\vdash$ extending **S4.3** by providing a class of modal algebras strongly adequate for $\vdash$, that is, the class determining the same consequence relation. The class consists of algebras of the form of the direct products $\mathcal{A} \times \mathfrak{n}^+$ of algebras $\mathcal{A}$ from a class of algebras adequate for the basic logic L and suitable finite Henle algebras $\mathfrak{n}^+$ (see Theorem 6.2(ii)); in algebraic terms: any quasivariety of S4.3-algebras is generated by such class. This means that finite Henle algebras (i.e., finite subdirectly irreducible monadic algebras; see Kagan and Quackenbush [15]) are used to semantically represent passive admissible rules. Hence the paper contributes to studies of admissible rules in modal logics (see, e.g., Jeřabek [14], Rybakov [22], [24]).

Finally, we show that the lattice **EXT(S4.3)** of all consequence relations extending S4.3 (in terms of algebra: the lattice of all quasivarieties of S4.3-algebras) is countable, complete, and distributive and that it forms a Heyting algebra.

2 Modal Logics

Results in this section are quoted without proofs and exact references. For details, we recommend some textbooks on modal logic, for example, Chagrov and Zakharyaschev [6], Blackburn, de Rijke, and Venema [2], and Kracht [16], [17], as well as some textbooks on logical matrices, for example, Wójcicki [26] and Pogorzelski and Wojtylak [20].

We consider the standard *modal language*. $\mathrm{Var} = \{p_1, p_2, \dots\}$ is the set of propositional variables and Fm is the set of all modal formulas built up from the variables by use of $\{\wedge, \neg, \Box, \top\}$. The remaining classical and modal operators $\rightarrow, \vee, \leftrightarrow, \Diamond, \bot$ are defined in the usual way; hence, we get the algebra of the language $(\mathrm{Fm}, \wedge, \neg, \Box, \top)$. $\mathrm{Var}(\alpha)$ denotes the set of variables occurring in the formula α and Fm_n is the set of all formulas α such that $\mathrm{Var}(\alpha) \subseteq \{p_1, \dots, p_n\}$.

By a *substitution* we mean any mapping $\varepsilon\colon \mathrm{Var} \rightarrow \mathrm{Fm}$; $\varepsilon(\alpha)$ denotes the result of the substitution in the formula α. Given two substitutions ε and σ, their composition $\varepsilon\sigma$ is defined in the usual way, $\varepsilon\sigma(\alpha) = \varepsilon(\sigma(\alpha))$, for each formula α.

A *modal logic*[2] is any subset of Fm containing all classical tautologies, the axiom $(K):\quad \Box(\alpha \rightarrow \beta) \rightarrow (\Box\alpha \rightarrow \Box\beta)$ and that is closed under substitutions and under

$$MP : \frac{\alpha \rightarrow \beta, \alpha}{\beta} \qquad \text{and} \qquad RG : \frac{\alpha}{\Box\alpha}.$$

The weakest modal logic is denoted by **K**, and any modal logic may be regarded as an extension of **K** with some axiom schemata. In particular, **S4**, a basic modal system for this paper, is the extension of **K** with $(T) : \Box\alpha \rightarrow \alpha$ and $(4) : \Box\Box\alpha \rightarrow \Box\alpha$. The logic **S4.3** contains additionally $(\mathbf{.3}) :\ \Box(\Box\alpha \rightarrow \Box\beta) \vee \Box(\Box\beta \rightarrow \Box\alpha)$ and, in **S5**, we also have $(5) : \Diamond\Box\alpha \rightarrow \Box\alpha$.

Given a modal logic L, we define its *global consequence relation* $\vdash_L$ admitting that $X \vdash_L \alpha$ means that α can be derived from $X \cup L$ using *MP* and *RG* as the only rules of inference. All consequence relations in this paper will be extensions of $\vdash_{S4}$, which is the global consequence relation of **S4**.

Theorem 2.1 (Deduction theorem) *If* **S4** $\subseteq L$, *then* $X, \beta \vdash_L \alpha$ *iff* $X \vdash_L \Box\beta \rightarrow \alpha$.

We will also consider extensions of **S4**, with extra inferential rules, which violate the above deduction theorem. By a *modal consequence relation* we mean any finitary consequence relation $\vdash$ which extends $\vdash_K$ and which is *structural*:

$$\text{if} \quad X \vdash \alpha, \quad \text{then } \varepsilon[X] \vdash \varepsilon(\alpha), \text{ for each substitution } \varepsilon.$$

$\mathrm{NExt}(L)$ or $\mathrm{NExt}L$ denotes the lattice of all (normal) extensions of L. $\mathbf{EXT}(L)$ is the lattice of all consequence relations extending the global consequence relation $\vdash_L$ of L. The order relation in $\mathrm{NExt}(L)$ and in $\mathbf{EXT}(L)$ is induced by inclusion. The set of theorems of $\vdash$ is the set $\{\alpha \in \mathrm{Fm} : \vdash \alpha\}$; for example, the set of theorems of $\vdash_L$ is L.

A *modal algebra* $\mathfrak{A} = (A, \wedge, \neg, \Box, \top)$ is a Boolean algebra $(A, \wedge, \neg, \top)$ with a unary operation $\Box$ on A which satisfies the following conditions:
(1) $\Box\top = \top$; (2) $\Box(a \wedge b) = \Box a \wedge \Box b$, for each $a, b \in A$.

Modal algebras are regarded as logical matrices with one designated element $\top$. Each valuation $v : \mathrm{Var} \to A$ extends to a homomorphism $v : \mathrm{For} \to A$, which is also denoted by v. If $v(\alpha) = \top$ for some v, we say that α is *satisfiable in* the algebra $\mathcal{A}$, in short, $\alpha \in \mathrm{Sat}(\mathcal{A})$. Let $\mathrm{Log}(\mathcal{A})$ be the set of all formulas valid in the algebra $\mathcal{A}$; that is, $\mathrm{Log}(\mathcal{A}) = \{\alpha : v(\alpha) = \top, \text{ for all } v : \mathrm{Var} \to A\}$. Given a class $\mathbb{K}$ of modal algebras, we put $\mathrm{Log}(\mathbb{K}) = \bigcap\{\mathrm{Log}(\mathcal{A}) : \mathcal{A} \in \mathbb{K}\}$.

Theorem 2.2 *For each modal logic L, there is a class $\mathbb{K}$ of modal algebras such that $L = \mathrm{Log}(\mathbb{K})$.*

The above completeness theorem says that each modal logic L has an *adequate* class of modal algebras. The main step in the proof is the construction of the *Lindenbaum–Tarski algebra* which results from the algebra of language Fm by identifying L-equivalent formulas:

$$\alpha =_L \beta \qquad \text{iff} \qquad \vdash_L \alpha \leftrightarrow \beta \qquad (\text{iff } \alpha \leftrightarrow \beta \in L).$$

The relation $=_L$ (we write $=$ if L is fixed) is a congruence on the algebra of the language and the quotient algebra turns out to be adequate for the logic L.

Each modal algebra $\mathcal{A}$ generates a consequence relation $\models_{\mathcal{A}}$ defined by

$$X \models_{\mathcal{A}} \alpha \qquad \text{iff} \qquad \big(v[X] \subseteq \{\top\} \Rightarrow v(\alpha) = \top, \text{for each } v\colon \mathrm{Var} \to A\big).$$

It is obvious that $\models_{\mathcal{A}} \alpha$ iff $\alpha \in \mathrm{Log}(\mathcal{A})$. For each class $\mathbb{K}$ of modal algebras,

$$X \models_{\mathbb{K}} \alpha \qquad \text{iff} \qquad (X \models_{\mathcal{A}} \alpha, \text{for each } \mathcal{A} \in \mathbb{K}).$$

One can prove that $\models_{\mathbb{K}}$ equals $\bigwedge\{\models_{\mathcal{A}}\colon \mathcal{A} \in \mathbb{K}\}$, where $\bigwedge$ is the meet operation in the lattice of all modal consequence relations.

Now, the completeness theorem easily extends on consequence relations, as for each (finitary) modal consequence relation one can find a *strongly adequate* class of modal algebras, that is, a class $\mathbb{L}$ such that, for each finite X and each $\alpha \in \mathrm{Fm}$,

$$X \vdash \alpha \qquad \text{iff} \qquad X \models_{\mathbb{L}} \alpha.$$

To this aim, it suffices to generalize the construction of the Lindenbaum–Tarski algebra on theories, that is, identify formulas such that $X \vdash \alpha \leftrightarrow \beta$. The problem arises, however, because the relations $\models_{\mathcal{A}}$ usually are not finitary. We have the following (see Łoś and Suszko [18]).

Theorem 2.3 *If the algebra $\mathcal{A}$ is finite, then the relation $\models_{\mathcal{A}}$ is finitary.*

A logic L has the *finite model property* (FMP for short) if it has an adequate family of finite modal algebras; a consequence relation $\vdash$ has the *strong finite model property* (SFMP for short) if there is a strongly adequate family of finite algebras for $\vdash$. If, additionally, the given family of finite algebras is finite, we say that L (or $\vdash$) is *finite (strongly finite)*. The finiteness of L is equivalent to its tabularity, where L is said to be *tabular* if $L = \mathrm{Log}(\mathcal{A})$ for some finite $\mathcal{A}$. In contrast, a strongly finite $\vdash$ may not have a strongly adequate family consisting of a single finite algebra.

The class $\mathbb{L}$ of all modal algebras for L is a variety and hence, if we deal with adequate families of algebras, we can always restrict ourselves to families of *subdirectly irreducible algebras* $\mathbb{L}_{\mathrm{SI}}$, that is, those algebras which are not subdirectly representable by their nonisomorphic quotients; we have $\mathrm{Log}(\mathbb{L}_{\mathrm{SI}}) = \mathrm{Log}(\mathbb{L})$ if $\mathbb{L}$ is a variety. If $\mathbb{K}$ is any class of modal algebras, then $H(\mathbb{K})$, $S(\mathbb{K})$, and $P(\mathbb{K})$ stand for the class of homomorphic images, subalgebras, and products, respectively, of algebras from $\mathbb{K}$. We have $\mathrm{Log}(HSP(\mathbb{K})) = \mathrm{Log}(\mathbb{K})$. For the consequence relation $\models_{\mathbb{K}}$ determined by a class $\mathbb{K}$, we get similar (though not the same) results. In particular, we get the following.

Theorem 2.4 *If $\mathcal{A} \in S(\mathcal{B})$, then $\models_{\mathcal{B}} \ \leq \ \models_{\mathcal{A}}$.*

Theorem 2.5

(i) If $\mathcal{A} = \mathbf{P}_{t\in T}\mathcal{A}_t$ and $X \in \mathrm{Sat}(\mathcal{A}_t)$, for each $t \in T$, then

$$X \models_{\mathcal{A}} \alpha \qquad \textit{iff} \qquad (X \models_{\mathcal{A}_t} \alpha, \textit{ for each } t \in T).$$

(ii) If $\mathcal{A} = \mathbf{P}_{t\in T}\mathcal{A}_t$ and $X \notin \mathrm{Sat}(\mathcal{A}_t)$, for some $t \in T$, then X is $\models_{\mathcal{A}}$-inconsistent; that is, $X \models_{\mathcal{A}} \alpha$, for each formula α.

Corollary 2.6 *If $\mathcal{A} = \mathcal{B} \times \mathcal{C}$, then $X \models_{\mathcal{A}} \alpha$ iff $X \models_{\mathcal{B}} \alpha$ and $X \models_{\mathcal{C}} \alpha$, provided that $X \in \mathrm{Sat}(\mathcal{B})$ and $X \in \mathrm{Sat}(\mathcal{C})$. Otherwise, $X \models_{\mathcal{A}} \alpha$ for each $\alpha \in \mathrm{Fm}$.*

From the well-known properties of consequence relations, it easily follows that $\models_{\mathbb{K}} \ \leq \ \models_{\mathcal{A}}$ if $\mathcal{A} \in SP(\mathbb{K})$. We also have the following (see [27] or [26, p. 303]).

Theorem 2.7 *Let $\mathbb{K}$ be a class of modal algebras, and let $\vdash$ be a modal consequence relation such that $\models_{\mathbb{K}} \ \leq \ \vdash$. Then there is a class $\mathbb{L} \subseteq SP(\mathbb{K})$ such that $\vdash = \ \models_{\mathbb{L}}$.*

Modal algebras which are models for **S4** are called *topological Boolean algebras* (TBAs for short), as they satisfy (3) $\Box a \leq a$ and (4) $\Box\Box a = \Box a$, for each $a, b \in A$. TBAs are also known as S4-algebras or interior (closure) algebras. For finite subdirectly irreducible TBAs, one defines the so-called *characteristic formulas* (see Bull and Segerberg [5] or [2, p. 223]) by means of which one shows the following.

Theorem 2.8 *Let $\mathbb{K}$ be a class of* TBAs*, and let $\mathcal{A}$ be a finite subdirectly irreducible* TBA*. Then $\mathrm{Log}(\mathbb{K}) \subseteq \mathrm{Log}(\mathcal{A})$ iff $\mathcal{A} \in SH(\mathbb{K})$.*

Let us recall that the correspondence between varieties and logics can be extended to quasivarieties and consequence relations. A class of algebras $\mathbb{K}$ is a *quasivariety* if it is closed under taking subalgebras, $S(\mathbb{K})$, direct products, $P(\mathbb{K})$, and ultraproducts, $P_U(\mathbb{K})$; that is, $\mathbb{K} = SPP_U(\mathbb{K})$. Equivalently, a class of algebras $\mathbb{K}$ is a quasivariety if it is axiomatized by a set of quasi-identities (or quasiequations), that is, by expressions of the form: $t_1 \approx t_1' \wedge \cdots \wedge t_n \approx t_n' \ \Rightarrow \ t_0 \approx t_0'$, where t_i, t_i' are terms. In the case of modal algebras $(A, \wedge, \neg, \Box, \top)$ the quasi-identities have the form $t_1 \approx \top \wedge \cdots \wedge t_n \approx \top \ \Rightarrow \ t_0 \approx \top$, and they correspond to rules of inference of the form $\alpha_1 \cdots \alpha_n/\alpha_0$ (where α_i is a formula corresponding to a term t_i). This correspondence was observed by Bloom [3].

A *frame* $\mathfrak{F} = (V, R)$ consists of a nonempty set V (of worlds) and a binary relation R on V. A subset C of V is a *cluster* in $\mathfrak{F}$) if it is a maximal subset of V such that xRy and yRx, for every $x, y \in C$. The *n-element cluster* is a pair $\mathfrak{n} = (V_n, R_n)$, where $V_n = \{1, \ldots, n\}$ and $R_n = V_n \times V_n$. We will use symbols 1, 2, 3, and so forth for 1-, 2-, 3-element, and so forth clusters, respectively.

A *Kripke model* $\mathfrak{M} = (V, R, v)$ is an extension of the frame (V, R) with a valuation $v : \mathrm{Var} \to P(V)$, where $P(V)$ is the power set of V. Kripke models will be rather noted in the form $(V, R, \Vdash)$, where $\Vdash \subseteq V \times \mathrm{Fm}$ is determined by the valuation v:

$$\begin{array}{lll}(\mathfrak{F}, x) \Vdash p & \text{iff} & p \in v(x),\\ (\mathfrak{F}, x) \Vdash \alpha \wedge \beta & \text{iff} & (\mathfrak{F}, x) \Vdash \alpha \text{ and } (\mathfrak{F}, x) \Vdash \beta,\\ (\mathfrak{F}, x) \Vdash \neg\alpha & \text{iff} & (\mathfrak{F}, x) \nVdash \alpha,\\ (\mathfrak{F}, x) \Vdash \Box\alpha & \text{iff} & (\mathfrak{F}, y) \Vdash \alpha, \text{ for each } y \in V \text{ such that } xRy.\end{array}$$

Now let $\mathrm{Log}(\mathfrak{F}) = \{\alpha : (\mathfrak{F}, x) \Vdash \alpha, \text{ for each } x \in V \text{ and each } \Vdash\}$ be the *logic of* $\mathfrak{F}$, that is, the set of all formulas that are true in $\mathfrak{F}$.

For a finite modal algebra $\mathcal{A} = (A, \wedge, \neg, \Box, \top)$, let V be the set of its Boolean atoms, and let R be a binary relation on V defined as follows: xRy iff $(x \le \Box a \Rightarrow y \le a$, for each $a \in A)$. For every valuation $v\colon \mathrm{Var} \to A$, one defines the relation $\Vdash$ as follows: $x \Vdash p$ iff $x \le v(p)$, for each $x \in V$ and each $p \in \mathrm{Var}$, and consequently one gets a Kripke model $\mathfrak{M} = (V, R, \Vdash)$ such that the following holds.

Theorem 2.9 *We have*

$$x \Vdash \alpha \quad \textit{iff} \quad x \le v(\alpha), \quad \textit{for each } x \in V\textit{, and each formula } \alpha.$$

Thus, finite modal algebras are tantamount to finite Kripke frames and each valuation $v : \mathrm{Var} \to A$ in $\mathcal{A}$ corresponds to a (forcing) relation $\Vdash$ or a valuation $v\colon \mathrm{Var} \to P(V)$ on (V, R), and vice versa, in such a way that the valuation fulfils α if and only if α is forced at each point of the frame. The proof requires that the algebra be atomic and complete, and $\Box \bigwedge_i a_i = \bigwedge_i \Box a_i$, for any set a_i of elements; these conditions are satisfied in all finite algebras (but not in all infinite ones).

Now suppose that a frame $\mathfrak{F} = (V, R)$ is given and that we define a modal algebra on the powerset $A = P(V)$ putting $\Box a = \{x \in V : R(x) \subseteq a\}$, where $R(x) = \{y \in V : xRy\}$. The modal algebra obtained in this way is denoted by $\mathfrak{F}^+$. It fulfills the condition $\Box \bigwedge_i a_i = \bigwedge_i \Box a_i$ and its frame on atoms (defined as above) coincides with the given frame $\mathfrak{F} = (V, R)$, if we identify elements of V with 1-element sets (atoms) in $P(V)$. Kripke frames can be identified with some special modal algebras. They are used as models for modal logics. Since not all modal algebras correspond to Kripke frames, some modal logics are *Kripke incomplete*. If a logic L has the finite model property, then L has an adequate family of finite Kripke frames (and vice versa).

Let $(V, R, \Vdash)$ and $(V', R', \Vdash')$ be two Kripke models. A map $f\colon V \to V'$ from V onto V' is called a *p-morphism* of the models if (i) $xRy \Rightarrow f(x)R'f(y)$, (ii) $f(x)R'u \Rightarrow \exists_y(xRy \wedge u = f(y))$, and (iii) $x \Vdash p \Leftrightarrow f(x) \Vdash' p$, for each variable p.

If f fulfills only (i) and (ii), then it is a called a *p-morphism of the frames* (V, R) and (V', R'). Note that, using (iii), for each forcing relation $\Vdash'$ on (V', R') one can define a corresponding relation $\Vdash$ on (V, R), and then we have the following.

Theorem 2.10 *If f is a p-morphism of models* $(V, R, \Vdash)$ *and* $(V', R', \Vdash')$*, then*

$$x \Vdash \alpha \quad \textit{iff} \quad f(x) \Vdash' \alpha, \quad \textit{for each formula } \alpha.$$

If (V, R) is a frame and V' is an upward closed subset of V (i.e., xRy and $x \in V'$ implies $y \in V'$), then (V', R) is called a *generated subframe* of (V, R).

Theorem 2.11 *Let $\mathfrak{F} = (V, R)$ and $\mathfrak{G} = (V', R')$ be finite Kripke frames.*

(i) There exists a p-morphism from $\mathfrak{F}$ onto $\mathfrak{G}$ if and only if $\mathfrak{G}^+$ is embeddable into $\mathfrak{F}^+$.

(ii) $\mathfrak{G}$ *is a generated subframe of* $\mathfrak{F}$ *if and only if* $\mathfrak{G}^+$ *is a homomorphic image of* $\mathfrak{F}^+$.

3 Projective Unification and Structural Completeness

Let L be a modal logic, and let α be a formula. A substitution ε is called a *unifier for* α *in* L if $\varepsilon(\alpha) \in L$. A formula α is said to be *unifiable* in L if there exists a unifier for α in L. Note that unifiable formulas always have many unifiers; in particular, if ε is a unifier for α and σ is any substitution, then $\sigma\varepsilon$ also is a unifier for α.

The constants $\{\bot, \top\}$ form a subalgebra of the Lindenbaum–Tarski algebra for L, where $L \supseteq$ **S4**, which is isomorphic to the 2-element modal algebra in which $\Box$ is an identity map; that is, $\Box a = a$ for each a. Let us denote the algebra by **2** and its modal logic, the *trivial* modal logic, by **Tr**. For each consistent $L \supseteq$ **S4**, we have $L \subseteq$ **Tr**.

Unifiers of the form $v\colon \mathrm{Var} \to \{\bot, \top\}$ are called *ground unifiers*. They can be identified with valuations in **2** which satisfy the formula α. Given a unifier ε for α and any substitution $\sigma\colon \mathrm{Var} \to \{\bot, \top\}$, we get the ground unifier $\sigma\varepsilon$ for α.

Corollary 3.1 *The following conditions are equivalent for each formula* α *and each modal logic* $L \supseteq$ **S4**:

(i) α *is* L*-unifiable;* *(ii) there is a ground unifier for* α *in* L*;*
(iii) α *is satisfiable in* **2**; *(iv)* $\sim\alpha \notin \mathbf{Tr} = \mathrm{Log}(\mathbf{2})$.

Hence unifiability of α in $L \supseteq$ **S4** does not depend on the logic L. For logics weaker than **S4**, such characterizations of unifiability are more complicated (see Gencer and de Jongh [12]).

A *projective unifier for* α *in* L is a unifier ε such that

$$\alpha \vdash_L \varepsilon(\beta) \leftrightarrow \beta, \quad \text{for each formula } \beta.$$

A notion which is close to this appeared in Wroński [28] for intermediate logics under the name of a transparent unifier (see also Dzik [9]). Projective unifiers (formulas, substitutions) were defined and extensively used by Silvio Ghilardi in his papers of 1997–2004 (see, e.g., [13]), though the term "projective unifier" did not appear until Baader and Ghilardi [1]. Projective unifiers are the *most general unifiers* and they have specific properties; hence they have many applications.

We say that a logic L enjoys *projective unification* if each L-unifiable formula has a projective unifier in L. Recall that **S4.3** is **S4** plus $\Box(\Box\alpha \to \Box\beta) \vee \Box(\Box\beta \to \Box\alpha)$.

Theorem 3.2 (see [10, Corollary 3.19]) *A modal logic* L *containing* **S4** *enjoys projective unification if and only if* **S4.3** $\subseteq L$.

Our proof in [10] showing that **S4.3** enjoys projective unification, provides an algorithm describing how one could construct a projective unifier for a given unifiable formula. Now we show that the mere fact that each formula unifiable in a logic L extending **S4.3** has a projective unifier in L can also be derived—in a shorter but less constructive way—from the following result by Ghilardi [13, Theorem 2.2].

Theorem 3.3 *If* $L \supseteq$ **K4** *has* FMP*, then the following conditions are equivalent:*

(i) α *has a projective unifier in* L*;*
(ii) $\mathrm{Mod}_{\mathrm{L}}(\alpha)$ *has the extension property.*

Let us recall that $\mathrm{Mod_L}(\alpha)$ denotes the class of all Kripke models for α, over finite rooted L-frames. $\mathrm{Mod_L}(\alpha)$ has the extension property if for every L-frame (V, R) and every $\Vdash$ on (V, R) such that $x \Vdash \alpha$ for each x above the root ρ (i.e., for each $x \notin \mathrm{cl}(\rho)$, where $\mathrm{cl}(\rho)$ is the cluster of the root), one can define $\Vdash'$ which coincides with $\Vdash$ for each $x \notin \mathrm{cl}(\rho)$ and $\rho \Vdash' \alpha$.

We prove the extension property for each unifiable formula in the next section. To see that formulas which are not unifiable do not have the extension property, it is enough to consider 1-element frame $\imath$ (note that $\imath^+ = \mathbf{2}$) and to note that each formula is satisfied "above" the root of $\imath$ (in the empty frame), but it is only for unifiable formulas that the empty valuation extends to a valuation which satisfies the formula.

In **S5** projective unifiers can be written in a simple and uniform way, depending on a ground unifier $\tau \colon \mathrm{Var} \to \{\top, \bot\}$ for a unifiable formula α (see Dzik [7]). Namely,

$$\varepsilon(x) = \begin{cases} \Box\alpha \to x & \text{if } \tau(x) = \top, \\ \Box\alpha \wedge x & \text{if } \tau(x) = \bot \end{cases}$$

is a projective unifier for α in **S5**. Note that (see [10, Examples 3.3–3.5]) this substitution is not a unifier for some formulas α in logics weaker than **S5**; in particular, this method of defining projective unifiers does not work for **S4.3**.

Let $\vdash$ be a consequence relation. Without loss of generality, we can consider here structural rules of the form $r : \alpha/\beta$, where α, β play the role of formula schemata. The rule $r : \alpha/\beta$ is *admissible* for $\vdash$, if $\vdash \varepsilon(\alpha)$ implies $\vdash \varepsilon(\beta)$, for every substitution ε and $r : \alpha/\beta$ is *derivable* for $\vdash$, if $\alpha \vdash \beta$. The relation $\vdash$ is *structurally complete* (see Pogorzelski [19]), $\vdash \in \mathrm{SCpl}$, if every admissible rule for $\vdash$ is derivable for $\vdash$.

Even strong modal systems may not be structurally complete because of the existence of the so-called *passive rules* (see [22]–[24]). The rule α/β is called *passive* in L, if α is not unifiable in L. The logic **S5** (and many others including **S4.3**) is not structurally complete, since the following rule is admissible but not derivable:

$$P_2 : \frac{\Diamond\alpha \wedge \Diamond{\sim}\alpha}{\beta}.$$

We say that $\vdash$ is *almost structurally complete*, $\vdash \in \mathrm{ASCpl}$, if every admissible rule for $\vdash$, which is not passive, is derivable for $\vdash$. From [10, Theorem 4.1], we have the following.

Theorem 3.4 *Every modal consequence relation $\vdash$ extending $S4.3$ is almost structurally complete.*

4 S4.3 and its Extensions; the Splitting of NExtS4.3

Much is known about **S4.3** and its extensions. Some of these results are celebrated in the literature (see, e.g., [2], [4], [5], [11]). We put them in one theorem.

Theorem 4.1

(i) Each logic in NExt**S4.3** *has the finite model property.*
(ii) Each logic in NExt**S4.3** *is finitely axiomatizable.*
(iii) The lattice NExt**S4.3** *is countable and distributive.*

Nevertheless, any description of the structure of NExt**S4.3** is "one of great complexity" as Kit Fine puts it in [11]. Subdirectly irreducible modal algebras for **S4.3**, *s.i.*

S4.3-algebras for short, are those TBAs in which all open elements (i.e., such a's that $\Box a = a$) form a chain (these algebras are called *well-founded algebras* in [4]). Since **S4.3** has the FMP, finite s.i. S4.3-algebras are sufficient to characterize this logic. These algebras are, in turn, tantamount to finite *quasichains*, which can be viewed as finite chains of finite clusters. Such frames can be represented as *lists of positive integers* $k_1, \ldots, k_n$; each positive integer in the list records the cardinality of the corresponding cluster. We will occasionally identify frames with their lists.

We say that a list $\mathfrak{f}$ *covers* a list $\mathfrak{g}$ if $\mathfrak{f}$ contains a sublist $\mathfrak{f}'$ of the same length as $\mathfrak{g}$ such that each item of $\mathfrak{f}'$ is greater than or equal to the corresponding item of $\mathfrak{g}$ and the last item of $\mathfrak{f}$ is greater than or equal to the last item in $\mathfrak{g}$. The following is well known (see [11, Lemma 6]).

Lemma 4.2 *Let $\mathfrak{F}$ and $\mathfrak{G}$ be finite S4.3 frames, and let $\mathfrak{f}$ and $\mathfrak{g}$ be their associated lists. Then $\mathfrak{f}$ covers $\mathfrak{g}$ if and only if there is a p-morphism from $\mathfrak{F}$ onto $\mathfrak{G}$ (which maps a sublist of $\mathfrak{f}$ onto $\mathfrak{g}$ and maps the last item of $\mathfrak{f}$ onto the last item in $\mathfrak{g}$).*

Now, it is obvious that we have the following (see comments after Theorem 3.3 above).

Theorem 4.3 $\mathrm{Mod}_{\mathrm{L}}(\alpha)$ *has the extension property in any modal logic $L \supseteq$ **S4.3** if and only if α is unifiable in L.*

Proof Let $\mathfrak{F}$ be a finite rooted L-frame, and let $\Vdash$ be a valuation on $\mathfrak{F}$ such that $x \Vdash \alpha$ for each x above the root ρ. We may assume that $\mathfrak{F} = \mathfrak{f}$ for some list $\mathfrak{f}$. Let us suppose the list contains at least two numbers (i.e., $\mathfrak{F}$ is not a cluster). Then $\mathfrak{f} = k, \mathfrak{g}$, where k represents the (number of elements in the) root cluster.

We may restrict the valuation $\Vdash$ to the frame $\mathfrak{g}$ and note that $(\mathfrak{g}, \Vdash)$ is a model for α. Without problems, one can define a p-morphism from $\mathfrak{f}$ onto $\mathfrak{g}$ which is an identity on $\mathfrak{g}$. Using the p-morphism, one can define a valuation $\Vdash'$ on $\mathfrak{f}$ which coincides with $\Vdash$ on $\mathfrak{g}$ and forces α.

The above argumentation is quite standard and does not depend on whether α is unifiable or not. A nonstandard case appears if $\mathfrak{F}$ is a cluster. Then each valuation on $\mathfrak{F}$ fulfills, trivially, the required condition that it forces α above the root (as there are no points above the root). Thus, our problem reduces to the question of whether α is satisfiable in $\mathfrak{F}$, which is really the case if α is unifiable. On the other hand, if α is not unifiable, then it is not satisfiable in any 1-element cluster and hence $\mathrm{Mod}_{\mathrm{L}}(\alpha)$ does not have the extension property.[3] $\Box$

Using Lemma 4.2 above (and some results from previous sections), we can also put Bull's theorem (see [4] and Theorem 4.1(i)) in a little more general form.

Theorem 4.4 *Let $\mathbb{K}$ be a set of finite s.i. S4.3-algebras, and let L be a modal logic. If* $\mathrm{Log}(\mathbb{K}) \subseteq L$, *then* $L = \mathrm{Log}(\mathbb{L})$ *for some* $\mathbb{L} \subseteq S(\mathbb{K})$.

Proof Let $\mathcal{A}$ be a finite s.i. $S4.3$-algebra. We prove

$$\mathrm{Log}(\mathbb{K}) \subseteq \mathrm{Log}(\mathcal{A}) \qquad \text{iff} \qquad \mathcal{A} \in S(\mathbb{K}). \tag{$\star$}$$

The implication ($\Leftarrow$) is obvious. If $\mathrm{Log}(\mathbb{K}) \subseteq \mathrm{Log}(\mathcal{A})$, then $\mathcal{A} \in SH(\mathbb{K})$ by Theorem 2.8, and hence by using Theorem 2.11 and the above lemma we get $\mathcal{A} \in S(\mathbb{K})$. $\Box$

Thus, there is a one-to-one correspondence between logics in NExt**S4.3** and sets of finite s.i. $S4.3$-algebras closed under subalgebras, or sets of lists closed (upwardly) under the covering relation. This correspondence is useful for identification of certain oversystems of **S4.3**. For instance, the logic **S5** is determined by all finite clusters.

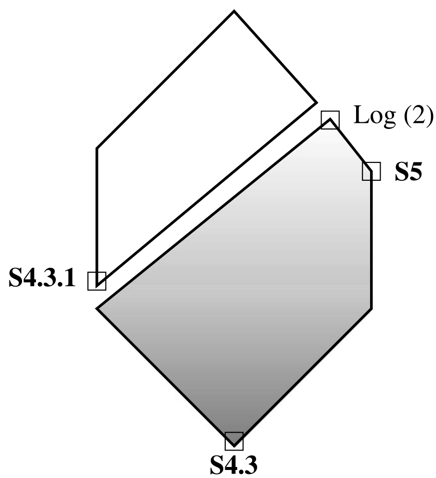
A modal algebra $\mathcal{A}$ is called a *Henle algebra* if $\Box a = \bot$ for each $a \neq \top$. Henle algebras[4] are known to be models for **S5**; in particular, they are simple monadic algebras (see [15]). Let us recall that the n-element cluster is denoted by $\mathfrak{n}$. Thus, $\mathfrak{n}^+$ is the Henle algebra with n atoms. Obviously, $\mathfrak{1}^+ = \mathbf{2}$.

The splitting pair (Log $\mathfrak{2}$, **S4.3.1**). Let Nsub(L) denote all normal modal sublogics of L (i.e., logics contained in L) extending **S4.3**, and let SUB(L) denote all consequence relations, together with all their extensions, determined by $L' \in$ Nsub(L). Recall that **S4.3.1**, or **S4.3M**, is **S4.3** plus the McKinsey axiom $M\colon \Box\Diamond x \to \Diamond\Box x$ and that the frame $\mathfrak{2}$ is a 2-element cluster.

A pair (L_1, L_2) of logics in NExtL_0 is a *splitting pair* in NExtL_0 if it divides the lattice into two disjoint parts: the filter NExtL_2 and the ideal determined by L_1. In this case, it is said that L_1 *splits* the lattice NExtL_0, and if $L_1 = \text{Log}\,\mathfrak{F}$, then it is said that $\mathfrak{F}$ *splits* the lattice NExtL_0 (see Rautenberg [21] and Zakharyaschev, Wolter, and Chagrov [29] for details).

Combining the fact that (Log $\mathfrak{2}$, **S4.3.1**) is the splitting pair in NExt**S4.3**, with projective unification and structural completeness in NExt**S4.3** (see [10]), we get the following.[5]

Theorem 4.5 *The frame $\mathfrak{2}$ splits the lattice* NExt**S4.3** *into two parts: the filter,* NExt($S4.3.1$)*, and the ideal,* Nsub(Log($\mathfrak{2}$))*. The filter of the splitting consists of all structurally complete logics of* NExt**S4.3***. The ideal,* Nsub(Log($\mathfrak{2}$))*, consists of all almost structurally complete but not structurally complete logics of* NExt**S4.3***.*



Splitting of NExt**S4.3**

By structural completeness, the consequence relation $\vdash_L$, for any L in NExt**S4.3.1**, is the greatest one among all consequence relations for which the set of theorems is L. Hence, a mapping: $L \mapsto \vdash_L$ within NExt**S4.3.1** is a bijection (even a lattice isomorphism).

Since logics in the ideal Nsub(Log($\mathfrak{2}$)) are not structurally complete, for each $L \in$ Nsub(Log($\mathfrak{2}$)) there are several (not one!) structural consequence relations $\vdash$

for which the set of theorems is L. They are extensions of $\vdash_L$ with passive rules and the maximal among them is the extension of $\vdash_L$ with the passive rule P_2 (see below).

Corollary 4.6 *The lattice of logics* (NExt**S4.3.1**, $\subseteq$) *and the lattice of consequence relations* (**EXT**(**S4.3.1**), $\leq$) *are isomorphic.*

The lattice of logics (Nsub(Log 2), $\subseteq$) *is embeddable into the lattice of consequence relations* (SUB(Log 2), $\leq$)*, but the embedding is not an epimorphism.*

The filter, NExt**S4.3.1**, contains well-known logics like two (out of five over **S4**; see [5]) pretabular logics: **S4.3Grz** and **S4.3MB$_2$** (**B$_2$** means height ≤ 2) as well as their extensions. The filter Nsub(Log(2)) contains, among others, only one pretabular logic **S5** and the chain of its extensions: $\cdots \subset \mathrm{Log}(\mathfrak{n}+1) \subset \mathrm{Log}(\mathfrak{n}) \subset \cdots \subset \mathrm{Log}(1)$.

5 Passive Rules

Any modal consequence relation over **S4.3** is an extension of $\vdash_L$, for some logic $L \in$ NExt**S4.3**, with some passive rules. So, passive rules and nonunifiable formulas require some study. In this section, we prove that in order to define a modal consequence relation over **S4.3**, we may restrict ourselves to passive rules of the form $\Diamond\,\mathrm{Ker}(h)/\Box\delta$, where $\mathrm{Ker}(h)$ is a formula given by a homomorphism from the algebra of the language onto a Henle algebra, and we may additionally assume that $\Diamond\,\mathrm{Ker}(h)$ and $\Box\delta$ have no variables in common. Our elaboration is tedious and laborious. We get to the required representation by considering many intermediate forms of passive rules, for example, $\Box\Diamond\alpha/\Box\beta$. Then we need to present any $\Box\Diamond$-formula as a disjunction of the formulas $\Diamond\,\mathrm{Ker}(h)$, and next, to reduce rules of the form $\Diamond\,\mathrm{Ker}(h_1) \vee \cdots \vee \Diamond\,\mathrm{Ker}(h_n)/\Box\delta$ to those where $n = 1$. At the end, we prove that $\Diamond\,\mathrm{Ker}(h)$ and $\Box\delta$ may have no variables in common.

Lemma 5.1 *If α is not unifiable and* $\mathrm{Var}(\alpha) \subseteq \{p_1, \dots, p_n\}$*, then*

$$\alpha \vdash_{S4} (\Diamond p_1 \wedge \Diamond \sim p_1) \vee \cdots \vee (\Diamond p_n \wedge \Diamond \sim p_n).$$

Proof We proceed by induction on n. It is true if $n = 0$, as α must be $\bot$. Suppose it holds for each formula in n variables, and let $\alpha(p_1, \dots, p_{n+1})$ be not unifiable. As

$$p_{n+1} \vdash \alpha(p_1, \dots, p_{n+1}) \leftrightarrow \alpha(p_1, \dots, p_n, \top)$$
$$\sim p_{n+1} \vdash \alpha(p_1, \dots, p_{n+1}) \leftrightarrow \alpha(p_1, \dots, p_n, \bot)$$

and neither $\alpha(p_1, \dots, p_n, \top)$ nor $\alpha(p_1, \dots, p_n, \bot)$ is unifiable, we get

$$p_{n+1}, \alpha(p_1, \dots, p_{n+1}) \vdash (\Diamond p_1 \wedge \Diamond \sim p_1) \vee \cdots \vee (\Diamond p_n \wedge \Diamond \sim p_n)$$
$$\sim p_{n+1}, \alpha(p_1, \dots, p_{n+1}) \vdash (\Diamond p_1 \wedge \Diamond \sim p_1) \vee \cdots \vee (\Diamond p_n \wedge \Diamond \sim p_n)$$

from which it follows that $\alpha \vdash (\Diamond p_1 \wedge \Diamond \sim p_1) \vee \cdots \vee (\Diamond p_{n+1} \wedge \Diamond \sim p_{n+1})$. $\Box$

Lemma 5.2 *If α is not unifiable, then $\alpha \vdash_{S4.3} \Diamond\beta \wedge \Diamond \sim \beta$ for some formula β.*

Proof By Lemma 5.1, it suffices to find, for each n, a formula β_n such that

$$(\Diamond p_1 \wedge \Diamond \sim p_1) \vee \cdots \vee (\Diamond p_n \wedge \Diamond \sim p_n) \vdash_{S4.3} \Diamond\beta_n \wedge \Diamond \sim \beta_n. \qquad (\circ)$$

Take $\beta_1 = p_1$ and $\beta_{n+1} = (p_{n+1} \wedge \Diamond \sim p_{n+1}) \vee (\Box p_{n+1} \vee \Box \sim p_{n+1}) \wedge \beta_n$. Then ($\circ$) holds for $n = 1$ by definition. Suppose that the formula β_n fulfills ($\circ$), and let us consider a finite Kripke model $\langle V, R, \Vdash\rangle$ for **S4.3** such that

$$x \Vdash \Box\big((\Diamond p_1 \wedge \Diamond \sim p_1) \vee \cdots \vee (\Diamond p_{n+1} \wedge \Diamond \sim p_{n+1})\big) \quad \text{for an element } x \in V.$$

There are two possibilities to consider.

(i) There exist y, z in the top cluster such that $y \Vdash p_{n+1}$ and $z \Vdash\sim p_{n+1}$. Then $y \Vdash p_{n+1} \wedge \Diamond \sim p_{n+1}$ and hence $\Vdash \Diamond\beta_{n+1}$. We get $\Vdash \Diamond\beta_{n+1} \wedge \Diamond \sim \beta_{n+1}$ as $z \Vdash\sim \beta_{n+1}$.

(ii) We have $y \Vdash \Box p_{n+1} \vee \Box \sim p_{n+1}$ for each element y in the top cluster. By our assumptions, $y \Vdash (\Diamond p_1 \wedge \Diamond \sim p_1) \vee \cdots \vee (\Diamond p_n \wedge \Diamond \sim p_n)$ and hence, by inductive hypothesis, $y \Vdash \Diamond\beta_n \wedge \Diamond \sim \beta_n$ for each y in the top cluster. Thus, there are y_1, y_2 in the top cluster such that $y_1 \Vdash \beta_n$ and $y_2 \Vdash\sim \beta_n$. By the definition of β_{n+1} we get $y_1 \Vdash \beta_{n+1}$ and $y_2 \Vdash\sim \beta_{n+1}$, which means that $(V, R) \Vdash \Diamond\beta_{n+1} \wedge \Diamond \sim \beta_{n+1}$. $\Box$

It follows from the above lemma that among all passive rules, the rule P_2 is the strongest one. All passive rules are consequences of P_2, and hence we have the following (see [22], [24]).

Corollary 5.3 *The modal consequence relation resulting by extending a modal logic $L \supseteq$ **S4.3** with the rule P_2 is structurally complete.*

So, we get a basis for the admissible rules of any logic L over **S4.3** (see [22]–[24]). Let us show that, apart from $L + P_2$, the logic L may have many extensions with passive rules. The notation introduced below will be used in the rest of the paper.

Let n be fixed, and consider Boolean atoms in Fm_n:

$$p_1^{\sigma(1)} \wedge \cdots \wedge p_n^{\sigma(n)},$$

where $\sigma: \{1, \ldots, n\} \to \{0, 1\}$, $p^0 = p$, and $p^1 =\sim p$.

There are 2^n Boolean atoms in Fm_n and suppose that they are denoted by $\theta_1, \ldots, \theta_{2^n}$. Let $\vdash_n$ be the extension of $\vdash_{S4.3}$ with the rule

$$\frac{\Diamond\theta_1 \wedge \cdots \wedge \Diamond\theta_{2^n}}{B}.$$

It is clear that the above rule is valid in (the algebra determined by) any $2^n - 1$ (or less) element cluster, and it is not valid in the 2^n element cluster. Thus, we have the following.

Corollary 5.4 *For every $n \in \omega$,*

$$\begin{array}{c} \vdash_{S4.3} < \quad \cdots \quad < \quad \vdash_n \quad < \quad \cdots \quad < \quad \vdash_1 \\ = \vdash_{S4.3} + P_2 \qquad \textit{and} \qquad \vdash_{S4.3} + P_2 \in \mathrm{SCpl}. \end{array}$$

We have infinitely many extensions of $\vdash_{\mathbf{S4.3}}$ (and many other logics above **S4.3**) with passive rules, that is, extensions preserving the set of theorems. The question appears if there are countably many such extensions, just as in the case of the usual extensions of **S4.3**. We settle this question using projective unification in **S4.3**.

Theorem 5.5 *Each passive rule is equivalent over* **S4.3** *to a subrule of P_2, that is, to a rule of the form*

$$\frac{\Diamond\gamma \wedge \Diamond\sim\gamma}{\delta} \quad \textit{for some } \gamma, \delta.$$

Proof Let us consider a passive rule α/β, where $\alpha = \Box\alpha$. By Lemma 5.2, $\alpha \vdash_{\mathbf{S4.3}} \Diamond\gamma \wedge \Diamond \sim \gamma$ for some γ and hence $\alpha = (\Diamond\gamma \wedge \Diamond \sim \gamma) \wedge (\Diamond\gamma \wedge \Diamond \sim \gamma \to \alpha)$. Note that $(\Diamond\gamma \wedge \Diamond \sim \gamma \to \alpha)$ is unifiable (in any logic extending **S4**) as this formula is valid in **2** (see Corollary 3.1) and hence, by [10], there is a *projective unifier* φ for this formula. We will show that the following two rules are equivalent:

$$r_1 : \frac{\alpha}{\beta} \qquad \text{and} \qquad r_2 : \frac{\Diamond\varphi(\gamma) \wedge \Diamond \sim \varphi(\gamma)}{\varphi(\beta)}.$$

($\Rightarrow$) First, let us prove that r_2 can be derived (over **S4.3**) from r_1. Suppose that $\vdash$ is any consequence relation over **S4.3** such that $\alpha \vdash \beta$. Then we also get $\Diamond\gamma \wedge \Diamond \sim \gamma$, $\Diamond\gamma \wedge \Diamond \sim \gamma \rightarrow \alpha \vdash \beta$ and, by structurality, $\Diamond\varphi(\gamma) \wedge \Diamond \sim \varphi(\gamma) \vdash \varphi(\beta)$ as φ is a unifier for $\Diamond\gamma \wedge \Diamond \sim \gamma \rightarrow \alpha$.

($\Leftarrow$) Let $\vdash$ be any consequence relation over **S4.3** with $\Diamond\varphi(\gamma) \wedge \Diamond \sim \varphi(\gamma) \vdash \varphi(\beta)$. Since $\vdash$ is an extension of **S4.3**, we get $\alpha \vdash \Diamond\gamma \wedge \Diamond \sim \gamma \rightarrow \alpha$ and hence, by the projectivity of φ, we have

$$\alpha \vdash \gamma \leftrightarrow \varphi(\gamma) \qquad \text{and} \qquad \alpha \vdash \beta \leftrightarrow \varphi(\beta). \qquad (\circ)$$

We have $\alpha \vdash \Diamond\gamma \wedge \Diamond \sim \gamma$, as it holds in **S4.3**, and hence $\alpha \vdash \Diamond\varphi(\gamma) \wedge \Diamond \sim \varphi(\gamma)$ by ($\circ$). Thus, $\alpha \vdash \varphi(\beta)$ by the assumption that $\Diamond\varphi(\gamma) \wedge \Diamond \sim \varphi(\gamma) \vdash \varphi(\beta)$ and consequently $\alpha \vdash \beta$ by ($\circ$). $\Box$

The above does not settle the problem of the cardinality of **EXT(S4.3)** as we can still have large families of subrules of P_2. However, each passive rule over **S4.3** reduces to

$$\frac{\Box(\Diamond\gamma \wedge \Diamond{\sim}\gamma)}{\Box\delta}$$

and we can use certain properties of $\Box\Diamond$-formulas for further reductions.

Since each finitely generated Boolean algebra is finite, there is a finite set Fm_n^b such that for each Boolean formula α in Fm_n we have $\alpha =_{S4.3} \beta$, for some β in Fm_n^b. We simply select a Boolean formula from each equivalence class in $\mathrm{Fm}_n / =_{S4.3}$ if there is at least one. The set Fm_n^b may be regarded as a freely generated Boolean algebra with n generators $p_1, \ldots, p_n$ (or their classical equivalents). The algebra has 2^{2^n} elements, and we let, as above, $\theta_1, \ldots, \theta_{2^n}$ be its atoms.

We define the modal operators $\Box$ and $\Diamond$, on the algebra Fm_n^b, as follows:

$$\Box\alpha = \begin{cases} \top & \text{if } \alpha = \top, \\ \bot & \text{if } \alpha \neq \top, \end{cases} \qquad \text{and} \qquad \Diamond\,\alpha = \begin{cases} \top & \text{if } \alpha \neq \bot, \\ \bot & \text{if } \alpha = \bot \end{cases}$$

and Fm_n^b with such a $\Box$ forms a Henle algebra. Each finite Henle algebra is a subalgebra of the just defined Fm_n^b, for a sufficiently big n. Thus, each homomorphism h of any algebra to any finite Henle algebra can be viewed as a homomorphism to some Fm_n^b. In what follows, we assume that $h\colon \mathrm{Fm}_n \to \mathrm{Fm}_n^b$. Let

$$\mathrm{Ker}(h) = \bigwedge\{\Box\beta : \beta \in \mathrm{Fm}_n^b \wedge h(\beta) = \top\} \wedge \bigwedge\{\Box\Diamond\beta : \beta \in \mathrm{Fm}_n^b \wedge h(\beta) \neq \bot\}.$$

Note that Fm_n^b is a set of randomly chosen representatives of equivalence classes but different representatives give the same, modulo equivalence in any modal logic, formula $\mathrm{Ker}(h)$.

Lemma 5.6 *We have* $h(\mathrm{Ker}(h)) = \top$.

Lemma 5.7 *For each* $\alpha \in \mathrm{Fm}_n$*, there is* $\alpha' \in \mathrm{Fm}_n^b$ *such that* $\mathrm{Ker}(h) \vdash \alpha \leftrightarrow \alpha'$.

Proof We prove it by induction on the length of α. If $\alpha \in \mathrm{Fm}_n^b$, then we take $\alpha' = \alpha$. Suppose that $\alpha = \alpha_1 \to \alpha_2$, and let $\alpha_1', \alpha_2' \in \mathrm{Fm}_n^b$ be such that

$$\mathrm{Ker}(h) \vdash \alpha_1 \leftrightarrow \alpha_1' \qquad \text{and} \qquad \mathrm{Ker}(h) \vdash \alpha_2 \leftrightarrow \alpha_2'.$$

Then $\mathrm{Ker}(h) \vdash \alpha \leftrightarrow (\alpha_1' \to \alpha_2')$ and $\alpha_1' \to \alpha_2' = \alpha'$ for some $\alpha' \in \mathrm{Fm}_n^b$.

Let $\alpha = \Diamond\alpha_1$, and suppose that $\mathrm{Ker}(h) \vdash \alpha_1 \leftrightarrow \alpha_1'$ for some $\alpha_1' \in \mathrm{Fm}_n^b$ and hence

$$\mathrm{Ker}(h) \vdash \Diamond\alpha_1 \leftrightarrow \Diamond\alpha_1'.$$

The problem is that $\Diamond\alpha_1'$ is not a formula in Fm_n^b. But we have the two possibilities:

(i) $h(\alpha_1') \neq \bot$; then $\mathrm{Ker}(h) \vdash \Diamond\alpha_1'$, which means that $\mathrm{Ker}(h) \vdash \alpha \leftrightarrow \top$;

(ii) $h(\alpha_1') = \bot$; then $\mathrm{Ker}(h) \vdash \Box \sim \alpha_1'$ and hence $\mathrm{Ker}(h) \vdash \alpha \leftrightarrow \bot$. □

Lemma 5.8 *We have* $\mathrm{Ker}(h) \vdash \alpha \leftrightarrow \beta$ *iff* $h(\alpha) = h(\beta)$.

Proof ($\Rightarrow$) Since each Henle algebra is a model for **S4**, we get $h(\alpha \leftrightarrow \beta) = \top$ by Lemma 5.6 and hence $h(\alpha) = h(\beta)$.

($\Leftarrow$) Suppose that $h(\alpha) = h(\beta)$, and let $\alpha', \beta' \in \mathrm{Fm}_n^b$ be such formulas that

$$\mathrm{Ker}(h) \vdash \alpha \leftrightarrow \alpha' \qquad \text{and} \qquad \mathrm{Ker}(h) \vdash \beta \leftrightarrow \beta'$$

(see above lemma). Then $h(\alpha') = h(\alpha) = h(\beta) = h(\beta')$ and hence $h(\alpha' \leftrightarrow \beta') = \top$. But $\alpha' \leftrightarrow \beta'$ is a formula in Fm_n^b and by the definition of $\mathrm{Ker}(h)$ we easily get $\mathrm{Ker}(h) \vdash \alpha' \leftrightarrow \beta'$, which also gives $\mathrm{Ker}(h) \vdash \alpha \leftrightarrow \beta$. □

Lemma 5.9 *We have* $\mathrm{Ker}(h) \vdash \alpha$ *iff* $h(\alpha) = \top$.

Proof Take $\beta = \top$ in the above lemma. □

Note that Lemmas 5.6–5.9 above are valid for **S4**. This assumption would be, however, insufficient to prove the next theorems which are the main results of this section.

Theorem 5.10 *Let* $\theta_1, \ldots, \theta_{2^n}$ *be the Boolean atoms in* Fm_n^b. *Then*

$$\Diamond\, \mathrm{Ker}(h) =_{S4.3} \bigwedge\{\Box \Diamond \theta_i : h(\Box \Diamond \theta_i) = \top\} \wedge \bigwedge\{\sim \Box \Diamond \theta_i : h(\Box \Diamond \theta_i) \neq \top\}.$$

Proof Note that $h(\Box \Diamond \theta_i) \neq \top$ iff $h(\Box \Diamond \theta_i) = \bot$ iff $h(\sim \Box \Diamond \theta_i) = \top$ and thus the implication $\Rightarrow$ follows from the above lemma. To prove ($\Leftarrow$) we need:

(i) $\Diamond\Box \Diamond \alpha =_{S4.3} \Box \Diamond \alpha$,

(ii) $\Box \Diamond \Box\alpha =_{S4.3} \Diamond\Box\alpha$,

(iii) $\Box \Diamond (\alpha \vee \beta) =_{S4.3} \Box \Diamond \alpha \vee \Box \Diamond \beta$,

(iv) $\Diamond\Box(\alpha \wedge \beta) =_{S4.3} \Diamond\Box\alpha \wedge \Diamond\Box\beta$.

Now, one can easily show that $\Diamond\, \mathrm{Ker}(h)$ is **S4.3**-equivalent to

$$\bigwedge\{\Diamond\Box\beta : \beta \in \mathrm{Fm}_n^b \wedge h(\beta) = \top\} \wedge \bigwedge\{\Box \Diamond \beta : \beta \in \mathrm{Fm}_n^b \wedge h(\beta) \neq \bot\}.$$

Suppose that $\beta \in \mathrm{Fm}_n^b$ and $h(\beta) \neq \bot$. Since $\beta = \theta i_1 \vee \cdots \vee \theta i_k$ for some $i_1, \ldots, i_k$, then $h(\theta_i) \neq \bot$ (and consequently $h(\Box \Diamond \theta_i) = \top$) and $\vdash \Box \Diamond \theta_i \to \Box \Diamond \beta$ for some i.

Suppose that $\beta \in \mathrm{Fm}_n^b$ and $h(\beta) = \top$. Since $\beta =\sim \theta i_1 \wedge \cdots \wedge \sim \theta i_k$ for some $i_1, \ldots, i_k$, then $\Diamond\Box\beta =\sim \Box\Diamond\theta i_1 \wedge \cdots \wedge \sim \Box\Diamond\theta i_k$, and we have $h(\sim \Box\Diamond\theta_{i_j}) = \top$ and hence $h(\Box \Diamond \theta_{i_j}) \neq \top$ for each i_j. □

So, $\Diamond\,\mathrm{Ker}(h) =_{S4.3} \bigwedge_{i \in I} \Box\Diamond\theta_i \wedge \bigwedge_{i \notin I} \sim \Box\Diamond\theta_i$ for some I, where the number $|I|$ shows the dimension of the target Henle algebra. Note that for each $I \subseteq \{1, \ldots, 2^n\}$, one finds a mapping $h\colon \mathrm{Var}_n \to \mathrm{Fm}_n^b$ such that the above $S4.3$-equation holds. The number $|I|$ will be called the *Henle rank* of the formula $\Diamond\,\mathrm{Ker}(h)$.

Corollary 5.11 *If* $\Diamond\,\mathrm{Ker}(h_1) \neq \Diamond\,\mathrm{Ker}(h_2)$, *then* $\Diamond\,\mathrm{Ker}(h_1), \Diamond\,\mathrm{Ker}(h_2) \vdash_{\mathbf{S4.3}} \bot$.

It may happen that $\Diamond\,\mathrm{Ker}(h_1) = \Diamond\,\mathrm{Ker}(h_2)$ and $h_1 \neq h_2$, but if $\Diamond\,\mathrm{Ker}(h_1) = \Diamond\,\mathrm{Ker}(h_2)$, then $h_1 = gh_2$, for an automorphism g on the target algebra.

Theorem 5.12 *We have* $\Box\Diamond\alpha =_{\mathbf{S4.3}} \bigvee\{\Diamond\,\mathrm{Ker}(h) : h(\Box\Diamond\alpha) = \top\}$, *for every* $\alpha \in \mathrm{Fm}_n$.

Proof The implication $\Leftarrow$ follows from Lemma 5.9. Note that $\mathrm{Ker}(h)$ is an open formula and hence $\vdash_{\mathbf{S4.3}} \mathrm{Ker}(h) \to \Box\Diamond\alpha$ if $h(\Box\Diamond\alpha) = \top$ and, consequently, we get $\vdash_{\mathbf{S4.3}} \Diamond\,\mathrm{Ker}(h) \to \Box\Diamond\alpha$ as $\Box\Diamond\alpha$ is a closed formula, as well.

To prove the reverse implication, let us assume that $\Vdash \Box\Diamond\alpha$ for some finite S4.3-model $(V, R, \Vdash)$. Let us consider the Henle algebra determined by the top cluster of (V, R), and let h_0 be the valuation in the algebra determined by $\Vdash$. Thus, if $\{y_1, \ldots, y_k\}$ is the top cluster, we take the power algebra of $\{y_1, \ldots, y_k\}$ as the universe of the Henle algebra, and we put $h_0(p_j) = \{y_i : y_i \Vdash p_j\}$ for each p_j.

It follows from our assumptions that $y_i \Vdash \alpha$ for some i and hence $h_0(\Box\Diamond\alpha) = \top$. It remains to note that the considered Henle algebra (the one determined by the top cluster of (V, R)) is embeddable in some Fm_n^b and hence the valuation h_0 must coincide with some $h\colon \mathrm{Var}_n \to \mathrm{Fm}_n^b$. Thus, we get $h(\Box\Diamond\alpha) = \top$ and $\Vdash \Diamond\,\mathrm{Ker}(h)$. $\Box$

To sum it up, in what follows we can consider passive rules in the form

$$\frac{\Diamond\,\mathrm{Ker}(h_1) \vee \cdots \vee \Diamond\,\mathrm{Ker}(h_k)}{\Box\delta} \qquad (\star)$$

where each $\Diamond\,\mathrm{Ker}(h_j)$ is $\bigwedge_{i \in I} \Box\Diamond\theta_i \wedge \bigwedge_{i \notin I} \sim \Box\Diamond\theta_i$ for some $I \subseteq \{1, \ldots, 2^n\}$ with $|I| \geq 2$.

It would be most convenient if the rule $(\star)$ were equivalent to the collection of its special cases $\Diamond\,\mathrm{Ker}(h_i)/\Box\delta$. However, it is not so.

Example 5.13 Let us show such α, β, γ that from the rules α/γ and β/γ one cannot derive $\alpha \vee \beta/\gamma$ in the ground of **S4.3** (or even **S5**). Take

$$\frac{\Box\Diamond p \wedge \sim \Box\Diamond \sim p}{\sim \Box\Diamond \sim p} \quad \text{and} \quad \frac{\Box\Diamond p \wedge \Box\Diamond \sim p}{\sim \Box\Diamond \sim p}$$

and note that both rules are admissible in **S4.3**; in fact, the first is derivable and the second is passive. Thus, both rules are derivable in **S4.3** $+ P_2$ though their "sum"

$$\frac{(\Box\Diamond p \wedge \sim \Box\Diamond \sim p) \ \vee \ (\Box\Diamond p \wedge \Box\Diamond \sim p)}{\sim \Box\Diamond \sim p}$$

is not. Indeed, the above rule is obviously equivalent to $\Box\Diamond p/ \sim \Box\Diamond \sim p$, which is not passive, and hence its admissibility in **S4.3** would give us its derivability. As a result, we would get $\Box\Diamond p \to \sim \Box\Diamond \sim p$, which is equivalent to M and is not **S4.3**-valid.

The above example is not quite satisfactory. It disqualifies the "summing" operation, but one of the rules involved was not passive, which would not happen if we decomposed $(\star)$ into its subrules. So, a more elaborate example is still required.

Example 5.14 Suppose that $n = 2$, and let $\theta_1, \theta_2, \theta_3, \theta_4$ be all atoms in Fm_2^b. Take

$$\Diamond\,\mathrm{Ker}(h_1) = \Box\Diamond\theta_1 \wedge \Box\Diamond\theta_2 \wedge \Box\Diamond\theta_3 \wedge \sim\Box\Diamond\theta_4 \quad \text{and}$$
$$\Diamond\,\mathrm{Ker}(h_2) = \Box\Diamond\theta_1 \wedge \Box\Diamond\theta_2 \wedge \sim\Box\Diamond\theta_3 \wedge \sim\Box\Diamond\theta_4.$$

Note that $\Diamond\,\mathrm{Ker}(h_1) \vee \Diamond\,\mathrm{Ker}(h_2) = \Box\Diamond\theta_1 \wedge \Box\Diamond\theta_2 \wedge \sim\Box\Diamond\theta_4$ and hence the rule $\Diamond\,\mathrm{Ker}(h_1) \vee \Diamond\,\mathrm{Ker}(h_2)/\sim\Diamond\,\mathrm{Ker}(h_1)$ reduces to $\Box\Diamond\theta_1 \wedge \Box\Diamond\theta_2 \wedge \sim\Box\Diamond\theta_4/\sim\Box\Diamond\theta_3$. On the other hand, $\Diamond\,\mathrm{Ker}(h_2)/\sim\Diamond\,\mathrm{Ker}(h_1)$ is derivable and $\Diamond\,\mathrm{Ker}(h_1)/\sim\Diamond\,\mathrm{Ker}(h_1)$ is equivalent to $\Diamond\,\mathrm{Ker}(h_1)/\perp$. It remains to show that the two rules

$$\frac{\Box\Diamond\theta_1 \wedge \Box\Diamond\theta_2 \wedge \Box\Diamond\theta_3 \wedge \sim\Box\Diamond\theta_4}{\perp} \quad \text{and} \quad \frac{\Box\Diamond\theta_1 \wedge \Box\Diamond\theta_2 \wedge \sim\Box\Diamond\theta_4}{\sim\Box\Diamond\theta_3}$$

are not equivalent. Consider the algebra $\mathcal{A} = 2^+ \times 3^+$, which is the product of two Henle algebras. Note that $\Box\Diamond\theta_1 \wedge \Box\Diamond\theta_2 \wedge \sim\Box\Diamond\theta_4$ is satisfiable in 2^+ whereas $\Box\Diamond\theta_1 \wedge \Box\Diamond\theta_2 \wedge \Box\Diamond\theta_3 \wedge \sim\Box\Diamond\theta_4$ is not. Thus, by Corollary 2.6, the first rule is valid in $\mathcal{A}$ and the second is not as it is not a rule valid in 3^+.

Remark We comment on the examples that, based on the remainder of the paper, one may prove that: the rule ($\star$) is equivalent to the collection of the rules $\Diamond\,\mathrm{Ker}(h_i)/\Box\delta$ if $\mathrm{Var}(\delta) \cap \mathrm{Fm}_n = \emptyset$ or if the Henle rank of all h_i's is the same.

The (tacit) assumption concerning passive rules ($\star$), that each of them is formulated in Fm_n and that all Henle homomorphisms are : $\mathrm{Fm}_n \to \mathrm{Fm}_n^b$ for a fixed number n, is violated in the next considerations where we assume that the conclusion δ has no variables in common with the premise. Since the rank of Henle homomorphisms is important, let us agree for the remaining text that h^s or h_i^s will always denote Henle homomorphisms of rank s.

Lemma 5.15 *Let $h^s : \mathrm{Fm}_n \to \mathrm{Fm}_n^b$ be a Henle homomorphism with $s \geq 2$, and let $\theta_1, \ldots, \theta_{2^n}$ be all Boolean atoms in Fm_n^b, for some number $n \geq 1$. If $\mathrm{Var}(\delta) \cap \mathrm{Fm}_n = \emptyset$, then the following two rules are equivalent over* **S4.3***:*

$$\frac{\Diamond\,\mathrm{Ker}(h^s)}{\Box\delta} \quad \textit{and} \quad \frac{\bigvee\{\Box\Diamond\theta_{i_1} \wedge \cdots \wedge \Box\Diamond\theta_{i_s} : 0 < i_1 < \cdots < i_s \leq 2^n\}}{\Box\delta}.$$

Proof Without loss of generality, we may assume that

$$\Diamond\,\mathrm{Ker}(h^s) = \Box\Diamond\theta_1 \wedge \cdots \wedge \Box\Diamond\theta_s \wedge \sim\Box\Diamond\theta_{s+1} \wedge \cdots \wedge \sim\Box\Diamond\theta_{2^n}.$$

Let us define a substitution

$$\varepsilon(p_j) = \begin{cases} (\bigwedge_{i>s} \sim\theta_i) \to p_j & \text{if } p_j \text{ appears in } \theta_1, \\ (\bigwedge_{i>s} \sim\theta_i) \wedge p_j & \text{if } \sim p_j \text{ appears in } \theta_1, \end{cases}$$

and note that

$$\varepsilon(\theta_j) = \begin{cases} (\bigvee_{i>s} \theta_i) \vee \theta_1 & \text{if } j = 1 \\ (\bigwedge_{i>s} \sim\theta_i) \wedge \theta_j & \text{if } j > 1 \end{cases} = \begin{cases} (\theta_1 \vee \bigvee_{i>s} \theta_i) & \text{if } j = 1, \\ \theta_j & \text{if } 1 < j \leq s, \\ \perp & \text{if } j > s, \end{cases}$$

$$\varepsilon(\Box\Diamond\theta_j) = \begin{cases} (\Box\Diamond\theta_1 \vee \bigvee_{i>s}\Box\Diamond\theta_i) & \text{if } j = 1,\\ \Box\Diamond\theta_j & \text{if } 1 < j \leq s,\\ \bot & \text{if } j > s. \end{cases}$$

Then

$$\varepsilon\big(\Diamond\,\mathrm{Ker}(h^s)\big) = (\Box\Diamond\theta_1 \wedge \cdots \wedge \Box\Diamond\theta_s) \vee \bigvee_{i>s}(\Box\Diamond\theta_i \wedge \Box\Diamond\theta_2 \wedge \cdots \wedge \Box\Diamond\theta_s),$$

and hence the following rules are equivalent:

$$\frac{\Diamond\,\mathrm{Ker}(h^s)}{\Box\delta} \qquad \text{and} \qquad \frac{\bigvee_{0<i_1<\cdots<i_s}\Box\Diamond\theta_{i_1}\wedge\cdots\wedge\Box\Diamond\theta_{i_s}}{\Box\delta},$$

where $\bigvee_{0<i_1<\cdots<i_s}\Box\Diamond\theta_{i_1}\wedge\cdots\wedge\Box\Diamond\theta_{i_s}$ contains (as disjuncts) some (not necessarily all) formulas $\Box\Diamond\theta_{i_1}\wedge\cdots\wedge\Box\Diamond\theta_{i_s}$. Our task is to extend this disjunction to a complete one, that is, one in which all $\Box\Diamond\theta_{i_1}\wedge\cdots\wedge\Box\Diamond\theta_{i_s}$ appear.

This extension must be done in several steps. Suppose we transformed our rule to

$$\frac{(\Box\Diamond\theta_p \wedge B) \;\vee\; (\Box\Diamond\theta_q \wedge \Box\Diamond\theta_{q_2} \wedge \cdots \wedge \Box\Diamond\theta_{q_s}) \vee C}{\Box\delta},$$

where C is $\bigvee_{0<i_1<\cdots<i_s}\Box\Diamond\theta_{i_1}\wedge\cdots\wedge\Box\Diamond\theta_{i_s}$ and $\Box\Diamond\theta_p$ does not occur in C, and B is $\bigvee_{0<i_2<\cdots<i_s}\Box\Diamond\theta_{i_2}\wedge\cdots\wedge\Box\Diamond\theta_{i_s}$ and $\Box\Diamond\theta_p$ does not occur in B, and $\Box\Diamond\theta_p$ does not occur in $\Box\Diamond\theta_q\wedge\Box\Diamond\theta_{q_2}\wedge\cdots\wedge\Box\Diamond\theta_{q_s}$ (and $|\{q, q_2, \ldots, q_s\}| = s$).

Let us consider the substitution

$$\sigma(p_j) = \begin{cases} (\sim\theta_p \vee B) \to p_j & \text{if } p_j \text{ appears in } \theta_q,\\ (\sim\theta_p \vee B) \wedge p_j & \text{if } \sim p_j \text{ appears in } \theta_q, \end{cases}$$

and note that

$$\sigma(\theta_j) = \begin{cases} (\theta_p \wedge \sim B) \vee \theta_q & \text{if } j = q,\\ B \wedge \theta_p & \text{if } j = p,\\ \theta_j & \text{if } j \neq p, q, \end{cases}$$

$$\sigma(\Box\Diamond\theta_j) = \begin{cases} (\Box\Diamond\theta_p \wedge \sim B) \vee \Box\Diamond\theta_q & \text{if } j = q,\\ B \wedge \Box\Diamond\theta_p & \text{if } j = p,\\ \Box\Diamond\theta_j & \text{if } j \neq p, q. \end{cases}$$

Thus, using structurality, we can derive the following rule (from the one above):

$$\frac{(\Box\Diamond\theta_p \wedge B) \;\vee\; (\Box\Diamond\theta_p \vee \Box\Diamond\theta_q) \wedge \Box\Diamond\theta_{q_2} \wedge \cdots \wedge \Box\Diamond\theta_{q_s} \;\vee\; C}{\Box\delta},$$

which is sufficient to complete our argument. $\Box$

Corollary 5.16 *Let $h^s, h^r : \mathrm{Fm}_n \to \mathrm{Fm}_n^b$ and $g^s : \mathrm{Fm}_m \to \mathrm{Fm}_m^b$ be Henle homomorphisms with some $r \geq s \geq 2$ and $m \geq n \geq 1$. If $\mathrm{Var}(\delta) \cap \mathrm{Fm}_m = \emptyset$, then*

$$\frac{\Diamond\,\mathrm{Ker}(h^r)}{\Box\delta} \quad \textit{can be derived (over } \mathbf{S4.3}\textit{) from} \quad \frac{\Diamond\,\mathrm{Ker}(h^s)}{\Box\delta};$$

$$\frac{\Diamond\,\mathrm{Ker}(g^s)}{\Box\delta} \quad \textit{is equivalent (over } \mathbf{S4.3}\textit{) to} \quad \frac{\Diamond\,\mathrm{Ker}(h^s)}{\Box\delta}.$$

Let us get back to passive rules of the form ($\star$), where $\delta \in \mathrm{Fm}_n$. We still need to reduce them to rules of a simpler form and it cannot be done without taking into account the structure of the conclusion $\Box\delta$. We could assume, for instance, that δ is a characteristic formula for some S4.3-algebra. But introducing characteristic formulas is laborious and they would, in fact, only complicate our approach. Instead, we (use a trick and) note that the rule ($\star$) is equivalent to the rule

$$\frac{\Diamond \mathrm{Ker}(h_1) \vee \cdots \vee \Diamond \mathrm{Ker}(h_m)}{\Diamond \mathrm{Ker}(h_1) \vee \cdots \vee \Diamond \mathrm{Ker}(h_m) \to \Box\delta},$$

which, in turn, can be replaced by the collection of rules

$$\frac{\Diamond \mathrm{Ker}(h_1) \vee \cdots \vee \Diamond \mathrm{Ker}(h_m)}{\Diamond \mathrm{Ker}(h_i) \to \Box\delta}. \tag{$\star\star$}$$

Lemma 5.17 *Let h^r, h^s be Henle homomorphisms : $\mathrm{Fm}_n \to \mathrm{Fm}_n^b$ with $2 \leq s \leq r \leq 2^n$, and let $\theta_1, \ldots, \theta_{2^n}$ be all Boolean atoms in Fm_n^b, for some number $n \geq 1$. Then the following two rules are equivalent over* **S4.3***:*

$$\frac{\Diamond \mathrm{Ker}(h^s) \vee \Diamond \mathrm{Ker}(h^r)}{\Diamond \mathrm{Ker}(h^r) \to \Box\delta} \quad \textit{and}$$

$$\frac{\bigvee\{\Box \Diamond \theta_{i_1} \wedge \cdots \wedge \Box \Diamond \theta_{i_s} : 0 < i_1 < \cdots < i_s \leq 2^n\}}{\Diamond \mathrm{Ker}(h^r) \to \Box\delta}.$$

Proof Without loss of generality, we may assume that

$$\Diamond \mathrm{Ker}(h^s) = \Box \Diamond \theta_1 \wedge \cdots \wedge \Box \Diamond \theta_s \wedge \sim \Box \Diamond \theta_{s+1} \wedge \cdots \wedge \sim \Box \Diamond \theta_{2^n}.$$

Our argument is quite similar to that of Lemma 5.15. Let us define a substitution

$$\varepsilon(p_j) = \begin{cases} (\Diamond \mathrm{Ker}(h^r) \vee \bigwedge_{i>s} \sim \theta_i) \to p_j & \text{if } p_j \text{ appears in } \theta_1, \\ (\Diamond \mathrm{Ker}(h^r) \vee \bigwedge_{i>s} \sim \theta_i) \wedge p_j & \text{if} \sim p_j \text{ appears in } \theta_1, \end{cases}$$

and note that

$$\varepsilon(\theta_j) = \begin{cases} \sim \Diamond \mathrm{Ker}(h^r) \wedge (\bigvee\limits_{i>s} \theta_i) \vee \theta_1 & \text{if } j = 1, \\ (\Diamond \mathrm{Ker}(h^r) \vee \bigwedge_{i>s} \sim \theta_i) \wedge \theta_j & \text{if } j > 1 \end{cases}$$

$$= \begin{cases} (\theta_1 \vee \bigvee_{i>s} \theta_i) \wedge (\theta_1 \vee \sim \Diamond \mathrm{Ker}(h^r)) & \text{if } j = 1, \\ \theta_j & \text{if } 1 < j \leq s, \\ \Diamond \mathrm{Ker}(h^r) \wedge \theta_j & \text{if } j > s, \end{cases}$$

and hence

$$\varepsilon(\Box \Diamond \theta_j)$$
$$= \begin{cases} (\Box \Diamond \theta_1 \vee \bigvee_{i>s} \Box \Diamond \theta_i) \wedge (\Box \Diamond \theta_1 \vee \sim \Diamond \mathrm{Ker}(h^r)) & \text{if } j = 1, \\ \Box \Diamond \theta_j & \text{if } 1 < j \leq s, \\ \Diamond \mathrm{Ker}(h^r) \wedge \Box \Diamond \theta_j & \text{if } j > s. \end{cases}$$

Suppose that $\mathrm{Ker}(h^s) \neq \mathrm{Ker}(h^r)$. Then, for some $j > s$, we have $\Box \Diamond \theta_j \vdash \Diamond \mathrm{Ker}(h^r)$ and hence $\varepsilon(\Diamond \mathrm{Ker}(h^r)) = \Diamond \mathrm{Ker}(h^r)$, which gives

$$\begin{aligned}
&\varepsilon\big(\Diamond \operatorname{Ker}(h^s) \vee \Diamond \operatorname{Ker}(h^r)\big) \\
&\quad = (\Box \Diamond \theta_1 \wedge \cdots \wedge \Box \Diamond \theta_s) \\
&\qquad \vee \bigvee_{i>s} (\Box \Diamond \theta_i \wedge \Box \Diamond \theta_2 \wedge \cdots \wedge \Box \Diamond \theta_s) \vee \Diamond \operatorname{Ker}(h^r).
\end{aligned}$$

One can also prove the same if $\operatorname{Ker}(h^s) = \operatorname{Ker}(h^r)$ (in this case $\Diamond \operatorname{Ker}(h^r)$ is absorbed by $\Box \Diamond \theta_1 \wedge \cdots \wedge \Box \Diamond \theta_s$).

Moreover, $\varepsilon(\Diamond \operatorname{Ker}(h^r) \to \Box\delta) \vdash \Diamond \operatorname{Ker}(h^r) \to \Box\delta$ as $\Diamond \operatorname{Ker}(h^r) \vdash \varphi \leftrightarrow \varepsilon(\varphi)$ for each formula φ. Hence,

$$\frac{\Diamond \operatorname{Ker}(h^s) \vee \Diamond \operatorname{Ker}(h^r)}{\Diamond \operatorname{Ker}(h^r) \to \Box\delta} \qquad \text{and}$$

$$\frac{\Diamond \operatorname{Ker}(h^r) \vee \bigvee_{0<i_1<\cdots<i_s} \Box \Diamond \theta_{i_1} \wedge \cdots \wedge \Box \Diamond \theta_{i_s}}{\Diamond \operatorname{Ker}(h^r) \to \Box\delta}$$

are equivalent, where $\bigvee_{0<i_1<\cdots<i_s} \Box \Diamond \theta_{i_1} \wedge \cdots \wedge \Box \Diamond \theta_{i_s}$ contains (as disjuncts) some (not necessarily all) formulas $\Box \Diamond \theta_{i_1} \wedge \cdots \wedge \Box \Diamond \theta_{i_s}$. Our task is to extend this disjunction to a complete one, that is, one in which all $\Box \Diamond \theta_{i_1} \wedge \cdots \wedge \Box \Diamond \theta_{i_s}$ appear. Note that if this is done, then $\Diamond \operatorname{Ker}(h^r)$ will be absorbed by the disjunction.

This extension must be done in several steps. Suppose that we transformed our rule to

$$\frac{\Diamond \operatorname{Ker}(h^r) \;\vee\; (\Box \Diamond \theta_p \wedge B) \;\vee\; (\Box \Diamond \theta_q \wedge \Box \Diamond \theta_{q_2} \wedge \cdots \wedge \Box \Diamond \theta_{q_s}) \vee C}{\Diamond \operatorname{Ker}(h^r) \to \Box\delta},$$

where C is $\bigvee_{0<i_1<\cdots<i_s} \Box \Diamond \theta_{i_1} \wedge \cdots \wedge \Box \Diamond \theta_{i_s}$ and $\Box \Diamond \theta_p$ does not occur in C, and B is $\bigvee_{0<i_2<\cdots<i_s} \Box \Diamond \theta_{i_2} \wedge \cdots \wedge \Box \Diamond \theta_{i_s}$ and $\Box \Diamond \theta_p$ does not occur in B, and $\Box \Diamond \theta_p$ does not occur in $\Box \Diamond \theta_q \wedge \Box \Diamond \theta_{q_2} \wedge \cdots \wedge \Box \Diamond \theta_{q_s}$ (and $|\{q, q_2, \ldots, q_s\}| = s$).

Let us consider the substitution

$$\sigma(p_j) = \begin{cases} (\Diamond \operatorname{Ker}(h^r) \vee \sim \theta_p \vee B) \to p_j & \text{if } p_j \text{ appears in } \theta_q, \\ (\Diamond \operatorname{Ker}(h^r) \vee \sim \theta_p \vee B) \wedge p_j & \text{if} \sim p_j \text{ appears in } \theta_q, \end{cases}$$

and note that

$$\sigma(\theta_j) = \begin{cases} (\sim \Diamond \operatorname{Ker}(h^r) \wedge \theta_p \wedge \sim B) \vee \theta_q & \text{if } j = q, \\ (\Diamond \operatorname{Ker}(h^r) \wedge \theta_p) \;\vee\; (B \wedge \theta_p) & \text{if } j = p, \\ \theta_j & \text{if } j \neq p, q, \end{cases}$$

and hence

$$\sigma(\Box \Diamond \theta_j) = \begin{cases} (\sim \Diamond \operatorname{Ker}(h^r) \wedge \Box \Diamond \theta_p \wedge \sim B) \vee \Box \Diamond \theta_q & \text{if } j = q, \\ (\Diamond \operatorname{Ker}(h^r) \wedge \Box \Diamond \theta_p) \;\vee\; (B \wedge \Box \Diamond \theta_p) & \text{if } j = p, \\ \Box \Diamond \theta_j & \text{if } j \neq p, q. \end{cases}$$

Thus, using structurality, we can derive the following rule (from the one above):

$$\frac{\Diamond \operatorname{Ker}(h^r) \;\vee\; (\Box \Diamond \theta_p \wedge B) \;\vee\; (\Box \Diamond \theta_p \vee \Box \Diamond \theta_q) \wedge \Box \Diamond \theta_{q_2} \wedge \cdots \wedge \Box \Diamond \theta_{q_s} \;\vee\; C}{\Diamond \operatorname{Ker}(h^r) \to \Box\delta},$$

which is sufficient to complete our argument. $\Box$

Note that $\bigvee\{\Box\Diamond\theta_{i_1}\wedge\cdots\wedge\Box\Diamond\theta_{i_s} : 0 < i_1 < \cdots < i_s \le 2^n\}$, which is the premise of one of the rules involved, can also be written in the form $\bigvee_{j\ge s}\bigvee_i \Diamond\,\mathrm{Ker}(h_i^j)$. The (double) disjunction $\bigvee_{j\ge s}\bigvee_i$ can range over some (not necessarily all) Henle homomorphisms h_i^j (of rank $\ge s$), but h^s and h^r must appear there.

Theorem 5.18 *Each consequence relation over* **S4.3** *can be given by extending a modal logic with a collection of passive rules of the form:*

$$\frac{\Diamond\,\mathrm{Ker}(h)}{\Box\delta}\quad \textit{where } \mathrm{Ker}(h) \textit{ and } \Box\delta \textit{ have no variables in common.} \qquad (\star\star\star)$$

Proof As it has been noted, it suffices to consider passive rules of the form

$$\frac{\Diamond\,\mathrm{Ker}(h_1)\vee\cdots\vee\Diamond\,\mathrm{Ker}(h_m)}{\Diamond\,\mathrm{Ker}(h_i)\to\Box\delta}, \qquad (\star\star)$$

which is a rule formulated in Fm_n, and all Henle homomorphisms occurring there are $h : \mathrm{Fm}_n \to \mathrm{Fm}_n^b$ for some number n.

If h_i has a minimal Henle rank among the homomorphisms $h_1,\ldots,h_m$, then by Lemma 5.15 (take $h^r = h^s = h_i$) the rules

$$\frac{\Diamond\,\mathrm{Ker}(h_i)}{\Box\delta}\quad\text{and}\quad \frac{\bigvee\{\Box\Diamond\theta_{i_1}\wedge\cdots\wedge\Box\Diamond\theta_{i_s} : 0 < i_1 < \cdots < i_s \le 2^n\}}{\Diamond\,\mathrm{Ker}(h_i)\to\Box\delta}$$

are equivalent. Since the rank of the homomorphisms $h_1,\ldots,h_m$ is at least s, the rule $(\star\star)$ is also equivalent to the above rules as

$$\begin{aligned}&\Diamond\,\mathrm{Ker}(h_i)\ \vdash\ \Diamond\,\mathrm{Ker}(h_1)\vee\cdots\vee\Diamond\,\mathrm{Ker}(h_m)\quad\text{and}\\ &\Diamond\,\mathrm{Ker}(h_1)\vee\cdots\vee\Diamond\,\mathrm{Ker}(h_m)\\ &\quad\vdash\ \bigvee\{\Box\Diamond\theta_{i_1}\wedge\cdots\wedge\Box\Diamond\theta_{i_s} : 0 < i_1 < \cdots < i_s \le 2^n\}.\end{aligned}$$

Suppose that $h^r = h_i$ and that h^s is one of the homomorphisms $h_1,\ldots,h_m$ with a minimal rank $s < r$. Then, in the same way as above, one shows that $(\star\star)$ is equivalent to

$$\frac{\Diamond\,\mathrm{Ker}(h^s)\vee\Diamond\,\mathrm{Ker}(h^r)}{\Diamond\,\mathrm{Ker}(h^r)\to\Box\delta}.$$

Thus, we have reduced all passive rules to the following form:

$$\frac{\Diamond\,\mathrm{Ker}(h^s)\vee\Diamond\,\mathrm{Ker}(h^r)}{\Diamond\,\mathrm{Ker}(h^r)\to\Box\delta}\quad\text{where } 2\le s\le r.$$

Suppose such a rule is given, and take $m = 2n$. To avoid any misunderstandings, Henle homomorphisms : $\mathrm{Fm}_n \to \mathrm{Fm}_n^b$ are denoted by $h, h^r, \ldots$ and so on, whereas $g, g^r, \ldots$ stand for Henle homomorphisms : $\mathrm{Fm}_m \to \mathrm{Fm}_m^b$. Moreover, let us make a copy of the algebra Fm_n^b on the variables $\{p_{n+1},\ldots,p_m\}$, and let the copies of $h, h^s, \ldots$ be denoted by $h_*, h_*^s, \ldots$, respectively.

Since $\Diamond\,\mathrm{Ker}(h^r)$ is a $\Box\Diamond$-formula in Fm_n, then by Theorem 5.12 we have

$$\Diamond\,\mathrm{Ker}(h^r) = \Diamond\,\mathrm{Ker}(g_{i_1})\vee\cdots\vee\Diamond\,\mathrm{Ker}(g_{i_k})$$

for some g_i of rank $\ge r$ (and in a similar way one can express $\Diamond\,\mathrm{Ker}(h^s)$). Thus, our rule is equivalent to a collection of rules

$$\frac{\Diamond\,\mathrm{Ker}(h^s)\vee\Diamond\,\mathrm{Ker}(h^r)}{\Diamond\,\mathrm{Ker}(g_i)\to\Box\delta}\quad\text{which are equivalent to}\quad\frac{\bigvee_{j\ge s}\bigvee_i\Diamond\,\mathrm{Ker}(g_i^j)}{\Diamond\,\mathrm{Ker}(h^r)\to\Box\delta}$$

by Lemma 5.15 above. In the same way, one shows the equivalence of

$$\frac{\Diamond \mathrm{Ker}(h^s_*) \vee \Diamond \mathrm{Ker}(h^r)}{\Diamond \mathrm{Ker}(h^r) \to \Box\delta} \qquad \text{and} \qquad \frac{\bigvee_{j \geq s} \bigvee_i \Diamond \mathrm{Ker}(g^j_i)}{\Diamond \mathrm{Ker}(h^r) \to \Box\delta}.$$

It remains to show the equivalence of

$$\frac{\Diamond \mathrm{Ker}(h^s_*)}{\Diamond \mathrm{Ker}(h^r) \to \Box\delta} \qquad \text{and} \qquad \frac{\Diamond \mathrm{Ker}(h^s_*) \vee \Diamond \mathrm{Ker}(h^r)}{\Diamond \mathrm{Ker}(h^r) \to \Box\delta}.$$

Note that the first rule has the form ($\star\star\star$), and it suffices to derive the second rule from the first one as the reverse derivation is obvious.

Since $\Diamond \mathrm{Ker}(h^r) \to \Box\delta$ and $\Diamond \mathrm{Ker}(h^s_*)$ have no variables in common, we can use Lemma 5.15 and, after renaming of variables, we get (from the first rule)

$$\frac{\bigvee\{\Box\Diamond\theta_{i_1} \wedge \cdots \wedge \Box\Diamond\theta_{i_s} : 0 < i_1 < \cdots < i_s \leq 2^n\}}{\Diamond \mathrm{Ker}(h^r) \to \Box\delta},$$

$$\text{which is} \quad \frac{\Diamond \mathrm{Ker}(h^s) \vee \Diamond \mathrm{Ker}(h^r)}{\Diamond \mathrm{Ker}(h^r) \to \Box\delta}. \qquad \Box$$

Clearly, we can take a sufficiently big m that both $\Diamond \mathrm{Ker}(h)$ and $\Box\delta$ belong to Fm_m, but h would not be a Henle homomorphism then; we could only express $\Diamond \mathrm{Ker}(h)$ as a disjunction of some $\Diamond \mathrm{Ker}(g)$ and it is less clear which g's are involved there.

Remark In algebraic terms, this result can be stated as follows. Each quasivariety of S4.3-algebras can be axiomatized by a collection of quasiequations of the form $\Diamond \mathrm{Ker}(h) \approx \top \Rightarrow \Box\delta \approx \top$, where $\mathrm{Ker}(h)$ and $\Box\delta$ have no variables in common.

6 Modal Consequence Relations

Now, we show that results which hold for modal logics in NExt**S4.3** (see Theorem 4.1) can be generalized to the consequence relations extending **S4.3**. In particular, each consequence relation in **EXT(S4.3)** is *finitely based*, that is, it can be given by extending the underlying modal logic with a finite number of passive rules. Our main concerns, however, are the algebraic operations needed to construct a strongly adequate class of algebras, for a given consequence relation, from any class of finite s.i. algebras adequate for the underlying logic. We prove that, in addition to the operation of generating subalgebras (see Theorem 4.4), direct products of algebras from the initial class with Henle algebras are sufficient for our purposes.

Suppose that $\vdash$ is a modal consequence relation over **S4.3**, and let L be the set of its theorems; that is, $L = \{\alpha : \vdash \alpha\}$. Since $L \in$ NExt**S4.3**, there is a class $\mathbb{K}$ of finite s.i. $S4.3$-algebras such that $L = \mathrm{Log}(\mathbb{K})$. We have $\vdash_L \leq \vdash$, and (if $\vdash_L \neq \vdash$) $\vdash$ is an extension of $\vdash_L$ with some passive rules. Our aim is to obtain a class $\mathbb{L}$ of algebras which is strongly adequate for $\vdash$, that is, such that for each finite X and each α

$$X \vdash \alpha \qquad \text{iff} \qquad X \models_{\mathbb{L}} \alpha \qquad (\text{ iff} \qquad X \models_{\mathcal{B}} \alpha, \text{for each } \mathcal{B} \in \mathbb{L}).$$

Let $\mathbb{K}^{\vdash} = \{\mathcal{B} \in \mathbb{K} : \vdash \leq \models_{\mathcal{B}}\}$. It follows from the next lemma that the class $\mathbb{K}^{\vdash}$ is not strongly adequate for $\vdash$ if $\vdash_L \neq \vdash$.

Lemma 6.1 *For any class $\mathbb{K}$ of s.i. S4.3-algebras, we have*

$$\alpha \models_{\mathbb{K}} \beta \qquad \textit{iff} \qquad \Box\alpha \to \beta \in \mathrm{Log}(\mathbb{K}), \quad \textit{for each } \alpha, \beta.$$

Proof As ($\Leftarrow$) is obvious, we prove only ($\Rightarrow$). Let $\Box\alpha \to \beta \notin \mathrm{Log}(\mathbb{K}) = L$, and hence $\alpha \nvdash_L \beta$. If α is unifiable, then by Theorem 3.4 the rule α/β is not admissible for $\vdash_L$ and hence it is not admissible for $\models_{\mathbb{K}}$; consequently, $\alpha \not\models_{\mathbb{K}} \beta$.

Suppose that α is not unifiable. By Theorem 5.5, the rule α/β is equivalent over **S4.3** to a subrule $\Diamond\gamma \wedge \Diamond \sim \gamma/\delta$ of P_2. As $\Box\alpha \to \beta \notin L$, both rules are not $\vdash_L$-derivable and hence there are an algebra $\mathcal{A}$ in $\mathbb{K}$ and a valuation $v\colon \mathrm{Var} \to A$ such that

$$v\big(\Box(\Diamond\gamma \wedge \Diamond \sim \gamma) \to \delta\big) \neq \top. \tag{1}$$

Note that the elements $v(\Box(\Diamond\gamma \wedge \Diamond \sim \gamma))$ and $v(\sim (\Diamond\gamma \wedge \Diamond \sim \gamma))$ are open in $\mathcal{A}$; the second element is open as it is equal to $\Box v(\sim (\gamma) \vee \Box v(\gamma))$, and the join of two open elements is open in any TBA. Since $\mathcal{A}$ is an s.i. modal algebra for **S4.3**, one of the following holds:

(i) $v(\Box(\Diamond\gamma \wedge \Diamond \sim \gamma)) \leq v(\sim (\Diamond\gamma \wedge \Diamond \sim \gamma))$; and then (as $\Box a \leq a$, for each element a) we get $v(\Box(\Diamond\gamma \wedge \Diamond \sim \gamma)) = v(\Box(\Diamond\gamma \wedge \Diamond \sim \gamma) \wedge v(\Diamond\gamma \wedge \Diamond \sim \gamma) = v(\bot) = \bot$ which contradicts (1);

(ii) $v(\Box(\Diamond\gamma \wedge \Diamond \sim \gamma) \geq v(\sim (\Diamond\gamma \wedge \Diamond \sim \gamma))$; and then we get $v(\Diamond\gamma \wedge \Diamond \sim \gamma) = v(\Box(\Diamond\gamma \wedge \Diamond \sim \gamma)) \vee v(\Diamond\gamma \wedge \Diamond \sim \gamma) = v(\top) = \top$ which gives $\Diamond\gamma \wedge \Diamond \sim \gamma \not\models_{\mathbb{K}} \delta$ by (1), and hence we get $\alpha \not\models_{\mathbb{K}} \beta$. $\Box$

The (weak) deduction theorem holds for the consequence relation $\models_{\mathbb{K}}$ generated by any class of s.i. $S4.3$-algebras and, consequently, $\mathbb{K}$, as well as $\mathbb{K}^{\vdash}$, is strongly adequate for $\vdash_L$ (but not for $\vdash$). No class of s.i. S4.3-algebras can be strongly adequate for any proper extension of $\vdash_L$ with passive rules. Thus, to get models for $\vdash$ one should consider products of s.i. algebras. In fact, products with Henle algebras are sufficient here.

Theorem 6.2 *Let $\vdash$ be an extension of $\vdash_L$, for $L \in \mathrm{NExt}\mathbf{S4.3}$, with some passive rules, and let $\mathbb{K}$ be a class of finite s.i. S4.3-algebras which is strongly adequate for $\vdash_L$. Then*

(i) $\vdash$ *is finitely based,*

(ii) $\mathbb{L} = \{\mathcal{A} \times \mathfrak{n}^+ : \mathcal{A} \in S(\mathbb{K})\ ,\ n \geq 1,\ \vdash \leq \models_{\mathcal{A}\times\mathfrak{n}^+}\}$ *is strongly adequate for* $\vdash$.

Proof Before proving the theorem, we will prove an auxiliary condition (E) below. Suppose that $n \geq 2$ and that R_n is a set of rules (of the form ($\star\star\star$)):

$$\frac{\Diamond\,\mathrm{Ker}(h^s)}{\Box\delta} \quad \text{with } 2 \leq s \leq n \text{ and } \mathrm{Var}\big(\mathrm{Ker}(h^s)\big) \cap \mathrm{Var}(\delta) = \emptyset.$$

Let $\vdash_n$ be the modal consequence relation extending $\vdash_L$ with the rules R_n, and let L_n be the modal logic extending L with $\{\alpha \to \beta : \alpha/\beta \text{ is a rule in } R_n\}$. Let us prove

$$\Diamond\,\mathrm{Ker}(h^n) \vdash_n \Box\delta \qquad \text{iff} \qquad \Diamond\,\mathrm{Ker}(h^n) \to \Box\delta \in L_n, \quad \text{for every } h^n, \delta. \quad (E)$$

The implication ($\Rightarrow$) is obvious as the rules R_n are derivable based on L_n, and we have the deduction theorem for $\vdash_{L_n}$. One easily shows that $\Box\Diamond\,\mathrm{Ker}(h^n) = \Diamond\,\mathrm{Ker}(h^n)$ using Theorem 5.10 and the known fact that $\Box\Diamond$-formulas are clopen in **S4.3**.

To prove ($\Leftarrow$), let us assume $\Diamond\,\mathrm{Ker}(h^n) \to \Box\delta \in L_n$. Then

$$\vdash_{S4.3} (\alpha_1 \to \beta_1) \wedge \cdots \wedge (\alpha_k \to \beta_k) \to \big(\Diamond\,\mathrm{Ker}(h^n) \to \Box\delta\big),$$

where $\alpha_1/\beta_1, \ldots, \alpha_k/\beta_k$ are instances of the rules R_n. Using our assumptions on the rules R_n and Corollary 5.16, we get $\Diamond \operatorname{Ker}(h^n) \vdash_n \beta_i$ for each $i \le k$ and hence $\Diamond \operatorname{Ker}(h^n) \vdash_n \delta$ by (MP).

(i) It has not been assumed (until now) that the rules R_n have anything to do with $\vdash$. Now, let $\vdash_1 = \vdash$ and $L_1 = L$, and take R_1 empty. As R_2, take the set of all $\vdash$-valid rules which have the required form with $n = 2$. The logic L_2 is finitely axiomatizable (see Theorem 4.1(ii)); hence we can choose from $\{\alpha \to \beta : \alpha/\beta$ is a rule in $R_2\}$ a finite subset of the set of its axioms. Let R_2' be the finite set of $\vdash$-valid rules which correspond to the mentioned finite set of axioms for L_2. It follows from (E) that R_2' and R_2 are equivalent, and thus we get a finite basis for $\vdash_2$.

We proceed as above with any $n \ge 2$. So, suppose that finite sets of passive rules $R_2' \subseteq \cdots \subseteq R_n'$ have been defined and that R_n' is equivalent to R_n, the set of all $\vdash$-rules having the appropriate form. Assume that R_{n+1} is the set of all $\vdash$-rules having the above form (where we put $n+1$ in place of n). Again, using Theorem 4.1(ii), we find a finite set $R_{n+1}' \supseteq R_n'$ which is a basis for $\vdash_{n+1}$.

It remains to take $\bigcup_{n=2}^{\infty} R_n'$ as a finite basis for $\vdash$. The set is finite as the sequence $L_2 \subseteq L_3 \subseteq \cdots$ must terminate, which means that $L_n = L_{n+1} = \cdots = L_m = \cdots$ for some n and each $m \ge n$. This termination is, again, a consequence of Theorem 4.1(ii) as the logic $\bigcup_{n=2}^{\infty} L_n$ is finitely axiomatizable and hence is equal to some L_n.

(ii) Let us consider algebraic models for the theories L_n. We can take $\mathbb{K}_1 = S(\mathbb{K})$ and, according to Theorem 4.4, we can find $\mathbb{K}_n = S(\mathbb{K}_n)$, for each n, such that $L_n = \operatorname{Log}(\mathbb{K}_n)$ and $\mathbb{K}_1 \supseteq \mathbb{K}_2 \supseteq \mathbb{K}_3 \supseteq \cdots$. Let

$$\begin{cases} \mathbb{L}_2 = \mathbb{K}_2 \cup ((\mathbb{K}_1 \setminus \mathbb{K}_2) \times \mathfrak{1}^+), \\ \mathbb{L}_{n+1} = \mathbb{K}_{n+1} \cup ((\mathbb{K}_n \setminus \mathbb{K}_{n+1}) \times \mathfrak{n}^+) \cup \cdots \cup ((\mathbb{K}_1 \setminus \mathbb{K}_2) \times \mathfrak{1}^+), \end{cases}$$

where $\mathbb{K}_i \times \mathcal{A} = \{\mathcal{B} \times \mathcal{A} : \mathcal{B} \in \mathbb{K}_i\}$.

Let us show by induction on n that $\mathbb{L}_n$ is a model for $\vdash_n$; that is, $\vdash_n \le \vdash_{\mathbb{L}_n}$. Take $n = 2$. The rules R_2 are derivable in L_2 and hence they are valid in the algebras $\mathbb{K}_2$. The rules R_2 are also valid in $(\mathbb{K}_1 \setminus \mathbb{K}_2) \times \mathfrak{1}^+$, as $\Diamond \operatorname{Ker}(h^2)$'s are not satisfiable in $\mathfrak{1}^+$ (see Corollary 2.6). In fact, all passive rules are valid in these algebras.

Suppose that $\vdash_n \le \vdash_{\mathbb{L}_n}$, which means that the rules R_n are valid in the algebras

$$\mathbb{K}_n \cup \big((\mathbb{K}_{n-1} \setminus \mathbb{K}_n) \times (\mathfrak{n}-\mathfrak{1})^+)\big) \cup \cdots \cup \big((\mathbb{K}_1 \setminus \mathbb{K}_2) \times \mathfrak{1}^+\big).$$

Let us consider any rule in R_{n+1} which has $\Diamond \operatorname{Ker}(h^{n+1})$ as the premise. By Corollary 2.6, the rule is valid in the algebras of

$$\big((\mathbb{K}_n \setminus \mathbb{K}_{n+1}) \times \mathfrak{n}^+\big) \cup \cdots \cup \big((\mathbb{K}_1 \setminus \mathbb{K}_2) \times \mathfrak{1}^+\big).$$

The rule is also valid in $\mathbb{K}_{n+1}$ by the definition of $\mathbb{K}_{n+1}$.

Since $\vdash = \vdash_m$, we get $\mathbb{L}_m \subseteq \mathbb{L}$ for any sufficiently big m. It is also clear that $\vdash \le \vdash_{\mathbb{L}}$. Thus, to complete our proof it suffices to show that $\vdash \ge \vdash_{\mathbb{L}}$. Since both consequence relations have the same sets of theorems (which is the logic L), we need to show that each passive rule ($\star\star\star$) of $\vdash_{\mathbb{L}}$ is also a (derivable) rule of $\vdash$.

Suppose that $\Diamond \operatorname{Ker}(h^n)/\Box\delta$ is not $\vdash$-derivable. Then $\Diamond \operatorname{Ker}(h^n) \to \Box\delta \notin L_n$, and hence there is an algebra $\mathcal{A} \in \mathbb{K}_n$ such that $\Diamond \operatorname{Ker}(h^n) \to \Box\delta \notin \operatorname{Log}(\mathcal{A})$. The consequence $\vdash_{\mathcal{A}}$ enjoys the deduction theorem (see Lemma 6.1) and hence $\Diamond \operatorname{Ker}(h^n) \nvdash_{\mathcal{A}} \Box\delta$. If one takes $m \ge n$, then we also have $\Diamond \operatorname{Ker}(h^n) \nvdash_{\mathcal{A} \times \mathfrak{m}^+} \Box\delta$. It means that $\Diamond \operatorname{Ker}(h^n)/\Box\delta$ is not $\mathbb{L}_m$-valid and hence it is not $\vdash_{\mathbb{L}}$-derivable. $\Box$

Corollary 6.3 *Every modal consequence relation extending* **S4.3** *has the strong finite model property and is finitely based.*

Proof If $\vdash$ is a modal consequence relation extending **S4.3**, then the set of its theorems $L = \{\alpha : \vdash \alpha\}$ is a modal logic over **S4.3**. Let $\mathbb{K}$ be a class of finite s.i. $S4.3$-algebras with $L = \mathrm{Log}(\mathbb{K})$. It follows from Lemma 6.1 that $\mathbb{K}$ is strongly adequate for $\vdash_L$. Since $\vdash_L$ is almost structurally complete (see Theorem 3.4), then $\vdash$ can be given as an extension of $\vdash_L$ with some passive rules. Thus, all assumptions of the above theorem are fulfilled, and we therefore conclude that $\vdash$ is finitely based and has the strong finite model property. □

Corollary 6.4 *Let $L \in$* NExt**S4.3**. *Then every quasivariety of L-algebras is generated by a class of the form*

$$\mathbb{K}_n \cup \big((\mathbb{K}_{n-1} \setminus \mathbb{K}_n) \times (\mathfrak{n}-1)^+)\big) \cup \cdots \cup \big((\mathbb{K}_1 \setminus \mathbb{K}_2) \times 1^+\big),$$

where each $\mathbb{K}_m$, $m = 1, \ldots, m$, is a class of some finite s.i. L-algebras, and $\mathfrak{m}^+$ denotes a simple monadic algebra with m atoms.

Hence, to describe quasivarieties of L-algebras, s.i. L-algebras are not sufficient—one needs direct products of them and of some simple monadic algebras.

Similarly, as for logics, we can use (a counterpart of) Harrop's theorem to obtain the following.

Corollary 6.5 *Every modal consequence relation extending* **S4.3** *is decidable.*

If $\vdash$ is structurally complete, then all passive rules are $\vdash$-derivable and hence L_2 is inconsistent, which means that $\mathbb{K}_2$ is empty. Thus, we obtain the following characterization of structurally complete modal logics over **S4.3**.

Corollary 6.6 *The structurally complete extension of $\vdash_{\mathbb{K}}$, that is, the extension of $\vdash_{\mathbb{K}}$ with P_2, is strongly complete with respect to the family $\{\mathcal{B} \times \mathbf{2} : \mathcal{B} \in \mathbb{K}\}$.*

Note that our formulation of Theorem 6.2(ii) is ineffective and that we actually prove more than is claimed there. For each $\vdash$ we define a sequence of modal logics $L = L_1 \subseteq L_2 \subseteq \cdots$ (the sequence terminates), and for each L_i there is given an adequate family $\mathbb{K}_i$ of algebras. Then we construct a class of algebras, for a sufficiently big number m, taking the direct products of some of the algebras with appropriate Henle algebras,

$$\mathbb{K}_m \cup \big((\mathbb{K}_{m-1} \setminus \mathbb{K}_m) \times (\mathfrak{m}-1)^+)\big) \cup \cdots \cup \big((\mathbb{K}_1 \setminus \mathbb{K}_2) \times 1^+\big),$$

which is strongly adequate for $\vdash$. This can be used for an effective construction of the class $\mathbb{L}$ that is strongly adequate for $\vdash$. We can also use it to show the following.

Theorem 6.7 *The lattice* **EXT**(**S4.3**) *is countable and distributive.*

Proof The lattice is countable by Theorem 6.2(i). To prove its distributivity, let us recall that the lattice NExt**S4.3** is distributive and, as previously stated, there is one-to-one correspondence between consequence relations in **EXT**(**S4.3**) and terminating sequences $L_1 \subseteq L_2 \subseteq \cdots$ of theories in NExt**S4.3**. Thus, we get an embedding of **EXT**(**S4.3**) into the direct product (= the power lattice) NExt$\mathbf{S4.3}^\omega$ = NExt**S4.3** × NExt**S4.3** × ⋯. It is easy to see that the embedding preserves the lattice operations (which means it is a lattice homomorphism). Since sublattices and products of distributive lattices remain distributive, we conclude that **EXT**(**S4.3**) is distributive. □

Since **EXT**(**S4**.**3**) is a complete algebraic lattice (see a more general case in [3]), which is distributive, the lattice satisfies the join-infinite distributive law; that is, finite meets distribute over arbitrary joins. Hence, a relative pseudocomplement can be defined in a standard way; therefore, we get the following.

Corollary 6.8 *The lattice* **EXT**(**S4**.**3**) *forms a countable complete Heyting algebra. The lattice of all quasivarieties of S4.3-algebras forms a countable complete Heyting algebra.*

We have shown that some known results concerning NExt**S4**.**3** extend to the lattice **EXT**(**S4**.**3**) of all modal consequence relations over **S4**.**3**. Even if one says that the result was expected, one would have to admit that the most essential part of our paper, that is, the reduction of passive rules to a more convenient form, was unexpectedly laborious and tedious. Projective unification was applied in it and we could not have got along without it. We also know that none of the results considered here extend to the case of infinitary rules, but that is another story.

Notes

1. As pointed out by an anonymous referee, it is also known that—according to Hemaspaandras's theorem—the satisfiability problem of every normal extension of **S4**.**3** is NP-complete (see [2, Theorem 6.41]).

2. What we call a *modal logic* is often called a *normal modal logic*.

3. This hint was suggested to us by Silvio Ghilardi in a private correspondence.

4. The name "Henle matrices," and later also "Henle algebras," was widely used in modal logic papers (see, e.g., Scroggs [25]).

5. A particular splitting of NExt**S4** and unification types in modal logics extending **S4** are related (see Dzik [8], [9]).

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Acknowledgments

The authors wish to thank the anonymous reviewers for their valuable comments.

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