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Note on Extending Congruential Modal Logics

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Abstract It is observed that a consistent congruential modal logic is not guaranteed to have a consistent extension in which the Box operator becomes a truth-functional connective for one of the four one-place (two-valued) truth functions.

1 Background

Let us fix the language of propositional (mono)modal logic as having countably many sentence letters ($p_1, \ldots, p_n, \ldots$) and primitive connectives $\to$, $\Box$, and $\bot$ of arities 2, 1, and 0, respectively, subject to the usual formation rules;[1] other connectives, in particular $\top$, $\neg$, $\wedge$, and $\leftrightarrow$, are taken as defined in the familiar ways. A *modal logic* is a set of formulas in this language which contains all truth-functional tautologies and is closed under uniform substitution and modus ponens, and is said to be *consistent* if it does not contain every formula (equivalently, does not contain $\bot$). A modal logic S is *monotone*, *antitone*, or *congruential*, respectively, when $A \to B \in S$ implies $\Box A \to \Box B \in S$ (for all formulas A, B), or $A \to B \in S$ implies $\Box B \to \Box A \in S$ (all formulas A, B), or $A \leftrightarrow B \in S$ implies $\Box A \leftrightarrow \Box B \in S$ (again: all A, B). (This terminology is taken from Makinson [14] and [15].) Finally, the term *truth function* refers throughout to two-valued truth functions (similarly with cognate vocabulary, such as *truth-functional*), though we sometimes include a parenthetical reminder to that effect.

Theorem 1 of Makinson [14] tells us that if S is any consistent congruential modal logic containing $\Box\top$ and $\neg\Box\bot$, then S can be consistently extended by the addition of all formulas $\Box A \leftrightarrow A$. Calling the modal logic containing all such formulas the *identity* logic, Makinson's formulation is that any consistent congruential S containing $\Box\top$ and $\neg\Box\bot$ is a sublogic of the identity logic—which is itself a consistent

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congruential logic. The "can be consistently extended" formulation has the advantage of being more directly suggestive of yet a further reformulation: the $\Box$ operator in any S meeting the stated conditions can be interpreted as the identity truth function, and on this interpretation every formula in S would be a truth-functional tautology (which is not to say in addition that every formula which is tautologous when so interpreted belongs to S). The equivalence of Makinson's formulation and that in terms of how S "can be extended" exploits the fact that if each of S, S' is consistently extended by all instances of the schema $\Box A \leftrightarrow A$, then the extensions coincide, being simply the identity logic, which is itself explained by the fact that, since the identity logic is just nonmodal classical propositional logic with a redundant additional primitive ($\Box$), this logic is Post-complete in the sense of having no consistent modal logics properly extending it, that is. (Without this, it could be that extending S by adding all these $\Box A \leftrightarrow A$ formulas gives a consistent logic distinct from that obtained by extending S' likewise.)

Theorems 2 and 3 of [14] tell us that if S is any consistent monotone or antitone modal logic, then $\Box$ can be given a similarly truth-functional interpretation, as the identity truth function, the (1-ary) constant true truth function or as the constant false truth function in the monotone case, and as the negation truth functions or one of the two constant functions in the antitone case. Put in terms of extensions, this means that these logics can be consistently extended by the addition of all formulas $\Box A \leftrightarrow A$, or all formulas $\Box A \leftrightarrow \neg A$, or all formulas $\Box A \leftrightarrow \top$, or all formulas $\Box A \leftrightarrow \bot$.[2] As with [14, Theorem 1], we can put this equally well by saying (as Makinson does) that under the conditions given, S is a sublogic of the identity logic, the complement logic, the unit logic, and the zero logic, respectively, where these last three are the (Post-complete, congruential) logics containing all instances of the last three schemata, respectively.[3]

One perspective on the congruentiality of these four logics is obtained by a consideration of their intersection, observed in Humberstone [6] (see also Humberstone [7] and [9, Section 3.2]) to be the smallest modal logic containing all instances of the "extensionality" schema $(A \leftrightarrow B) \rightarrow (\Box A \leftrightarrow \Box B)$; this is the intersection of the four Makinson logics (of the preceding paragraph), and these instances belong to each of them, and congruentiality following by appeal to modus ponens. We return to this schema in the Coda section below.

2 A Question and Its Answer

Since all monotone or antitone modal logics are congruential, though not conversely, Humberstone [8, pp. 449–451] asks in passing whether the result just described—on the amenability of $\Box$ to a (two-valued) truth-functional interpretation in any consistent monotone or antitone modal logic—applies more generally to the case of arbitrary consistent congruential modal logics. The discussion there is inconclusive, confining itself to noting the failure of one line of thought to yield a correct proof. A partial answer is of course provided by Makinson's Theorem 1, reported above, but one would like to know whether or not this is the case universally, rather than just for the case of the consistent congruential modal logics containing $\Box\top$ and $\neg\Box\bot$. Here we show with a simple example that the answer to the general question is negative.

To that end we recapitulate the basics of the neighborhood semantics for modal logic. A *neighborhood frame* is a pair $\langle W, \mathcal{N} \rangle$ with W a nonempty set and $N : W \longrightarrow \wp(\wp(W))$. For $x \in W$, the sets $X \in N(x)$ are spoken of as "neighborhoods" of the point x (generalizing the notion of a neighborhood from metric spaces—or indeed topology—suggested in Scott [17] for a temporal reading of $\Box$ as a marker of progressive aspect, the elements of W here being thought of as moments of time with the structure of the real numbers). A *(neighborhood) model* is obtained by expanding such a frame with a function V assigning arbitrary subsets of W to the sentence letters, with $x \in V(p_i)$ thought of as stipulating that p_i is true at $x \in W$. The inductive definition of truth of a formula A at a point $x \in W$ in a neighborhood model $\mathcal{M} = \langle W, N, V\rangle$ (written as "$\mathcal{M} \models_x A$") is given by the following, in which $\|A\|^{\mathcal{M}}$ denotes $\{y \in W \mid \mathcal{M} \models_y A\}$:

- $\mathcal{M} \models_x p_i$ if and only if $x \in V(p_i)$;
- $\mathcal{M} \models_x A \to B$ if and only if either $\mathcal{M} \not\models_x A$ or $\mathcal{M} \models_x B$;
- $\mathcal{M} \not\models_x \bot$;
- $\mathcal{M} \models_x \Box A$ if and only if $\|A\|^{\mathcal{M}} \in N(x)$.

If, for $\mathcal{M} = \langle W, N, V\rangle$, we have $\mathcal{M} \models_x A$ for all $x \in W$, then A is *true throughout* $\mathcal{M}$, notated as "$\mathcal{M} \models A$," and if $\mathcal{M} \models A$ for every model $\mathcal{M}$ expanding a given neighborhood frame $\langle W, \mathcal{N}\rangle$, then A is said to be *valid on* the frame $\langle W, \mathcal{N}\rangle$. Well-known facts about this style of semantics include the fact that the smallest congruential modal logic contains exactly those formulas valid on every neighborhood frame, that for every consistent congruential modal logic S there is a model the formulas true throughout which are exactly those A for which $A \in S$, and that it is not true that for every such logic S there is a neighborhood frame (or even a class of such frames) such that the formulas valid on that frame (on each frame in the class) are exactly those $A \in S$.[4] But all we need here is this easily checked fact: for any neighborhood frame, the class of formulas valid on that frame is a consistent congruential logic.

With this fact in mind, take two objects a, b ($a \neq b$) and consider the neighborhood frame $\langle W, N\rangle$ with $W = \{a, b\}$, $N(a) = \{\varnothing\}$, $N(b) = \{\varnothing, \{a, b\}\}$. From now on, let S_0 be the set of formulas valid on this frame. We are concerned with the formula $\Box\bot \wedge \neg\Box\Box\Box\bot$. Since this formula contains no sentence letters, its truth throughout one model on the frame is equivalent to its truth throughout any other model on the frame, and thus to its validity on the frame. To record this lack of dependence on a given model, we use the $\|\cdot\|$ notation without the superscript "$\mathcal{M}$" when considering this formula and its subformulas. Since $\|\bot\| = \varnothing$ and $\varnothing$ belongs to each of $N(a)$ and $N(b)$, $\|\Box\bot\| = W$. Thus $\|\Box\Box\bot\| = \{x \mid W \in N(x)\} = \{b\}$ (as $W \notin N(a)$). But $\{b\}$ is not a neighborhood of a or of b, so $\|\Box\Box\Box\bot\| = \varnothing$, and so $\|\neg\Box\Box\Box\bot\| = W$. As already noted, we have $\|\Box\bot\| = W$, so $\|\Box\bot \wedge \neg\Box\Box\Box\bot\| = W$, and thus $\Box\bot \wedge \neg\Box\Box\Box\bot \in S_0$. This all but completes the demonstration that $\Box$, as it behaves in the consistent congruential logic S_0, cannot be given a truth-functional interpretation, the proof below hammering in the final nail.

Proposition 2.1 *There is no truth function f with the property that every $A \in S_0$ is a tautology when $\Box$ is interpreted as f.*

Proof We need the well-known (and of course easily checked) fact that every one-place truth function f satisfies $f = f^3$. Thus for any formula B, $\Box B \to \Box\Box\Box B$ (as well as its converse) belongs to the extension of S_0 by all instances of any one of the schemata, $\Box A \leftrightarrow A$, $\Box A \leftrightarrow \neg A$, $\Box A \leftrightarrow \top$, $\Box A \leftrightarrow \bot$; each such extension is

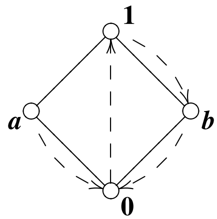

Figure 1 Our neighborhood frame as an expanded Boolean algebra.

accordingly inconsistent, as we see by taking $B = \bot$, since we have already noted that $\Box\bot \wedge \neg\Box\Box\Box\bot \in S_0$ (and thus $\neg(\Box\bot \to \Box\Box\Box\bot) \in S_0$). □

Let us pause to place the well-known fact cited in this proof into broader perspective: $F(F(F(x))) \approx F(x)$ is a hyperidentity of Boolean algebras, in the sense of Taylor [18].[5]

We can reconstrue the neighborhood frame of the above example as a Boolean algebra expanded by an operation interpreting $\Box$, in the sense that for any model $\mathcal{M}$ on the frame, $\|\Box A\|^{\mathcal{M}}$ is the result of applying this operation to $\|A\|^{\mathcal{M}}$. The algebra in question is depicted in Figure 1, with a solid-line Hasse diagram for its Boolean reduct and dashed arrows indicating the action of the $\Box$-operation. Also, **1**, ***a***, ***b***, and **0** represent the subsets $\{a, b\}$ $(=W)$, $\{a\}, \{b\}$, and $\varnothing$, respectively. Such algebras—*modal algebras* in a suitably general sense, not building in normality—are called "Boolean frames" in Hansson and Gärdenfors [5], where it is observed that in the finite case they correspond one-to-one with neighborhood frames, corresponding structures validating the same formulas.[6]

We prefer to stress the semantic characterization in terms of neighborhood frames, however, since not every consistent congruential logic is determined by a class of such frames (see note 4); thus we can say something more informative than just that we have in S_0 an example of a consistent congruential logic with no consistent extension in which $\Box$ admits of a truth-functional interpretation, since S_0 serves as a witness to the following corollary of Proposition 2.1, for whose formulation we use the phrase *Makinson logic* as above to describe the logics in which $\Box$ is interpreted by one of the four truth functions.

Corollary 2.2 *There are consistent neighborhood-complete congruential modal logics which are not sublogics of any of the four Makinson logics.*

One notices that neither of the conditions $\Box\top \in S$ nor $\neg\Box\bot \in S$ on Makinson's Theorem 1 (from [14]) is satisfied when S is taken as our illustrative S_0. The algebraic format (of Figure 1) is probably easier than the neighborhood frames format for experimenting with the possibility of dropping satisfying one but not the other of those conditions; but since here we are interested more in $\Box$'s amenability to some truth-functional interpretation or other rather than specifically as the identity truth function in particular, let us note that [14, Theorem 1], recalled in our Background section above, admits of a very straightforward generalization, proved in the same way as that result is proved in [14];[7] for stating this theorem we say that a logic S *decides* a formula A if either $A \in S$ or $\neg A \in S$.

Theorem 2.3 *If S is a consistent congruential modal logic which decides each of the formulas $\Box\top$ and $\Box\bot$, then S is a sublogic of one of the Makinson logics.*

Thus the important feature of our counterexample to the hypothesis that every consistent congruential modal logic can be extended to one of the four $\Box$-as-truth-functional logics is that the logic concerned, S_0, does not decide both $\Box\top$ and $\Box\bot$ (though it does decide $\Box\bot$). Note also in passing that for congruential modal logics S the condition that S decides $\Box\top$ and $\Box\bot$ is equivalent to the condition that S decides all pure formulas (where a *pure* formula is a formula containing no propositional variables).

Of course one could be more specific than Theorem 2.3 is about how the way the formulas mentioned get decided fixes what the relevant extension is, as Makinson's result did for the case in which $\Box\top$ and $\Box\bot$ are, respectively, decided positively and negatively; if both are decided positively, then the logic in question is a sublogic of the unit logic, for example.[8] While on the subject of this example, in fact, we note that it shows how the present generalized version of Makinson's Theorem 1 (from [14]) gives information on logics not covered by his version or by his Theorems 2 and 3, which pertain to monotone and antitone modal logics: consider the case of noncontingency, traditionally denoted by Δ, with ΔA understood as $\Box A \vee \Box\neg A$ for $\Box$ from any consistent normal modal logic, except that here we take Δ as primitive and write it as "$\Box$": the resulting modal logic is consistent and congruential but neither monotone nor antitone, while falling under the present result as containing both $\Box\top$ and $\Box\bot$.[9]

Finally, we include an alternative example to that used to illustrate for Proposition 2.1 and Corollary 2.2 which is simpler in one respect and more complex in another. One measure of simplicity is low modal degree, and the case of S_0 fares rather poorly in this respect, since both the general principle $\Box B \to \Box\Box\Box B$ (ignoring any modality that might arise in instantiating this schema to a particular B) and the specific S_0 formula $\Box\bot \wedge \neg\Box\Box\Box\bot$ inconsistent with the schema are of modal degree 3. The general principle, despite its high modal degree, involves only a single schematic letter (or, if we are using a representative instance of the schema, say, $\Box p_1 \to \Box\Box\Box p_1$, only a single sentence letter/propositional variable). Our second example, given in the following paragraph, is worse in this respect, in that the general principle used to make problems—the Aggregation schema below—involves two schematic letters, but better in respect of modal degree in that it has modal degree 1.

Again we give a two-element neighborhood frame $\langle W, N\rangle$ with $W = \{a, b\}$, but this time $N(a) = \{\{a\}, \{b\}\}$ and $N(b) = \{\{a\}, \{b\}, \{a, b\}\}$. Denote by S_1 the set of formulas valid on this frame. We have $\|\Box\top\| = \{b\}$, so $\|\neg\Box\top\| = \{a\}$. (Again we use $\|\cdot\|$ to indicate independence from any choice of V in a particular model on the frame.) Since $\{a\}$ and $\{b\}$ are neighborhoods of both points, $\|\Box\neg\Box\top\| = W$ and $\|\Box\Box\top\| = W$, and since $\varnothing$ is a neighborhood of no point, $\|\neg\Box\bot\| = W$. Thus the formula $\varphi = (\Box\Box\top \wedge \Box\neg\Box\top) \wedge \neg\Box\bot$ is valid on the frame and so belongs to S_1. Now suppose that S_1^+ is the extension of S_1 by one of the four truth-functionalizing schemata. Let us recall that on any truth-functional interpretation of $\Box$ all instances of the following "Aggregation" schema are tautologous:[10]

$$(\Box A \wedge \Box B) \to \Box(A \wedge B).$$

So, in particular, taking A, B, as $\Box\top, \neg\Box\top$, respectively, in the first conjunct of φ, we conclude that

$$\Box(\Box\top \wedge \neg\Box\top) \wedge \neg\Box\bot \in S_1^+.$$

By congruentiality we can replace the contradiction in the scope of the $\Box$ on the first conjunct here by $\bot$, giving that $\Box\bot \wedge \neg\Box\bot \in S_1^+$: S_1^+ is inconsistent.

One could also give a description in the style of Figure 1 of the induced modal algebra, though here we are keen to stress that S_1 does as well as S_0 does to establish Corollary 2.2, with its built-in neighborhood completeness rider. Observe also that, apropos of Theorem 2.3, S_1, like S_0, does not decide $\Box\top$ but does decide $\Box\bot$—though deciding it negatively rather than, as S_0 does, positively.

3 Coda

Since every consistent modal logic has at least one Post-complete extension, this is so for the logics S_0 and S_1 determined by the two-element neighborhood frames presented in the preceding section. And since this is not one of the four $\Box$-truth-functionalizing logics we have called *Makinson logics* in honor of their role in Makinson [14], there are further Post-complete extensions of congruential logics, and indeed by similar (Lindenbaum-style) reasoning further *congruential* Post-complete modal logics. It would be interesting to have some idea as to how extensive these two classes are ([19] gives references to the literature on Post-completeness in modal logic, but the case of congruential logics and their Post-complete extensions does not seem to have been specifically addressed), as well as to the question as to whether the classes in question are in fact distinct. Concerning the analogous issue apropos of normal modal logics, Goldblatt and Kowalski [4, Abstract] write:

> Monomodal logic has exactly two maximally normal logics, which are also the only quasi-normal logics that are Post complete.

By "maximally normal" the authors really mean "maximally consistent normal" (alternatively, as in [19], "maximal consistent normal"). The tradition is being followed here that for Φ picking out, by means of some closure conditions, a collection of (modal) logics, by a quasi-Φ logic is meant one extending the smallest logic satisfying those conditions, whether or not the logic is itself so closed. In the present case, then, the question is about the relation between the maximally consistent congruential modal logics and the Post-complete quasicongruential modal logics.[11]

Whatever the answer to that question may be, let us close with a contrast between the normal and congruential cases. The specifically modal closure conditions in these two cases require closure under the rules of necessitation ($A \,/\, \Box A$) and replacement ($A \leftrightarrow B \,/\, \Box A \leftrightarrow \Box B$), respectively. In the former case the corresponding "theorem form" of the rule, the schema $A \to \Box A$, when its instances are added to the smallest normal modal logic, gives us the intersection of the two logics referred to in the passage quoted from Goldblatt and Kowalski (the unit logic and the identity logic, in Makinson's terminology). In the latter case the corresponding theorem form is the extensionality schema $(A \leftrightarrow B) \to (\Box A \leftrightarrow \Box B)$ from the end of our opening section above. The case of S_0 shows that the smallest congruential modal logic containing all instances of this schema is, by contrast, neither the intersection of the Post-complete congruential modal logics nor the intersection of the maximally consistent congruential modal logics (whether or not these classes turn out to coincide).

Notes

1. It is convenient to have $\bot$ as primitive so as to make available formulas containing no sentence letters; there is no untoward dependence on the choice of primitives of the kind noted in Makinson [15] because only logics which are congruential in the sense of that paper—defined immediately below—are under consideration.

2. Of course, one could simply write $\Box A$ and $\neg\Box A$ for these last two, but the uniformly biconditional formulation emphasizes the provision of a Boolean translation for $\Box$ in each case.

3. The identity logic and the unit logic—the only ones of the four to constitute normal modal logics—are frequently called the *trivial logic* and the *Verum logic*.

4. For details see [5], in which neighborhood models are called *Scott–Montague models*, and [1], in which they are called *minimal models*, and [3] for congruential (in fact, normal) modal logics not determined by—that is, sound and complete with respect to—any class of neighborhood frames; such logics are called *neighborhood incomplete*. Subsequent refinements appear in Litak [13].

5. In fact, this hyperidentity is the $n = 1$ case of a sequence referred to as $(0.n)$ of Movsisyan [16, p. 609], where the author refers to hyperidentities in the sense of Taylor [18]—and much subsequent literature—as "polynomial hyperidentities."

6. In the algebraic case, the validity of a formula A is a matter of having $h(A) = \mathbf{1}$ for each assignment h mapping formulas homomorphically into the algebra concerned.

7. This involves considering the Lindenbaum algebra for the logic in question and noting the behavior of (the operation corresponding to) $\Box$ on the top and bottom elements and then using the fact that the two of them comprise a subalgebra in which every formula valid in the original algebra is valid (see note 6), and the set of whose valid formulas coincides with one of the four $\Box$-as-truth-functional logics. (In fact, Makinson [14] calls these four two-element modal algebras the "identity algebra," the "unit algebra," and so on.)

8. In fact, a more explicit formulation on these lines appears in [8, Theorem 2.3].

9. See Kuhn [12] and references, if this topic is not familiar.

10. More generally, in classical propositional logic, every 1-ary context exhibits this same behavior (see [11]); this means that for any formula C containing the sentence letter p_1, perhaps among others, and denoting by $C(A)$ the result of substituting A for p_1 in C, the formula $(C(A) \wedge C(B)) \to C(A \wedge B)$ is a tautology. In fact, all such conditionals are theorems of any extension of the intermediate logic LC (see [10]). The same goes for $(C(A) \wedge C(B)) \to C(A \vee B)$ and various further schemata given in [10].

11. Instead, [8] favored using the Post-completeness terminology across the board, relativized to lattices of logics and picking out the dual atoms of any such lattice. This policy is followed, for example, in French [2].

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