Notre Dame Journal of Formal Logic
Volume 56, Number 1, 2015

Varieties of Indefinite Extensibility

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Abstract We look at recent accounts of the indefinite extensibility of the concept *set* and compare them with a certain linguistic model of indefinite extensibility. We suggest that the linguistic model has much to recommend over alternative accounts of indefinite extensibility, and we defend it against three prima facie objections.

1 Introduction

Michael Dummett has famously argued that one of the morals of the set-theoretic antinomies is that the concept *set* is indefinitely extensible. Very roughly, for Dummett, a concept is indefinitely extensible if given a definite totality of instances, we are in a position, by reference to them, to characterize a further instance of the concept that lies beyond the initial totality of objects. This broad characterization of indefinite extensibility is not meant as a definition, as it takes for granted we understand what it is for a domain to be *definite*, but it is nevertheless very suggestive. Take the case of the concept *set*. Given a definite totality of sets, we are in a position to characterize a further set that is not in the given totality. We do this by considering the set of non-self-membered sets in the totality. The argument from Russell's paradox is supposed to tell us that this set cannot be in the initial totality on pain of contradiction. You may of course be tempted to respond that the proper moral of Russell's paradox is that we cannot form a set of all the antecedently given non-self-membered sets, but this is only to postpone the problem. Absent some independent reason to doubt that we can collect the non-self-membered sets in the totality into a set, to claim that there is no such set is merely "to wield the big stick," not to provide an explanation.[1]

Since the purpose of this paper is not exegetical, I suggest we replace Dummett's talk of a totality with plural talk and that we assume that no matter what some objects may be, they constitute a "definite totality." When we dispense with "definite totality" in favor of plural talk, the indefinite extensibility of the concept *set* becomes

Received April 27, 2012; accepted December 9, 2013
2010 Mathematics Subject Classification: Primary 00A30; Secondary 03E30, 03A05
Keywords: indefinite extensibility, set, plural quantification
 10.1215/00294527-2835056

the thesis that no matter what some sets may be, we are in a position, by reference to them, to characterize a further set that is not one of them. We do this by considering the set of antecedently given sets that are not elements of themselves. We may now be tempted to take as a corollary of Russell's paradox that this set of non-self-membered sets is not one of the antecedently given sets on pain of contradiction.[2]

We work in a two-sorted first-order language in which we supplement the usual individual variables such as x and y with plural variables such as xx and yy. These plural variables are bound by plural quantifiers, where as usual, $\exists xx$ is read as "there are some objects xx" and $\forall xx$ is read as "no matter what some objects xx are...." While the plural quantifier $\exists xx$ is often glossed as "there are one or more objects xx," it is important for present purposes to rely on a wider interpretation on which it is true, for example, that some objects are all and only non-self-identical objects. In other words, we read $\exists xx$ as "there are zero or more objects xx."[3] The plural quantifiers are still governed by the standard rules of inference for first-order quantifiers. Standard plural languages include a one-many predicate $x \prec yy$, which is to be read as "x is one of yy." We take the language to contain a primitive set-theoretic predicate, $x \equiv xx$, read "x is a set of xx." In the presence of this predicate, we may define more familiar set-theoretic predicates as follows:

- $x \in y$ abbreviates $\exists xx(y \equiv xx \wedge x \prec xx)$;
- $\text{Set}(x)$ abbreviates $\exists xx\, x \equiv xx$.

The theory of plural quantification is governed by at least two schematic principles. We take x to occur free in the formula $\varphi(x)$, which contains no free occurrences of the variable yy in the first schema:

$$\exists yy \forall x \left(x \prec yy \leftrightarrow \varphi(x)\right), \qquad \text{(Comprehension)}$$

$$\forall xx \forall yy \forall x(x \prec xx \leftrightarrow x \prec yy) \rightarrow \left(\varphi(xx) \leftrightarrow \varphi(yy)\right). \qquad \text{(Extensionality)}$$

The axiom schema of plural comprehension tells us that given a condition $\varphi(x)$, some objects are all and only the objects that satisfy the condition. The axiom schema of extensionality states that whatever is true of some objects is true of any objects with exactly the same members.

In this framework, we may initially rephrase the indefinite extensibility of the concept *set* as the thesis

$$\forall xx\left(\forall x\left(x \prec xx \rightarrow \text{Set}(x)\right) \rightarrow \exists y\left(\text{Set}(y) \wedge y \not\prec xx\right)\right). \qquad \text{(IE-Set)}$$

No matter what some sets may be, there is a further set which is not one of them. Note that on this interpretation, the indefinite extensibility of the concept *set* is plainly inconsistent with a simple instance of (Comprehension):

$$\exists xx \forall x\left(x \prec xx \leftrightarrow \text{Set}(x)\right). \qquad (1)$$

But how could there not be such objects as all and only those objects which are sets? Yablo [16] has recently called attention to this option as a live answer to the set-theoretic antinomies. The view that emerges is one in which the blame for the contradiction is placed squarely on the principle of plural comprehension. The moral of the paradox is that some conditions fail to determine some objects as all and only the objects that satisfy the condition. There is, for example, no reason to think that there are some sets which are all and only the non-self-membered sets. Indeed, Russell's paradox is an important reason to the contrary. But without such an assurance, we should not have expected the instance of naive comprehension to be true in the first place.

How plausible is this response to Russell's paradox? Not only is the failure of (Comprehension) a high price to pay, but notice that the model of indefinite extensibility remains incomplete unless it is supplemented with some account of the difference between true and untrue instances of plural comprehension. Otherwise, to claim that some conditions fail to determine some objects as all and only the objects that satisfy the condition is no better than "to wield the big stick" without offering an explanation.[4]

We do better if we conceive of the indefinite extensibility of the concept *set* as a byproduct of the potential character of the set-theoretic universe. Since the existence of a set is merely potential relative to the existence of its elements, we should reframe the indefinite extensibility as the thesis that no matter what some sets may be, they *can* form a set:

$$\forall xx\big(\forall x\big(x \prec xx \rightarrow \mathrm{Set}(x)\big) \rightarrow \Diamond\exists y\big(\mathrm{Set}(y) \wedge y \not\prec xx\big)\big). \qquad (\mathrm{IE}^{\Diamond}\text{-Set})$$

This is very much in line with recent proposals defended by Linnebo [8], [9] and Hellman [7], and perhaps Studd [13]. In the case of [8] and [9] at least, the relevant modality is not metaphysical, but rather whatever modality is involved in the thesis that the existence of a set is potential relative to the existence of its elements. This fine-grained interpretation of the modality, which is often taken as primitive, is sometimes taken to underwrite the metaphor of set formation in the iterative conception. On this broad picture, to claim that p is possible—relative to a certain stage—is to claim that p is true at a subsequent stage of the cumulative process of set formation; likewise, p is necessary relative to a stage if and only if p is true at all subsequent stages in the cumulative hierarchy.

The thought expressed by the modal version of indefinite extensibility is clear enough for present purposes: no matter that some sets may be formed at a given stage in the cumulative hierarchy; they *can* form a set at a later stage in the cumulative process of set formation. Whatever the stage in the process of set formation, there is never an end to it; subsequent stages in the process will contain sets not available at earlier stages. The modal regimentation of the indefinite extensibility of *set* vindicates the thought that the cumulative hierarchy has an inherently potential character that accounts for the open-ended nature of the set-theoretic universe.

The modal formulation of indefinite extensibility is consistent with (Comprehension). In this context, the import of (1) above is that within each stage, there are some sets which are all and only those sets formed at the relevant stage. Nonetheless, $\mathrm{IE}^{\Diamond}$-Set is still inconsistent with certain modal versions of plural comprehension. Consider, for example,

$$\Diamond\exists xx\,\Box\forall x\big(x \prec xx \leftrightarrow \mathrm{Set}(x)\big). \qquad (2)$$

On the intended interpretation of the modality, (2) tells us that some stage contains all sets formed at every stage of the cumulative hierarchy, which would require an end to the cumulative process of set formation. Notice that the failure of modal versions of plural comprehension extends to instances generated by conditions formulated without the help of a distinctively set-theoretic vocabulary:

$$\Diamond\exists xx\,\Box\forall x(x \prec xx \leftrightarrow x = x). \qquad (3)$$

This points to an important cost associated with the present model of indefinite extensibility: plural quantification cannot be used to provide a plural interpretation of a theory of classes such as Morse–Kelley set theory. Such an interpretation would, for example, require us to make sense of a proper class of all sets, no matter where they may be located in the cumulative hierarchy, in plural terms, which would in turn require (2) to be true. For similar reasons, plural quantification cannot be used to provide a perfectly general model theory for plural set theory.[5] As mentioned above, the other important cost has to do with the reliance on the ontological conception of a set as merely potential with respect to its elements. It is not clear that such a conception of set is required for an explanation of set-theoretic practice, and to the extent to which one must rely on a primitive understanding of the relation a set bears to its elements, this reliance strikes us as another cost for the model.

2 The Linguistic Turn

The purpose of this paper is to explore an alternative model of indefinite extensibility, one which conceives of indefinite extensibility as a feature of the set-theoretic vocabulary and not the concepts they are supposed to express. Earlier accounts of indefinite extensibility tacitly assume the set-theoretic vocabulary to univocally express a battery of set-theoretic concepts, which are themselves indefinitely extensible. Instead, we now take different uses of the set-theoretic vocabulary by different speakers to express different set-theoretic concepts and even belong to different members in a hierarchy of ever more comprehensive languages. This model of indefinite extensibility has a precedent in Williamson [15]. In a paper primarily concerned with the semantic paradoxes and the indefinite extensibility of semantic predicates such as "say," "true," and "false," Timothy Williamson outlines a reconstruction of the indefinite extensibility of the predicate *set* in terms of correlative reinterpretations of the set-theoretic vocabulary:

> For given any reasonable assignment of meaning to the word 'set' we can assign it a more inclusive meaning while feeling that we are going on in the same way, and make correlative changes to the words in an iterative account of sets, to preserve it too. The inconsistency is not in any one meaning we assign the iterative account; it is in the attempt to combine all the different meanings that we could reasonably assign it into a single super-meaning. ([15, p. 20])

We would like to combine this linguistic model of indefinite extensibility with a certain common conception of set described, for example, by Kurt Gödel:

> The concept of set, however, according to which a set is anything obtainable from the integers (or some other well-defined objects) by iterated application of the operation "set of" and not something obtained by dividing the totality of all existing things into two categories, has never led to any antinomy whatsoever; that is, the perfectly "naive" and uncritical working with this concept of set has so far proved self-consistent. ([6, p. 180])

We make use of the *set of* operation in order to expand an antecedently given domain to a more comprehensive domain, which contains not only them but whatever sets we have been able to form from them. We later iterate this operation. In what follows, we aim to reframe Gödel's procedure in terms of a cumulative process of reinterpretation of the primitive set-theoretic vocabulary.

First, we introduce a new primitive predicate, α, to apply to all and only objects that are *available* for collection into a set. The primitive vocabulary of the language will now include α, which is read "available for collection," and ≡, which is read "is a set of." The two primitive predicates are governed by principles of collection and extensionality:

$$\forall xx\big(\forall x\big(x \prec xx \rightarrow \alpha(x)\big) \rightarrow \exists x\, x \equiv xx\big), \qquad \text{(Collection)}$$

$$\forall xx \forall yy \forall x \forall y\big(x \equiv xx \wedge y \equiv yy \leftrightarrow \big(x = y \leftrightarrow \forall z(z \prec xx \leftrightarrow z \prec yy)\big)\big). \qquad \text{(Extensionality)}$$

The principle of collection tells us that no matter what some objects may be, if they are available for collection, then they have been collected into a set. The principle of extensionality tells us that no matter what some objects may be, there is at most one set of them. We may now reframe the procedure described by Gödel as a cumulative process of reinterpretation of the set-theoretic vocabulary. At the initial stage, we may, for example, interpret α to apply to the domain of integers. We may subsequently interpret ≡ in terms of the *set of* operation on the domain of integers. The initial interpretation of ≡ relates sets of integers into its elements. In the next stage of reinterpretation, we take α to apply to the expanded domain of integers and sets thereof. This reinterpretation of α forces a reinterpretation of ≡ in terms of the *set of* operation on the expanded domain: ≡ will now relate sets of integers to their elements, sets of integers and sets of integers to their elements, and finally, it will relate sets of sets of integers to their elements.

At no point in the cumulative process of reinterpretation do we find ourselves in a position to interpret α to apply to all the sets there are. Indeed, (Collection) and (Extensionality) are plainly inconsistent with the thesis that all sets are available for collection:

$$\forall x\big(\text{Set}(x) \rightarrow \alpha(x)\big). \qquad \text{(Availability)}$$

The contradiction is not hard to find. An instance of (Comprehension) allows us to consider some sets rr, which are all and only non-self-membered sets:

$$\forall x\big(x \prec rr \leftrightarrow \big(\text{Set}(x) \wedge x \notin x\big)\big). \qquad (4)$$

Since all sets are, by (Availability), available for collection, we infer

$$\forall x\big(x \prec rr \rightarrow \alpha(x)\big). \qquad (5)$$

By (Collection), they are collected into a set:

$$\exists x\, x \equiv rr. \qquad (6)$$

Let r be the set of rr, $r \equiv rr$. By definition of ∈, (Extensionality) gives us

$$r \in r \leftrightarrow \big(\text{Set}(r) \wedge r \notin r\big). \qquad (7)$$

But we know of course that Set(r), whence

$$r \in r \leftrightarrow r \notin r. \qquad (8)$$

But this is no reason for concern. There is nothing in Gödel's conception of set that may be used to motivate the thought that all sets are available for collection.

What is certainly implicit in Gödel's conception of set is that there is no end to the iteration of the *set of* operation. In the current framework, this amounts to the claim that the cumulative process of reinterpretation is indefinitely extensible: no matter how we interpret the primitive set-theoretic predicates, α and ≡, we *can*, if we like, move to a more comprehensive interpretation, α^+ and $\equiv^+$, on which α^+ is taken to extend α to include sets of whatever objects were available by the lights of

α; $\equiv^+$ would likewise extend $\equiv$ because it would be interpreted in terms of the *set of* operations on the extended domain of objects available for collection. Note, however, that in the present account of indefinite extensibility, we will be interested not only in whatever is true on one or another interpretation of the set-theoretic lexicon but rather on what remains true no matter how we reinterpret the primitive vocabulary. This suggests a modal formulation of indefinite extensibility.

2.1 A modal formulation of indefinite extensibility If we supplement the language with a modal operator, $\Diamond$, aimed to express the interpretational modality involved in the cumulative process of reinterpretation, we may give a modal formulation of the indefinite extensibility of the set-theoretic vocabulary:

$$\forall xx\big(\forall x\big(x \prec xx \rightarrow \alpha(x)\big) \rightarrow \exists x\ x \equiv xx\big), \tag{Collection}$$

$$\forall xx \forall yy \forall x \forall y\big(x \equiv xx \wedge y \equiv yy \leftrightarrow \big(x = y \leftrightarrow \forall z(z \prec xx \leftrightarrow z \prec yy)\big)\big), \tag{Extensionality}$$

$$\forall x\big(\mathrm{Set}(x) \rightarrow \Diamond\alpha(x)\big). \tag{Availability$^\Diamond$}$$

These axioms give expression to the thought that no matter how we interpret the primitive predicates "available for collection" and "set of," we find ourselves in a position to assign to them a more comprehensive interpretation that preserves the thought that no matter what some available objects are, there is a set of them. In particular, (Availability$^\Diamond$) tells us that no matter what set x may be, there is a reinterpretation of the predicate α on which it is true that $\alpha(x)$.

Two clarificatory remarks are in order. First, notice that these axioms do not, by themselves, tell us how far we should proceed in the cumulative process of reinterpretation of the set-theoretic vocabulary. They would need to be supplemented in order to make sure, for example, that we reach a transfinite stage. Likewise, they would need to be supplemented in order to enforce the thought that sets are formed in a well-founded process of reinterpretation. For now, however, we will focus on the question of whether the present three axioms form the core of an internally coherent conception of set on which we conceive of indefinite extensibility generally associated to the set-theoretic universe as a feature of the set-theoretic vocabulary.

Second, notice that unlike [8] or [7], we take the modality to be merely interpretational: $\Diamond$ is a sentential operator, which combines with a formula φ to produce another sentence, $\Diamond\varphi$, which is true (relative to an assignment of values to the variables) if and only if there is some reinterpretation of the primitive set-theoretic vocabulary in φ on which φ comes out true (relative to the assignment of values to the variables). Thus $\Diamond\varphi$ tells us that φ is true on some subsequent reinterpretation of the set-theoretic vocabulary—while $\Box\varphi$ tells us that φ remains true on all subsequent reinterpretations. The successive interpretations of the vocabulary are intended of course to correspond to successive stages of the cumulative hierarchy generated by a set of urelements U:

$$\begin{aligned} U_0 &= U, \\ U_{\alpha+1} &= U \cup \mathcal{P}(U_\alpha), \\ U_\lambda &= \bigcup_{\alpha<\lambda} U_\alpha \quad \text{for } \lambda \text{ a limit ordinal.} \end{aligned}$$

The cumulative hierarchy gives rise to a cumulative hierarchy of successive admissible interpretations of the primitive predicate α:

$$\begin{aligned} \alpha_0 &= U, \\ \alpha_{\alpha+1} &= U \cup \mathcal{P}(U_\alpha), \\ \alpha_\lambda &= \bigcup_{\alpha<\lambda} U_\alpha \quad \text{for } \lambda \text{ a limit ordinal.} \end{aligned}$$

Admissible interpretations of α are in turn correlated with admissible interpretations of $\equiv$, whereby objects in the extension of one predicate are collected into sets in the extension of the other in accordance with (Collection). These interpretations give rise to admissible interpretations of each of the defined predicates Set and $\in$. In particular, we come to the following cumulative hierarchy of interpretations of Set and $\in$:

$$\begin{aligned} \text{Set}_0 &= \mathcal{P}(U), \\ \text{Set}_{\alpha+1} &= \text{Set}_\alpha \cup \mathcal{P}(\text{Set}_\alpha), \\ \text{Set}_\lambda &= \bigcup_{\alpha<\lambda} \text{Set}_\alpha \cup \mathcal{P}\Big(\bigcup_{\alpha<\lambda} \text{Set}_\alpha\Big) \quad \text{for } \lambda \text{ a limit ordinal.} \end{aligned}$$

Likewise, successive reinterpretations of α and $\equiv$ result in reinterpretations of the predicate $\in$:

$$\in_\alpha = \in \cap \langle U_\alpha, \text{Set}_\alpha \rangle.$$

You may have noticed an omission. We offered no explicit specification of admissible interpretations of $\equiv$, which would form a cumulative hierarchy of one-many relations. But while we have singular and plural variables in the metalanguage, we have no variables that would allow us to range over one-many relations of the appropriate sort. No matter, we can still encode $\equiv^\alpha$ by means of $\in_\alpha$, which is itself a binary relation. For after all,

$$\forall xx \forall x \big(x \equiv_\alpha xx \leftrightarrow \forall x (y \in_\alpha x \leftrightarrow y \prec xx)\big).$$

The cumulative hierarchy of pure sets corresponds to a cumulative hierarchy of reinterpretations of the set-theoretic vocabulary in which $U = \emptyset$ and there are no available objects in the first stage of interpretation.[6]

2.2 Which modal logic? We have described a process of reinterpretation of the set-theoretic vocabulary of the expanded language, and we have suggested a modal formulation of the indefinite extensibility of Set in which the modality is merely interpretational. We are now in a position to outline the modal logic that is appropriate for the interpretational modality which concerns us. The intended interpretation of the modality involves reinterpretations of the set-theoretic vocabulary over the domain of all objects. Thus we are interested in a *constant domain* model theory in which a "world" is a formal counterpart of an interpretation of the primitive set-theoretic vocabulary $\mathcal{I}_\alpha$, given by an ordered pair $\langle U_\alpha, \equiv_\alpha \rangle$.[7]

We may think of a constant domain model for the modal extension of the language of plural set theory as given by an ordered quadruple $\langle W, D, \leq, I \rangle$, where as usual, I specifies the interpretation of the nonlogical vocabulary at each world, W is a set of "possible worlds," and D is a domain of quantification over which singular and plural quantifiers are supposed to range. We know of course that D must be a set and cannot contain all objects on pain of contradiction. Still, you may think of each stage of the cumulative hierarchy described above as the domain of a constant domain model of the appropriate sort in which each "world" corresponds to a stage in the process of reinterpretation.

The accessibility relation, $\leq$, is the relation an interpretation $\mathcal{I}_\alpha$ bears to $\mathcal{I}_\beta$ whenever $\mathcal{I}_\beta$ extends $\mathcal{I}_\alpha$; that is, $U_\alpha \subseteq U_\beta$ and $\equiv_\alpha \subseteq \equiv_\beta$. Thus $\leq$ is a partial order of the set of worlds: reflexive, antisymmetric, and transitive, which suggests that the appropriate modal logic should extend S4, which is the minimal normal modal logic equipped with the following modal axioms:

$$\varphi \to \Diamond\varphi \tag{Axiom T}$$

$$\Diamond\Diamond\,\varphi \to \Diamond. \tag{Axiom 4}$$

Moreover, it is reasonable to assume that $\leq$ is directed: no matter what two interpretations may be, there is a more comprehensive interpretation which extends them both. In this respect at least, the framework is similar to the one outlined in [13] and [9].[8] This makes $\leq$ at least a directed partial order governed by the axioms of S4.2, which is the normal logic extending S4 by means of the axiom

$$\Diamond\Box\varphi \to \Box\,\Diamond\,\varphi. \tag{G}$$

Since we work in a constant domain model theory, the range of models we are interested in will validate singular and plural formulations of the Barcan (BF) and converse Barcan formula (CBF):

$$\Box\forall x\varphi \to \forall x\Box\varphi, \qquad \Box\forall xx\varphi \to \forall xx\Box\varphi, \tag{CBF}$$

$$\forall x\Box\varphi \to \Box\forall x\varphi, \qquad \forall xx\Box\varphi \to \Box\forall xx\varphi. \tag{BF}$$

The validity of singular and plural versions of (CBF) and (BF) illustrates the fact that the phenomenon of indefinite extensibility is concerned not with ontology but rather with language. In this framework, we expect to validate the necessity of identity and the *one of* relation

$$\forall x\forall y(x = y \to \Box x = y), \tag{$\Box=$}$$

$$\forall xx\forall y(x \prec xx \to \Box x \prec xx). \tag{$\Box\prec$}$$

The necessity of identity is derivable from the interaction of propositional modal logic and the logic of identity.[9] Since each reinterpretation of the set-theoretic vocabulary is meant to extend earlier interpretations, we expect to validate similar principles for two primitive predicates:

$$\forall x\big(\alpha(x) \to \Box\alpha(x)\big), \tag{$\Box\alpha$}$$

$$\forall xx\forall x(x \equiv xx \to \Box x \equiv xx). \tag{$\Box\equiv$}$$

If we take ($\Box\alpha$) and ($\Box\equiv$) as axioms, we can derive similar principles for Set and $\in$:

$$\forall x\big(\mathrm{Set}(x) \to \Box\mathrm{Set}(x)\big), \tag{$\Box$Set}$$

$$\forall x\forall y(x \in y \to \Box x \in y).^{10} \tag{$\Box\in$}$$

The formalism invites some apparently uncomfortable questions. One may wonder at this point what interpretation of the set-theoretic vocabulary should be counted as "actual" with respect to the interpretational modality. This is tantamount to the question of what exactly is the current interpretation of α and $\equiv$ and, likewise, what is the current interpretation of Set and $\in$. There is, however, an air of parochiality to such questions. For given the indefinite extensibility of the process of interpretation, it may be more fruitful to focus not on what is the case relative to one interpretation or another, but rather on what remains the case no matter how far we ascend in the process of reinterpretation of the set-theoretic vocabulary.

This problem is completely parallel to one arising for philosophers who take modal formulation of indefinite extensibility to be grounded on the potential nature of the set-theoretic universe. And the answer is parallel to the response they offer. Whatever the current stage of interpretation, it is more fruitful to think of the axioms of set theory as being concerned not with a specific interpretation of the set-theoretic vocabulary, but rather with successive reinterpretations of the vocabulary. On this point of view, we need not locate ourselves in the cumulative hierarchy of reinterpretation but rather would aim to make perfectly general claims that apply, regardless of one's position in the process of reinterpretation. The intended generality of a set-theoretic assertion in the plural language of set theory is best captured by its modalization in the modal extension of this language. To be more precise, let the *modalization* of a well-formed formula φ of the language of plural set theory be the well-formed formula $\varphi^{\Diamond}$ of the modal extension of the language that results when atomic subformulas of the form $\alpha(t)$ and $t \equiv tt$ are prefixed with an occurrence of $\Diamond$ as in $\Diamond\alpha(t)$ and $\Diamond t \equiv tt$. Once we do this, we can check that $(x \in y)^{\Diamond}$ becomes $\Diamond x \in y$ and $(\mathrm{Set}(x))^{\Diamond}$ becomes $\Diamond\,\mathrm{Set}(x)$.[11]

Moreover, we can now check that if $\varphi^{\Diamond}$ is the modalization of a well-formed formula φ, then $\varphi^{\Diamond}$, $\Diamond\varphi^{\Diamond}$, and $\Box\varphi^{\Diamond}$ will all be equivalent. This is proved by induction on the complexity of well-formed formulas of the language of plural set theory. In the context of T, we need only check that $\varphi^{\Diamond} \to \Box\varphi^{\Diamond}$. If φ is an atomic well-formed formula, then ($\Box\alpha$) and ($\Box \equiv$) will allow us to derive $\Diamond\varphi^{\Diamond} \to \Diamond\Box\varphi^{\Diamond}$. By (G), we conclude $\Diamond\varphi^{\Diamond} \to \Box\Diamond\varphi^{\Diamond}$. For $\neg\psi^{\Diamond}$, we apply the inductive hypothesis to $\psi^{\Diamond}$ and derive the conditional. For $(\psi_1^{\Diamond} \wedge \psi_2^{\Diamond})$, we argue $\Diamond(\psi_1^{\Diamond} \wedge \psi_2^{\Diamond}) \to (\Diamond\psi_1^{\Diamond} \wedge \Diamond\psi_2^{\Diamond})$. By inductive hypothesis, $\Diamond(\psi_1^{\Diamond} \wedge \psi_2^{\Diamond}) \to (\Box\psi_1^{\Diamond} \wedge \Box\psi_2^{\Diamond})$, whence $\Diamond(\psi_1^{\Diamond} \wedge \psi_2^{\Diamond}) \to \Box(\psi_1^{\Diamond} \wedge \psi_2^{\Diamond})$. For $(\exists x\varphi)^{\Diamond}$, note that by (BF), $\Diamond\exists x\varphi^{\Diamond} \to \exists x \Diamond \varphi^{\Diamond}$. By inductive hypothesis, $\Diamond\exists x\varphi^{\Diamond} \to \exists x\Box\varphi^{\Diamond}$. But since $\exists x\Box\varphi^{\Diamond} \to \Box\exists x\varphi^{\Diamond}$, we have $\Diamond(\exists x\varphi)^{\Diamond} \to \Box(\exists x\varphi)^{\Diamond}$. A completely parallel argument takes care of the case of the plural quantifier.

The point of this result is to guarantee that the modalization of a formula receives the same evaluation no matter where one is located in the cumulative process of reinterpretation. If we interpret set-theoretic assertions to be implicitly modalized statements, then the question of what is the current interpretation of the set-theoretic vocabulary falls out as completely irrelevant for the purposes of set theory. From this perspective, one could take the assertions made by set theorists to correspond to the modalizations of perfectly general claims, whose quantifiers are implicitly restricted to the domain of all objects that eventually become available for collection. For example, the pair set axiom becomes

$$\forall x\forall y\big(\Diamond\alpha(x) \wedge \Diamond\alpha(y) \to \exists z\forall u(\Diamond u \in z \leftrightarrow u = x \vee u = y)\big).$$

When we take the quantifiers to be restricted to objects that eventually become available for collection, this claim is tantamount to the fact that whatever x and y may be, we may eventually come to an interpretation of the set-theoretic vocabulary on which a certain z becomes the pair set $\{x, y\}$.

3 Three Prima Facie Objections

Much work remains to be done, but hopefully, we have a sufficiently detailed outline of the linguistic model of indefinite extensibility to begin a preliminary discussion of its merits. There may appear to be decisive objections to the linguistic model and drawbacks. There are, in particular, three prima facie objections, which are sufficiently serious to suggest that the linguistic model of indefinite extensibility may not be more than an intellectual curiosity that deserves attention only for purposes of bookkeeping as a somewhat extravagant vision of the subject matter of set theory. The remainder of the paper will attempt to dispel that impression and urge us to take seriously the linguistic model of indefinite extensibility outlined in Section 2.

3.1 What ontological conception of set? One immediate problem with the linguistic model is that it must be supplemented with some ontological conception of set in order to replace the conception of set as constituted by its elements. On this conception of set, the elements of the set are ontologically prior to them. Indeed, it is precisely this relation of priority that lies at the heart of the modality involved in the modal conception of indefinite extensibility outlined at the outset. The elements are prior to the set in the sense that their existence is all it takes for the potential existence of the set. This picture naturally invites us to think that the formation of a set of non-self-membered sets in an antecedently given domain calls for an expansion of the initial domain into a strictly more comprehensive one, which includes a further *sui generis* object constituted by the objects in the initial domain. But since we wanted to avoid any reliance on a primitive understanding of the relation such sets are supposed to bear to its elements, we owe an alternative characterization of what a set is and what it is for a set to have some objects as elements.

But whatever ontological conception of set we bring to the table, it should be able to accommodate the *invalidity* of each of the following modal principles:

$$\forall x(\Diamond x \equiv xx \rightarrow x \equiv xx). \qquad (\Diamond\equiv)$$

This principle tells us that if an object x is not a set of some objects xx, then x will never become a set of xx no matter how far we ascend in the cumulative process of set formation. Successive stages in the cumulative hierarchy involve the formation of new sets, never the transformation of nonsets into sets. Two more invalid modal principles are the following:

$$\forall x\big(\Diamond\, \mathrm{Set}(x) \rightarrow \mathrm{Set}(x)\big), \qquad (\Diamond\mathrm{Set})$$

$$\forall x \forall y(\Diamond x \in y \rightarrow x \in y). \qquad (\Diamond\in)$$

Now, we must reject all three principles when we take the modality involved in them to be merely interpretational. What is a set of some objects at a more comprehensive interpretation of $\equiv$ can still be counted as a nonset by an earlier interpretation of the set-theoretic vocabulary. And likewise, what is a nonset on one interpretation of the set-theoretic vocabulary can become a set on a more comprehensive interpretation of Set. Finally, an object can not be an element of another on one interpretation of $\in$, but it may become one on a more comprehensive interpretation.

The challenge at this point is to come up with an ontological conception of set on which to make sense of the failure of the preceding modal principles. Perhaps we should think of a set as a mere node in a structure that satisfies certain formal conditions imposed by the axioms given at the outset. The set-theoretic universe could perhaps be reduced to a domain of objects related by a formally appropriate relation that satisfies the relevant axioms. The indefinite extensibility of *set* amounts to the availability of a more comprehensive interpretation of $\in$ on which more objects stand in a relation that satisfies the axioms of set theory. But one may well object to this that there is much more to the element-set relation than to stand in a relation that satisfies certain structural conditions; one may be tempted to dismiss the linguistic model of indefinite extensibility as a nonstarter.

3.2 A proper class of *nonsets* A second important problem for the linguistic model of indefinite extensibility is related to the first difficulty. Familiar cardinality considerations tell us that no matter how far we find ourselves in the cumulative process of reinterpretation, the ability to provide a strictly more comprehensive interpretation of α and $\equiv$ requires the existence of an immensely abundant stock of nonsets, which may eventually be recast as further sets. In particular, the linguistic model appears to require a proper class of nonsets at each stage of reinterpretation.[12] By replacement, we infer that the nonsets cannot form a set. But since every nonset is routinely taken to be an urelement, we find that there is no reinterpretation of the set-theoretic vocabulary on which a set may contain all the urelements as members. This is supposed to be in conflict with the iterative conception of set on which the urelements are supposed to form a set at the very first stage of the cumulative hierarchy and to deprive us from certain uses of the assumption that the urelements form a set.[13]

3.3 A revenge problem? The third and final objection is that once we realize that each reinterpretation of the set-theoretic vocabulary extends earlier available interpretations, one may be tempted to combine all the interpretations of α and $\equiv$ into a *final* interpretation of the set-theoretic vocabulary. You may, for example, want to interpret α to apply to all objects that eventually become available in the extension of one or another reinterpretation of the predicate. And you may similarly want to take $\equiv$ to relate an object with some objects if and only if they are eventually so related by one or another reinterpretation of the predicate. But we can of course specify the putative interpretations mentioned above with the help of a modal operator:

- $\alpha^{\Diamond}(x)$ abbreviates: $\Diamond\alpha(x)$;
- $\equiv^{\Diamond}(x)$ abbreviates: $\Diamond x \equiv xx$.

If the newly defined predicates are admissible reinterpretations of the set-theoretic vocabulary, then they will automatically induce a reinterpretation of the defined predicates Set and $\in$.[14] For example,

- $\text{Set}^{\Diamond}(x)$ abbreviates $\Diamond\,\text{Set}(x)$;
- $x \in^{\Diamond} y$ abbreviates $\Diamond x \in y$.

But recall that the indefinite extensibility of the set-theoretic vocabulary would immediately commit us to the following instance of (Availability$^{\Diamond}$):

$$\forall x\big(\text{Set}^{\Diamond}(x) \to \Diamond\alpha^{\Diamond}(x)\big). \tag{9}$$

Since we have explicitly taken the interpretational modality to be governed by (Axiom 4) ($\Diamond\Diamond\varphi \to \Diamond\varphi$), we are on the verge of a contradiction. For the presence of this axiom forces us to collapse (Availability$^{\Diamond}$) into (Availability):

$$\forall x\big(\text{Set}^{\Diamond}(x) \to \alpha^{\Diamond}(x)\big). \tag{10}$$

And we know that (Availability) is inconsistent with (Collection) and (Extensionality), which in this context reads

$$\forall xx(\forall x(x \prec xx \rightarrow \alpha^{\diamond}(x)) \rightarrow \exists x\, x \equiv^{\diamond} xx), \tag{11}$$

$$\forall xx \forall yy \forall x \forall y(x \equiv^{\diamond} xx \wedge y \equiv^{\diamond} yy \leftrightarrow (x = y \leftrightarrow \forall z(z \prec xx \leftrightarrow z \prec yy))). \tag{12}$$

To sum up the objection, note that if we let all the successive interpretations of α and ≡ combine into a *final* admissible interpretation of the set-theoretic vocabulary, $\alpha^{\diamond}$ and $\equiv^{\diamond}$, then we will generate a contradiction. But what could possibly prevent us from reinterpreting α, read "available for collection," and ≡, read "set of," to mean, respectively, "eventually available for collection" and "eventually a set of"? Unless we find a principled reason to resist the move, it will not even be clear that we have described an internally coherent vision of indefinite extensibility.

4 In Defense of the Linguistic Model

There is no denying that the preceding considerations appear as powerful reasons to resist the proposal now on the table. The aim of this section is to outline an answer to each of the three important objections faced by the linguistic model of indefinite extensibility.

4.1 In response to the first concern: Representation Let us focus on the relation a set bears to its elements. The set is not identical to its elements. The set is one, but the elements are generally many. The set can enter into the element-set relation, sometimes as a set and sometimes as an element, but the elements of the set are generally neither a set nor an element. But now the question arises of what it is for one to be a *set of* the others. It is not uncommon to respond that the set is *constituted* by its elements and that the elements are ontologically *prior* to the set. Since it is difficult to analyze the relation of priority in more basic terms, it is often taken to be primitive and unanalyzable. All this may be taken to suggest that there is much more to the relation a set bears to its elements than the mere satisfaction of whatever structural constraints have fallen out of the axioms of set theory. Instead, sets are often viewed as sui generis objects, and the set-theoretic domain provides the subject matter of set theory. What sets there are depends on what is the nature of the relation a set bears to its elements and how it interacts with the domain of nonsets. We begin with an antecedently given domain of objects and ask what sets must be admitted into our ontology in accordance with the nature of the "set of" relation.

But maybe there is an alternative model of the relation a set bears to its elements. The suggestion, instead, is that for an object to be a set of some objects is merely for one to represent—stand in for—the others.[15] One important difference between this and the previous model of the relation a set bears to its elements is that it places no apparent constraints on the nature of a set; there is no requirement that sets be sui generis or that they bear some primitive and unanalyzable metaphysical relation to its elements. We are free to choose any representative for some given objects, provided the choices are globally sensitive to further considerations such as, for example, making sure that the resulting element-set relation verifies the axioms of set theory.

But what exactly is it for a one-many relation to be a relation of representation over a given domain of objects? Very roughly, a relation of representation R over a given domain of objects is a one-many relation that assigns at most one representative to any objects in the domain. Of special interest are what we will call strict relations of representation over a domain of available objects. We will write that a one-many relation R is a *strict* relation of representation over a domain of objects characterized by some condition $\varphi(x)$ if and only if R meets two specific constraints:

$$\forall xx\,\big(\forall x\,\big(x \prec xx \rightarrow x \prec \varphi(x)\big) \leftrightarrow \exists x\, xRxx\big), \tag{13}$$

$$\forall xx\;\forall yy\;\forall x \forall y\big(xRxx \wedge yRyy \leftrightarrow \big(x = y \leftrightarrow \forall z(z \prec xx \leftrightarrow z \prec yy)\big)\big). \tag{14}$$

Consider the first stage of the process described by Gödel. We begin with a domain of integers and form sets thereof. Now, consider the relation a set of integers bears to some integers; it represents them. The relation in which a set of integers stands with respect to its elements is a relation of strict representation over the domain of integers. Thus if the integers exhaust the range of initially available objects, then a relation of strict representation over them will include all sets of integers in its domain.

Notice, however, that there may be a multiplicity of strict relations of representation over the same domain. For we could instead consider an assignment of real numbers to sets of integers that satisfies conditions (13) and (14) above, and it would still count as a relation of representation over the domain of integers. The thought is that what matters is less the identity of the object we assign to some integers and more the fact that the assignment meets two merely structural constraints. Notice, in particular, that we do not require the objects that represent objects in the domain to lie themselves within the initial domain. If we start with a domain of natural numbers, there is no presumption that the objects that represent different collections of natural numbers should themselves be numbers. In fact, we will generally assume that not all of the objects in the domain of a strict relation of representation lie in the domain over which R is a relation of representation.

The thought should be clear by now. An interpretation of the primitive set-theoretic vocabulary corresponds to a relation of strict representation, R, which constitutes the interpretation of $\equiv$, over a domain of available objects, U, which constitutes the interpretation of α. (Collection) and (Extensionality) do indeed amount to the requirement that R be a relation of strict representation over U. In contrast to them, the point of (Availability$^{\Diamond}$) is to make sure we continue the iteration of the "set of" operation.[16]

The moral of Russell's paradox as developed in the last section is that the domain of all sets does not support a strict relation of representation over its members. For if all sets could be contained in the interpretation of α, then the interpretation of $\equiv$ would constitute a relation of strict representation R over the domain of sets, which would, in turn, require some set to represent all and only non-self-membered sets. But we should not overstate the significance of this observation. For we are often interested in the availability of strict relations of representation over an antecedently given domain of objects. We should not forget, for example, that Cantor arrived at his theory of sets only by reflection on an earlier theory of sets of points on the line of real numbers. Sets of points became important for purposes of generalizing results primarily concerned with points in the real line. Sets are thus obtained by iterations of the "set of" operation as famously described by Kurt Gödel in [6]. Gödel's conception of set may be recast in terms of representation. The "set of" operation on an antecedently given domain corresponds to the formation of a *strict* relation of representation over the domain. Gödel's thought becomes the suggestion that we begin with an antecedently given domain of individuals and proceed to iterate the formation of strict relations of representations over the successive domains of sets we have generated.

The model of representation, in other words, fits well with the linguistic model of indefinite extensibility. The subject matter of set theory on the present view is no longer constituted by a domain of sui generis objects but rather is to be identified with relations of strict representation over a domain. Perhaps more importantly for present purposes, there is no longer reason to think of the ontology of set theory as being constrained by the nature of the "set of" relation; quite the opposite. We are in a position to take the ontology of set theory as antecedently given at the outset and ask what relations on the domain are suitable interpretations of the "set of" predicate.[17] Before we move on, however, let me note that the model of representation is not unprecedented. John L. Bell and Richard Cartwright explore a similar thought in [1] and [3], respectively. Erik Stenius provides an earlier precedent for the view in [12].[18] In this light, it is not difficult to recast the axioms of standard set theory as general principles concerning representation. For example, the axiom of extensionality tells us that no matter what some objects are, they are represented by at most one set. The pair set axiom tells us that no matter what two objects a and b are, they are represented by a set, and power set tells us that the subsets a set are themselves represented by a set. A similar interpretation is available for axioms such as union and replacement.

4.2 In response to the second concern: Urelements versus nonsets We have introduced a model of a set as a representative for certain objects. To be a set is to represent some objects. To be an element of a set is to be one of some objects represented by the set. Finally, to be an element is to be an element of some set.[19] The definitions of Set and $\in$ are only to be expected, but they suggest a significant departure from the standard characterization of *urelement* as a nonset. For in the present picture, there is no reason to think that a nonset must itself be an element of some set or another. In contrast, it is presumably more appropriate to think of an *urelement* as a nonset that is itself an element of some set or another. This suggests the following definition:

- υx abbreviates $(\exists y\ x \in y \wedge \neg\mathrm{Set}(x))$, read "$x$ is an urelement."

In this view, we have reason to think that nonsets will generally outstrip urelements: a tree may neither be a set nor an element relative to a relation of representation over the domain of integers. Alternatively, you may have wanted trees to be available at the first stage of the process of set formation, in which case a tree would have been an urelement. Quite generally, urelements are available at the first stage of the cumulative process of set formation, whereas many nonsets eventually become sets at later stages in the cumulative hierarchy.

4.3 In response to the third concern: Strict representation Only the last concern remains. What is there to prevent us from combining the successive interpretations of the primitive set-theoretic predicates α and $\equiv$ into an ultimate interpretation of the set-theoretic vocabulary, which we could specify with the help of a modal operator:

- $\alpha^{\Diamond}(x)$, which abbreviates: $\Diamond\alpha(x)$;
- $x \equiv^{\Diamond} xx$, which $\Diamond\, x \equiv xx$.

This putative interpretation of the primitive vocabulary takes an object to be available if and only if it ever falls under an interpretation of α in the cumulative process of reinterpretation. Likewise, an object represents some objects if and only if it represents them under some reinterpretation of $\equiv$.

We have taken the interpretational modality involved in our account of indefinite extensibility to be governed by the principle by axiom 4

$$\Diamond\Diamond\varphi \rightarrow \Diamond\varphi. \qquad \text{(Axiom 4)}$$

But in the presence of (Axiom 4), (Availability$^{\Diamond}$) collapses into (Availability), which is inconsistent with (Collection) and (Extensionality).

One option at this point of course is to backtrack and revise the commitment to S4.2 as the appropriate modal logic for interpretational modality. This may seem counterintuitive at first, but I think it could be motivated by an open-ended view of the cumulative process of reinterpretation.[20] It is perhaps tempting to assume at the outset that there is a perfectly delimited range of ever more comprehensive interpretations of the set-theoretic vocabulary that remains invariant no matter where we locate ourselves in the cumulative process of reinterpretation. But this suggestive image may in fact distort the open-ended nature of the cumulative process of reinterpretation; not only does the mechanism of reinterpretation transcend every candidate interpretation of the set-theoretic vocabulary, the very idea of a candidate interpretation may be sensitive to our location in the cumulative hierarchy. Maybe the range of candidate interpretations evolves as we ascend the ladder of ever more comprehensive interpretations and is as open-ended as the interpretation of the set-theoretic vocabulary itself. This would give us a rationale for the rejection of (Axiom 4) for the interpretational modality involved in the account. What constitutes an admissible reinterpretation of the set-theoretic vocabulary at one point may not have constituted an admissible interpretation at an earlier point in the cumulative process of reinterpretation.

The suggestion is that we are not entitled to assume the existence of a fixed range of admissible interpretations over which we make generalizations. Note that on this view, one should take the interpretation of $\equiv^{\Diamond}$ to be as open-ended as the interpretation of $\equiv$ is supposed to be: maybe further candidate interpretations emerge only as we consider more and more comprehensive interpretations of the predicate $\equiv$. There is, at this point, no ontological indeterminacy, only linguistic indeterminacy. There is a perfectly comprehensive domain of all objects, and we may avail ourselves of plural quantification over them with the assurance that it will be governed by the usual principles of the theory of plural quantification, including plural comprehension. What there is not is a perfectly delimited and invariant range of candidate interpretations of the set-theoretic vocabulary, only ever more comprehensive classes of candidate interpretations suggested by our progression through the cumulative process of reinterpretation.

But if there is no invariant, perfectly delimited range of candidate interpretations for the set-theoretic vocabulary, then there is no reason to think that $\equiv^{\Diamond}$ univocally expresses a single concept; instead, much like $\equiv$, we should treat it as an open-ended predicate which admits of different interpretations at different stages in the cumulative process of reinterpretation. To the extent to which a candidate interpretation for the set-theoretic vocabulary ought to provide a perfectly delimited extension for them, that is, they ought not to be open-ended, we would seem to have a reason of principle for rejecting $\equiv^\Diamond$ as a candidate interpretation of $\equiv$. The open-ended character of $\in^\Diamond$ is supposed to originate from the speakers' inability to anticipate all candidate interpretations of the set-theoretic vocabulary; as they inadvertently move up in the cumulative hierarchy, they begin to anticipate more and more candidate interpretations, but there is no ultimate interpretation of $\equiv^\Diamond$ just like there is no ultimate interpretation of $\equiv$.

This general line of response is not without consequence. Part of the reason the modal formulation of the linguistic model of indefinite extensibility recommended itself had to do with the ability to give a more general formulation of the view. In addition to this, set-theoretic axioms and other assertions were supposed to tacitly involve modal versions of the standard set-theoretic predicates. Otherwise, set-theoretic statements might turn out to be too parochial to be of interest. If the pair-set axiom is only concerned with the current interpretation of the set-theoretic vocabulary, then it will remain an open question whether the axiom will still obtain when we move to more comprehensive interpretations of the language. One of the costs which we incur if we respond to the third concern by insisting on the open-ended character of $\equiv^\Diamond$ is to make the modalization of set-theoretic statements enjoy a measure of parochiality, since there is no guarantee that we will not at some point come to occupy a perspective on which the range of candidate interpretations of $\equiv$ becomes richer and more varied.

Fortunately, there is no need to pay the high costs associated with the response. For we can just point out that there is a principled reason why $\alpha^\Diamond$ and $\equiv^\Diamond$ do not, by themselves, constitute an admissible interpretation of the primitive set-theoretic vocabulary. There is a principled reason for this. The extension of $\equiv^\Diamond$ is *not* a strict relation of representation over the domain provided by $\alpha^\Diamond$. This is a crucial difference between the putative interpretations $\alpha^\Diamond$ and $\equiv^\Diamond$, on the one hand, and successive interpretations of the form α_α and $\equiv_\alpha$, on the other, for each ordinal α. Whether α is 0 or a successor ordinal or a limit, it is invariably the case that $\equiv_\alpha$ is a relation of strict representation over the domain afforded by the extension of α_α. This is what explains the fact that they satisfy the axioms (Collection) and (Extensionality), whereas there is no reason to think that $\alpha^\Diamond$ and $\equiv^\Diamond$ are governed by them. Indeed, the paradox shows that they are not.

The main advantage of this line of response is that it enables us to assume that there is a perfectly delimited range of ever more comprehensive interpretations of the set-theoretic vocabulary that remain invariant no matter where we locate ourselves in the cumulative process of reinterpretation. This, in turn, allows us to make use of the modalization of various set-theoretic statements to express perfectly general claims to do with the entire sequence of admissible reinterpretations of the set-theoretic vocabulary.

5 Conclusion

Much work remains to be done. In the best-case scenario, we have taken a vision of indefinite extensibility that many may initially have regarded as extravagant, and we have responded to three seemingly powerful considerations against it. The core of this indefinite extensibility vision is encapsulated by three axioms phrased in a modal extension of the language of plural set theory. The axioms give expression to the thought that there is no final interpretation of the set-theoretic vocabulary, but as stated, they do not even require the existence of a transfinite stage in the cumulative process of reinterpretation. The next step in the journey would be to offer a modal presentation of set theory that is in line with the linguistic model of indefinite extensibility and provides more insight into the height of the cumulative hierarchy. This would require the supplementation of the three axioms with further axioms as well as a comparison with nonmodal formulations of set theory. This work transcends the aim of this paper, which has merely been to remove a variety of seemingly unsurmountable roadblocks that stand in the way of the project.

Once we do, the motivation for the linguistic model of indefinite extensibility is not very different from one of the motivations for the extant forms of indefinite extensibility. We find ourselves in a position to assert that no matter what some objects may be, they form a set if they are available for collection. While they deny that it may make sense to speak of all and only objects, which may potentially be sets, we have no objection to such talk. We object instead to the idea that they can all ever be available for collection.

The linguistic model of indefinite extensibility has two main advantages over other modal formulations of indefinite extensibility. First, note that unlike them, we can take the modality to be interpretational: to claim that an object is potentially a set is just to claim that there is a reinterpretation of Set on which the object falls under the predicate. I take this to be uncontroversial, and we are therefore under no obligation to rely on a primitive conception of the modality.

Second, and perhaps more importantly, the fact that we are in a position to make sense of modal versions of plural comprehension on which there are, for example, some objects, which are all and only potential sets, opens the way to make sense of proper class talk in terms of plural quantification over objects across stages of the cumulative hierarchy. Likewise, we can, if we like, put plural quantification to use in the development of a perfectly general model theory for the language of plural set theory in which we allow for the domain of a model to consist of all objects whatever. This is an important application of plural quantification that is not available to proponents of extant modal accounts of indefinite extensibility.

Notes

1. See Dummett [4, p. 316].

2. A similar reconstruction applies to the indefinite extensibility of the concept *ordinal*.

3. Whatever plural quantifier is taken as primitive, the other can be defined in terms of it.

4. The answer suggested by [16] is intimately related to the iterative conception of set. For the suggestion is that a set is determined by its elements and once we specify a condition that determines some objects, we have thereby specified a set with them by elements.

5. These applications are discussed at length in Uzquiano [14] and Burgess [2].

6. Note that it is crucial at this point to have a liberal interpretation of the plural quantifier $\exists xx$ in terms of "there are zero or more objects" as opposed to "there are one or more objects."

7. As mentioned above, we have to be devious when it comes to $\equiv_\alpha$, which we will code by means of the converse of the binary relation $\in_\alpha$. An open formula of the form $x \equiv xx$ will be evaluated as true at a "world" $\langle U_\alpha, \equiv_\alpha \rangle$ relative to an assignment β if and only if for every $y \prec \beta(xx)$, the ordered pair $\langle \beta(x), y \rangle$ lies in the binary relation $\equiv_\alpha$, which is merely the converse of $\in_\alpha$.

8. The crucial difference between the two frameworks lies in the choice of a constant domain model theory that validates both the Barcan and converse Barcan formulas in contrast to a variable domain model theory with expanding domains in which only the converse Barcan formula is validated.

   In a bimodal language, we can impose further important requirements such as the condition that the cumulative process of reinterpretation be well founded. Studd [13] discusses a formulation of bimodal set theory in which he exploits this feature of the language.

9. The necessity of *one of* may be assumed as an axiom for present purposes.

10. From $\exists xx\ x \equiv xx$, we infer $\exists xx\ \Box x \equiv xx$, which, in the present framework, yields $\Box \exists xx\, x \equiv xx$. And similarly, by (($\Box \prec$) and ($\Box \equiv$)), from $\exists xx(x \prec xx \wedge y \equiv xx)$, we move to $\exists xx(\Box x \prec xx \wedge \Box y \equiv xx)$, which entails $\exists xx \Box(x \prec xx \wedge y \equiv xx)$ and $\Box \exists xx(x \prec xx \wedge y \equiv xx)$.

11. Notice that $(x \in y)^\Diamond = \exists xx(\Diamond y \equiv xx \wedge x \prec xx)$. But given (T), $\exists xx(\Diamond y \equiv xx \wedge x \prec xx)$ is equivalent to $\exists xx(\Diamond y \equiv xx \wedge \Diamond x \prec xx)$, and by ($\Box \prec$, $\exists xx \Diamond (y \equiv xx \wedge x \prec xx)$). So, by the plural version of (CBF), $\Diamond \exists xx(y \equiv xx \wedge x \prec xx)$, which is $\Diamond x \in y$. Likewise, $(\mathrm{Set}(x))^\Diamond = \exists xx\ \Diamond x \prec xx$, which by the plural version of (CBF), amounts to $\Diamond \exists xx\ x \equiv xx$, that is, $\Diamond\, \mathrm{Set}(x)$.

12. This problem is extensively discussed by Shapiro [11].

13. I have in mind, for example, McGee's categoricity result for second-order ZFCU supplemented with the axiom that the urelements form a set in McGee [10], though strictly speaking, McGee's result can be obtained from a weaker assumption according to which there are no more urelements than there are pure sets.

14. We assume that $\Diamond$ commutes with $\exists$.

15. This conception of the "set of" relation may resonate with the idea of a Fregean extension, though there are important differences.

16. We need further axioms in order to make sure the iteration is continued into the transfinite, but this is a topic for another occasion.

17. This is akin to the "Copernican revolution" outlined by Kit Fine in [5].

18. Bell writes in [1, p. 586]: "Now while we shall require a set to be a class of some kind, construing the class concept as 'class as many' entails that sets can no longer literally

be taken as individuals. So instead we shall take sets to be classes that are represented, or labelled, by individuals in an appropriate way." As for Cartwright, he writes in [3, p. 38]: "Although no set is a collection, every set represents a collection, in the sense that its members are precisely the things each of which is one of the collection. On the other hand, not every collection is represented by a set: witness the non-self-members."

19. One could have defined $\in$ differently:

$$x \in y \leftrightarrow \forall xx\,(y \equiv xx \to y \prec xx).$$

However, the extensional character of $\equiv$ guarantees the equivalence of the two definitions.

20. I am grateful to Tim Williamson for suggesting this move in discussion.

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Acknowledgments

I am grateful to audiences at Birkbeck College, Oxford, Cambridge, and the University of California at Irvine. I owe special thanks to Andrew Bacon, Øystein Linnebo, and James Studd for discussions and detailed comments on earlier versions of this paper. Author's work supported by Arts and Humanities Research Council grant AH/H039791/1.

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