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Introduction

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This special issue of the *Notre Dame Journal of Formal Logic* features work related to the talks and presentations from the summer school "Set Theory and Higher-Order Logic: Foundational Issues and Mathematical Developments," which took place at the Institute of Philosophy in London, August 1–6, 2011. The local organizers were part of Øystein Linnebo's European Research Council-funded project Plurals, Predicates, and Paradox. Further organization and support was generously provided by the New Frontiers of Infinity Project (European Science Foundation), the Ideals of Proof Project (Agence nationale de la recherche), the Philosophy Department and Logic Group of the University of Notre Dame, the Munich Center for Mathematical Philosophy (Alexander von Humboldt Stiftung), and the Philosophy Department of Birkbeck, University of London.

The summer school was well attended (circa 80 total participants) and was split into two components, with five days of introductory and advanced tutorials, followed by a two-day workshop. We are thankful to the speakers for the large amounts of time that they devoted to the summer school, and we are very happy that so many of them could contribute to this special issue. The other contributors spoke on separate occasions to the Plurals, Predicates, and Paradox project on the topic of set theory and higher-order logic. One can find the slides for the tutorials and talks for the summer school at http://www.bbk.ac.uk/philosophy/our-research/ppp/summer-school and http://www.oysteinlinnebo.org/ppp/summer-school.

The topic of set theory and higher-order logic has of course been at the heart of classical developments in the foundations of set theory. For instance, there is Gödel's famous idea that one can conceive of the cumulative hierarchy of sets as an extension of second-order logic, third-order logic, and so on, into the transfinite. Likewise, there is Kreisel's important observation that Zermelo's quasi-categoricity theorem indicates that the continuum hypothesis is decided by certain varieties of the semantics for second-order logic, in spite of its deductive independence from the first-order axioms of set theory. Finally, there is the old idea that large cardinal axioms themselves can be motivated by so-called reflection principles, which posit that the universe cannot be distinguished from its initial segments by various first- or higher-order resources.

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While ideas such as these are staples of classical foundations of set theory, it is our hope that this special issue and the summer school help to underscore that there is still much left to say on the topic of set theory and higher-order logic. For instance, the special issue includes an introduction to large cardinals as strong axioms of infinity and sketches how they imply increasing levels of the axiom of definable determinacy. Likewise, one can consult the above websites for the slides from the tutorials and talks of Antonelli, Bagaria, Horsten, Incurvati, Leitgeb, and our contributors on a multitude of recent topics like Omega-logic, the multiverse conception of set theory, groundedness, and formal theories of truth.

Many of the papers in this special issue—like much work in the philosophy of set theory—involves grappling in one way or another with the paradoxes. Russell originally used his paradox to demonstrate the inconsistency of Frege's *Grundgesetze*, and two of our contributions pertain to the *Grundgesetze*. Payne articulates a consistent modal rendition of a variant of Frege's system, and Cruz-Filipe and Ferreira study the strength of a version of the *Grundgesetze* in which the comprehension schema for the ambient higher-order logic is restricted to its so-called predicative instances. Russell, and Dummett after him, suggested that the general mechanism behind many of the paradoxes was that of indefinite extensibility. Uzquiano, in his contribution, proposes a new way of understanding the Russellian and Dummettian notion of indefinite extensibility. While it has long been noted that Russell's paradox is structurally similar to Cantor's theorem on the cardinality of the power set, the article by Meadows points out a similarity between the proof of Cantor's theorem and certain elements of Cohen's method of forcing. The status of forcing extensions is also at the center of Hamkins' essay: while many of the models of fragments of ZFC come equipped with philosophical accounts, such as the stage axioms or a limitation of size principle, forcing extensions do not. Given this, two different attitudes have emerged: (i) forcing extensions are just a technical tool used to illustrate the deductive weakness of first-order formalisms, and (ii) forcing extensions are genuine models of set theory and indeed partially constitutive of the subject matter of contemporary set theory. Hamkins' essay can be viewed as a defense of this second position, from which he further argues that it is unrealistic to expect certain kinds of resolutions to the continuum hypothesis.

Higher-order logic also features prominently in several of the other more technical papers in this special issue. Philosophical logicians and set theorists have long used logics wherein the space of truth values is a Boolean algebra with more than just two elements, and the paper by Ikegami and Väänänen devises a related Boolean-valued semantics for second-order logic. In their joint paper, Väänänen and Wang note that the classical categoricity theorems of Dedekind and Zermelo can be done in a weak background second-order logic. Visser, in his tutorial, discussed the general question of how much consistency strength is added by including a layer of predicative classes on top of a theory, and, in his contribution, he studies another measure of the strength of a theory—namely, the arithmetics of the theory. Like Visser's tutorial, the paper of Mathias and Bowler is a contribution to the study of the fine-grained properties of weak theories of classes and set. In particular, Mathias and Bowler here lay the groundwork for a weak set theory that is yet still strong enough to do some forcing.