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Revisiting $\mathbb{Z}$

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Abstract Béziau developed the paraconsistent logic $\mathbb{Z}$, which is definitionally equivalent to the modal logic $\mathbb{S}5$, and gave an axiomatization of the logic $\mathbb{Z}$: the system HZ. Omori and Waragai proved that some axioms of HZ are not independent and then proposed another axiomatization for $\mathbb{Z}$ that includes two inference rules and helps to understand the relation between $\mathbb{S}5$ and classical propositional logic. In the present paper, we analyze logic $\mathbb{Z}$ in detail; in the process we also construct a family of paraconsistent logics that are characterized by different properties that are relevant in the study of logics.

1 Introduction

The present work is a theoretical contribution to the areas of modal logics, paraconsistent logics, and nonmonotonic reasoning. Briefly speaking, following Béziau [7], a logic is paraconsistent if it has a negation ($\neg$) which is paraconsistent in the sense that the relation $a, \neg a \vdash b$ does not always hold for arbitrary formulas a, b and at the same time has strong properties that justify calling it a negation. Nevertheless, there is no paraconsistent logic that is unanimously recognized as a good one (see [7]), and there are different proposals for what a paraconsistent logic should be (see Carnielli, Marcos, and de Amo [12]). In this work, we discuss some paraconsistent logics and relevant properties that can be considered as "desirable" in any logic. Our study is limited to logics that are closed under modus ponens and substitution; some are multivalued, some are defined in terms of an axiomatic system, and some can be defined either way.

Motivation The convenience of accepting local inconsistencies is supported by Minsky's comment[1] in [22, p. 76]:

> But I do not believe that consistency is necessary or even desirable in a developing intelligent system. No one is ever completely consistent. What is important

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> is how one handles paradox or conflict, how one learns from mistakes, how one turns aside from suspected inconsistencies.

We think that paraconsistent logics could help to give an answer to this important issue addressed by Minsky. In fact, in [25] an interesting approach for knowledge representation (KR) was proposed. This approach can be supported by any paraconsistent logic stronger than or equal to C_ω, the weakest paraconsistent logic introduced by da Costa [13].

A second criticism that Minsky made of the logistic approach is that logic is monotonic:

> Because Logicians are not concerned with systems that will later be enlarged, they can design axioms that permit only the conclusions they want. In the development of Intelligence the situation is different. One has to learn which features of situations are important, and which kinds of deductions are not to be regarded seriously [22, p. 75].

Nonmonotonic logics were developed as an attempt to solve this problem. Actually the research community has long recognized the study of nonmonotonic reasoning (NMR) as a promising approach to model features of commonsense reasoning. On the other hand, monotonic logics have been successfully applied as a basic building block in the formalization of nonmonotonic reasoning. The original idea, suggested by McDermott and Doyle [20], was to use well-known modal logics. McDermott, in [19], attempted to define nonmonotonic logics based on logics T, $\mathbb{S}4$, and $\mathbb{S}5$. But he observed that, unfortunately, the nonmonotonic version of $\mathbb{S}5$ collapses to the ordinary logic $\mathbb{S}5$. *Grounded* nonmonotonic logic is a proposed solution to this problem.

In [7], the author establishes a strong relation between two logics that are treated here, the paraconsistent logic $\mathbb{Z}$ and the modal logic $\mathbb{S}5$. The interest of modal logics is well known (see, e.g., Blackburn, van Benthem, and Wolter [8]). Several authors have applied logic $\mathbb{S}5$ in the modeling of nonmonotonic reasoning by means of completions (see [15], [20], [29]).

The logic $\mathbb{Z}$, introduced by Béziau [7], is a paraconsistent logic with a very intuitive semantics and a negation with strong properties. $\mathbb{Z}$ is of particular interest because of the properties it possesses; for instance, it is equivalent to the modal logic $\mathbb{S}5$ under a translation, it satisfies the substitution theorem, it has a bottom particle, and also it can be defined in terms of axiomatic systems (see [7], [23]) as well as in terms of bivaluations (see [7]). However, $\mathbb{Z}$ is not a maximal paraconsistent logic in the following sense: for some authors a useful paraconsistent logic should be maximal in the sense of containing as much of classical logic as possible while still allowing nontrivial inconsistent theories (see [1], [13]).

In [28] the authors observed that an expressive fragment of a grounded version of $\mathbb{S}5$ can be captured by a 4-valued logic that later was called *MFOUR*. (The commented fragment consists of sentences with modalities applied only to literals.) Gelfond proposes the interpretation of *not a* as $\neg\Box a$. In fact, Baral [3] explains that the definition of stable models by Gelfond and Lifschitz [16] was inspired by this transformation. This line of research was continued and developed in further detail in [29]. It has been observed that logics whose negation operator is based on some modal logics (such as $\mathbb{S}5$ or *MFOUR*) using Gelfond's idea correspond to paraconsistent logics (see [6]). This means that by considering standard monotonic paraconsistent logics one can construct nonmonotonic logics (see [30], [25]).

Understanding nonmonotonic reasoning in the context of inconsistent knowledge bases is still an open problem; however in [24], [25], [29], [30], and [31], evidence is presented that suggests that paraconsistent logics could support the definition of robust nonmonotonic approaches in order to deal with inconsistent knowledge bases. So, it is important to study/explore paraconsistent logics in order to understand them better and see how such logics can be used to solve the problem of providing a useful nonmonotonic logic for this application.

Contributions The contributions we present here are the following.

- We have a comparative table with some relevant paraconsistent logics that share some desirable properties with $\mathbb{Z}$. Some of these logics are multivalued, and some are defined in terms of axiomatic systems. Among the desirable properties one would like a logic to have are the substitution property, the $\neg\neg$-necessitation rule, the De Morgan laws, and some other properties related to the behavior of the connective $\neg$. When comparing $\mathbb{Z}$ with other paraconsistent logics, it turns out that the paraconsistent 3-valued logic G'_3 shares more properties with $\mathbb{Z}$ than most of the other logics. Both of them are extensions of C_ω; the substitution property as well as the $\neg\neg$-necessitation rule are valid in both of them. However $\mathbb{Z}$ is not multivalued, does not satisfy all of the De Morgan laws, and counts among its axioms the Pierce formula, which is not valid in G'_3. One more difference between these two logics is that $\mathbb{Z}$ (see [7]) is originally presented in terms of an axiomatic system with two inference rules, modus ponens and the $\neg\neg$-necessitation rule, whereas G'_3, can also be defined by an axiomatic system with only modus ponens as an inference rule.
- One of the results we present is that in any paraconsistent logic that has modus ponens and the $\neg\neg$-necessitation rule as inference rules and for which some rather relevant formulas are valid, the $\neg\neg$-necessitation rule can be replaced by an extension of the axiomatic system.
- Starting from C_ω, we define an increasing family of paraconsistent logics by adding at each step one more axiom or property, usually representing a desirable property in any logic, like the weak principle of explosion, the weak contrapositive rule, the $\neg\neg$-necessitation rule, and so forth.
- We prove that in a logic that contains all of the axioms of C_ω as theorems, the substitution property is equivalent to the weak substitution property.
- We prove that a paraconsistent logic that extends C_ω and has only modus ponens as inference rule cannot satisfy double negation, standard De Morgan laws, and the substitution property at the same time.
- We prove that adding double negation and weak contrapositive as axioms to C_ω destroys paraconsistency.
- The first paraconsistent logic in our family, which we call L_1, is defined in terms of the axioms of C_ω plus the formula that expresses the weak contrapositive rule. We show that in L_1 one can define a constructive negation, and we prove that any theorem in intuitionistic logic is a theorem in the paraconsistent logic L_1.
- In the process of defining our family of paraconsistent logics, we show that the axiom or property added to each of them to obtain the next one is a valid formula in $\mathbb{Z}$. As a result of our observations we offer a very interesting result;

namely, $\mathbb{Z}$ can be defined in terms of an axiomatic system with only modus ponens as inference rule.

- We show that the only 3-valued paraconsistent logic that extends C_ω and in which the substitution property is valid is G'_3.
- We show that neither $\mathbb{Z}$ nor G'_3 are maximal paraconsistent logics: we prove that the valid formulas of $\mathbb{Z}$ are tautologies in the 4-valued logic *P-FOUR* which is also paraconsistent, but not all of the tautologies in *P-FOUR* are valid in $\mathbb{Z}$. We also show that the consequence relation in G'_3 can be extended to define another paraconsistent logic CG'_3 with the property that the set of theorems of G'_3 is properly contained in the set of tautologies of CG'_3.
- We prove with the help of the contrapositive rule, which is valid in $\mathbb{Z}$, that adding double negation as an axiom schema to $\mathbb{Z}$ results in a logic that is no longer paraconsistent. This result provides an answer to the problem posted in [12] of characterizing the exact relation between $LF2$ and the logic that results when adding the axiom schema $A \to \neg\neg A$ to $\mathbb{Z}$.

The structure of our paper is as follows. In Section 2, we present the necessary background to read the paper. In Section 3, we present some multivalued logics and look into some of their properties. We also review an axiomatization of $\mathbb{Z}$. We close the section by summarizing the properties of these logics in a table. In Section 4, we examine the substitution property along with some other important properties in the context of paraconsistent logics. In Section 5, we define in terms of axiomatic systems a family of paraconsistent logics that share some interesting properties among those gathered in Table 8, in such a way that at each step the resulting logic is closer to $\mathbb{Z}$. Finally, we arrive at two axiomatizations of $\mathbb{Z}$, one of them with only modus ponens as inference rule. In Section 6, we compare $\mathbb{Z}$ to some important multivalued logics. Finally, in Section 7, we present our conclusions and ideas for future work. Proofs are presented in the appendix.

2 Background

We first introduce the syntax of logical formulas considered in this paper. Then we present a few basic definitions of how logics can be built to interpret the meaning of such formulas. Finally, we give a brief introduction to several of the logics that are relevant for the results of later sections. We assume that the reader has some familiarity with basic logic such as [21, Chapter 1].

2.1 Syntax of formulas We consider a formal (propositional) language built from an enumerable set $\mathcal{L}$ of elements called *atoms* (denoted a, b, c, ...); the binary connectives $\wedge$ (conjunction), $\vee$ (disjunction), and $\to$ (implication); and the unary connective $\neg$ (negation). Formulas (denoted A, B, C, ...) are constructed as usual by combining these basic connectives together with the help of parentheses. We also use $A \leftrightarrow B$ to abbreviate $(A \to B) \wedge (B \to A)$ and $A \leftarrow B$ as another way of writing $B \to A$. Finally, it is useful to agree on some conventions to avoid the use of so many parentheses when writing formulas in order to make the reading of complicated expressions easier. First, we may omit the outer pair of parentheses of a formula. Second, the connectives are ordered as follows: $\neg$, $\wedge$, $\vee$, $\to$, $\leftrightarrow$, and parentheses are eliminated according to the rule that, first, $\neg$ applies to the smallest formula following it, then $\wedge$ is to connect the smallest formulas surrounding it, and so on.

2.2 Logic systems We consider a *logic* simply as a set of formulas that (i) is closed under modus ponens (i.e., if A and $A \to B$ are in the logic, then so is B) and (ii) is closed under substitution (i.e., if a formula A is in the logic, then any other formula obtained by replacing all occurrences of an atom b in A with another formula B is also in the logic). The elements of a logic are called *theorems*, and the notation $\vdash_X A$ is used to state that the formula A is a theorem of X (i.e., $A \in X$). We say that a logic X is *weaker than or equal to* a logic Y if $X \subseteq Y$; similarly we say that X is *stronger than or equal to* Y if $Y \subseteq X$.

2.2.1 Hilbert-style proof systems There are many different approaches that have been used to specify the meaning of logic formulas or, in other words, to define *logics*. In Hilbert-style proof systems, also known as *axiomatic systems*, a logic is specified by giving a set of axioms (which is usually assumed to be closed under substitution). This set of axioms specifies, so to speak, the "kernel" of the logic. The actual logic is obtained when this "kernel" is closed with respect to some given inference rules which include modus ponens. The notation $\vdash_X F$ for provability of a logic formula F in the logic X is usually extended within Hilbert-style systems; given a theory T, we use $T \vdash_X F$ to denote the fact that the formula F can be derived from the axioms of the logic and the formulas contained in T by a sequence of applications of the inference rules.[2] For any pair of theories T and U, we use $T \vdash_X U$ to state the fact that $T \vdash_X F$ for every formula $F \in U$.

Since we will be working with logics that extend some other logics, we provide the next definition.

Definition 2.1 Let X and Y be two logics. We say that X *extends* Y if every theorem in Y is a theorem in X. We say that X *AX-extends* Y if both logics have modus ponens as their unique rule of inference, and every axiom in Y is a theorem in X.

Note that if X AX-extends Y, then X extends Y.

We consider the *standard* substitution, here represented with the usual notation: $\varphi[\alpha/p]$ will denote the formula that results from substituting the formula α in place of the atom p wherever it occurs in φ. Recall the recursive definition: if φ is atomic, then $\varphi[\alpha/p]$ is α when φ equals p, and φ otherwise. Inductively, if φ is a formula of the form $\varphi_1 \# \varphi_2$ for any binary connective #, then $\varphi[\alpha/p]$ will be $\varphi_1[\alpha/p] \# \varphi_2[\alpha/p]$. Finally, if φ is a formula of the form $\neg\varphi_1$ (resp., $\sim\varphi_1$), then $\varphi[\alpha/p]$ will be $\neg\varphi_1[\alpha/p]$ (resp., $\sim\varphi_1[\alpha/p]$).

As an example of a Hilbert style system we present next a logic that is relevant for our work.

C_ω (see [13]) is defined by the following set of axiom schemata:

Pos1: $a \to (b \to a)$,
Pos2: $(a \to (b \to c)) \to ((a \to b) \to (a \to c))$,
Pos3: $a \wedge b \to a$,
Pos4: $a \wedge b \to b$,
Pos5: $a \to (b \to (a \wedge b))$,
Pos6: $a \to (a \vee b)$,
Pos7: $b \to (a \vee b)$,
Pos8: $(a \to c) \to ((b \to c) \to (a \vee b \to c))$,
C_ω1: $a \vee \neg a$,
C_ω2: $\neg\neg a \to a$.

Note that the first eight axiom schemata somewhat constrain the meaning of the $\to$, $\wedge$, and $\vee$ connectives to match our usual intuitions. It is a well-known result that in any logic satisfying **Pos1** and **Pos2**, and with *modus ponens* as its unique inference rule, the *deduction theorem* holds (see [21]).

We also have the following defined connective: $A \leftrightarrow B := (A \to B) \wedge (B \to A)$.

It is always useful to know that some strong inference rules hold in a given logic. In this sense, a simple but useful and well-known result is the following (see Carnielli and Marcos [11]).

Theorem 2.2 *Let Γ and Δ be two sets of formulas. Let θ, θ_1, θ_2, α, and ψ be arbitrary formulas. Let $\vdash$ be the deductive inference operator of C_ω. Then the following basic properties hold:*

1. *$\Gamma \vdash \alpha$ implies $\Gamma \cup \Delta \vdash \alpha$ (monotonicity);*
2. *$\Gamma \vdash \alpha$ and $\Delta, \alpha \vdash \psi$, then $\Gamma \cup \Delta \vdash \psi$ (cut);*
3. *$\Gamma, \theta \vdash \alpha$ iff $\Gamma \vdash \theta \to \alpha$ (deduction theorem and its reciprocal);*
4. *$\Gamma \vdash \theta_1 \wedge \theta_2$ iff $\Gamma \vdash \theta_1$ and $\Gamma \vdash \theta_2$ ($\wedge$-rules);*
5. *$\Gamma, \theta \vdash \alpha$ and $\Gamma, \neg\theta \vdash \alpha$ iff $\Gamma \vdash \alpha$ (strong proof by cases).*

Note that the cut rule, when $\Delta = \emptyset$, has a form of transitivity, namely, $\Gamma \vdash \alpha$ and $\alpha \vdash \psi$, then $\Gamma \vdash \psi$. We point out that Theorem 2.2 holds in any logic that extends C_ω logic by adding new axioms.

2.2.2 Multivalued systems An alternative way to define a logic is with the use of truth values and interpretations. Multivalued systems generalize the idea of using truth tables that are used to determine the validity of formulas in classical logic. It has been suggested that multivalued systems should not count as logics; on the other hand, pioneers such as Lukasiewicz considered such multivalued systems as alternatives to the classical framework. Like other authors do, we prefer to give multivalued systems the benefit of the doubt about their status as logics.

The core of a multivalued system is its *domain* of values $\mathcal{D}$, where some of such values are special and identified as *designated*. Connectives (e.g., $\wedge$, $\vee$, $\to$, $\neg$) are then introduced as operators over $\mathcal{D}$ according to the particular definition of the logic. An *interpretation* is a function $I\colon \mathcal{L} \to \mathcal{D}$ that maps atoms to elements in the domain. The application of I is then extended to arbitrary formulas by mapping first the atoms to values in $\mathcal{D}$ and then evaluating the resulting expression in terms of the connectives of the logic. A formula is said to be a *tautology* if, for every possible interpretation, the formula evaluates to a designated value. The most simple example of a multivalued logic is classical logic where $\mathcal{D} = \{0, 1\}$, 1 is the unique designated value, and connectives are defined through the usual basic truth tables.

Note that in a multivalued system, so that it can truly be a *logic*, the implication connective has to satisfy the following property: for every value $x \in \mathcal{D}$, if there is a designated value $y \in \mathcal{D}$ such that $y \to x$ is designated, then x must also be a designated value. This restriction enforces the validity of modus ponens in the logic. The inference rule of substitution holds without further conditions because of the functional nature of interpretations and how they are evaluated.

2.3 Maximality Next we present two nonequivalent definitions of maximality for paraconsistent logics. The first definition refers to maximality of a logic relative to another logic, whereas the second one is absolute in the sense that it is not defined with respect to some other given logic.

Definition 2.3 (see [11]) A logic L_2 is said to be *maximal relative to a logic* L_1 if (i) both are written in the same language (so that they can be deductively compared); (ii) all theorems of L_2 are provable by L_1; (iii) given a theorem D of L_1 which is not a theorem of L_2, if D is added to L_2 as a new axiom schema, then all theorems of L_1 turn out to be provable.

A paraconsistent logic L_2 is *maximally paraconsistent in the strong sense* (see [2]) if every logic L_1 in the language of L_2 that properly extends L_2 (i.e., $\vdash_{L2} \subset \vdash_{L1}$) is no longer paraconsistent.

The idea of the first definition, of course, is that any deductive extension of L_2 contained in L_1 and obtained by adding a new axiom schema to L_2 would turn out to be identical to or stronger than L_1. Examples of maximal logics in this sense abound in the literature. The second definition takes into account any possible extension of the underlying consequence relation of a logic, not just its set of logically valid sentences.[3]

Some of the many-valued paraconsistent logics presented in this work (**Civw**, some of the logics LFI's, the *P-FOUR* logic) are maximally paraconsistent relative to classical logic (see [12], [18]).

In what follows, we will be specific when we refer to the term *maximal paraconsistent logic*.

3 Some Paraconsistent Logics

In this section, we review some well-known multivalued logics. We start with the 3-valued logics, among them logics that are maximal paraconsistent with respect to classical logic, $\mathbf{P}^1$, $\mathbf{P}^2$, $\mathbf{P}^3$, **LFI2**, and **LFI1**. We remark that the matrices of $\mathbf{P}^1$ are axiomatized by **Civw**, the matrices of $\mathbf{P}^2$ by **Cive**, the matrices of $\mathbf{P}^3$ by **Ciorw**, the matrices of **LFI2** by **Ciore**, and the matrices of **LFI1** by **Cije** (see [11]). We continue with the review of 4-valued logics, and then we summarize some important facts and results about ℤ.

Finally, as a result of this review, we present a comparative table showing some of the important properties of these logics as well as some of their differences.

3.1 3-valued logics: Civw, Cive, Ciorw, Ciore, Cije, G'_3, and CG'_3

3.1.1 The logic $\mathbf{P}^1$ *(also called* ***Civw****)* The maximal 3-valued logic $\mathbf{P}^1$ proposed by Sette in [33] is paraconsistent. The truth values of logic $\mathbf{P}^1$ are in the domain $D = \{0, 1, 2\}$ where 1 and 2 are the designated values. The $\wedge$, $\vee$, $\rightarrow$, and $\neg$ connectives are defined according to the truth tables given in Table 1.

3.1.2 The logic $\mathbf{P}^2$ *(also called* ***Cive****)* It is possible to modify the matrix of negation of $\mathbf{P}^1$ in order to obtain a new and interesting maximal 3-valued paraconsistent logic called $\mathbf{P}^2$. In this case, we change the matrix of negation, setting the value of $\neg 1$ as 1, instead of 2.

Table 1 Truth tables of connectives $\wedge$, $\vee$, $\rightarrow$, and $\neg$ in $\mathbf{P}^1$.

$\wedge$	0	1	2
0	0	0	0
1	0	2	2
2	0	2	2

$\vee$	0	1	2
0	0	2	2
1	2	2	2
2	2	2	2

$\rightarrow$	0	1	2
0	2	2	2
1	0	2	2
2	0	2	2

x	$\neg x$
0	2
1	2
2	0

Table 2 Truth tables of connectives $\wedge$, $\vee$, $\rightarrow$, and $\neg$ in $\mathbf{P}^3$.

$\wedge$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	2

$\vee$	0	1	2
0	0	2	2
1	2	1	2
2	2	2	2

$\rightarrow$	0	1	2
0	2	2	2
1	0	1	2
2	0	2	2

x	$\neg x$
0	2
1	2
2	0

Table 3 Truth tables of connectives $\wedge$, $\vee$, $\rightarrow$, and $\neg$ in **LFI1**.

$\wedge$	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

$\vee$	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

$\rightarrow$	0	1	2
0	2	2	2
1	0	1	2
2	0	1	2

x	$\neg x$
0	2
1	1
2	0

Table 4 Truth tables of connectives $\neg$ and $\rightarrow$ in G'_3.

x	$\neg x$
0	2
1	2
2	0

$\rightarrow$	0	1	2
0	2	2	2
1	0	2	2
2	0	1	2

3.1.3 The logic $\mathbf{P}^3$ *(also called* ***Ciorw****)* The 3-valued logic $\mathbf{P}^3$ is a logic which is paraconsistent. The truth values of logic $\mathbf{P}^3$ are in the domain $D = \{0, 1, 2\}$ where 1 and 2 are the designated values. The $\wedge$, $\vee$, $\rightarrow$, and $\neg$ connectives are defined according to the truth tables given in Table 2.

3.1.4 The logic ***LFI2*** *(also called* ***Ciore****)* It is possible to modify the matrix of negation of $\mathbf{P}^3$ in order to obtain a new and interesting maximal 3-valued paraconsistent logic called **LFI2** (investigated in [12]). In this case, we change the matrix of negation, setting the value of $\neg 1$ as 1, instead of 2.

3.1.5 The logic ***LFI1*** *(also called* ***Cije****)* The 3-valued logic **LFI1** is a logic which is paraconsistent. The truth values of logic **LFI1** are in the domain $D = \{0, 1, 2\}$ where 1 and 2 are the designated values. The $\wedge$, $\vee$, $\rightarrow$, and $\neg$ connectives are defined according to the truth tables given in Table 3.

3.1.6 The logic G'_3 The G'_3 logic (see [30]) is a 3-valued logic with truth values in the domain $D = \{0, 1, 2\}$ where 2 is the designated value. The evaluation function of the logic connectives is then defined as follows: $x \wedge y = \min(x, y)$, $x \vee y = \max(x, y)$; the $\neg$ and $\rightarrow$ connectives are defined according to the truth tables given in Table 4. G'_3 can also be defined in terms of an axiomatic system (see [26]); that is, a theorem in such a system is a tautology in the 3-valued definition, and vice versa.

3.1.7 The logic CG'_3 We introduce a new logic which is useful for our purposes; as far as we know it has not been studied before. The logic CG'_3 is defined with the truth tables of G'_3, except that 1 and 2 are the designated values.

3.2 4-valued logics: *P-FOUR*, M4$_p$, M4′

3.2.1 The logic P-FOUR *P-FOUR* logic is a 4-valued logic with truth values in the domain $D = \{0, 1, 2, 3\}$ where 3 is the designated value. This logic is defined in [28] and studied in further detail in [30]. The connectives $\wedge$, $\vee$, $\neg$, and $\rightarrow$ are defined according to the truth tables given in Table 5.

Table 5 Truth tables of connectives $\wedge$, $\vee$, $\rightarrow$, and $\neg$ in *P-FOUR*.

$\wedge$	0	1	2	3
0	0	0	0	0
1	0	1	0	1
2	0	0	2	2
3	0	1	2	3

$\vee$	0	1	2	3
0	0	1	2	3
1	1	1	3	3
2	2	3	2	3
3	3	3	3	3

$\rightarrow$	0	1	2	3
0	3	3	3	3
1	2	3	2	3
2	1	1	3	3
3	0	1	2	3

x	$\neg x$
0	3
1	3
2	3
3	0

Table 6 Truth tables of connectives $\wedge$, $\vee$, $\rightarrow$, and $\neg$ in **M4** and $\sim$ in $\mathbf{M4}_p$.

$\wedge$	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	0	1	2	2
3	0	1	2	3

$\vee$	0	1	2	3
0	0	1	2	3
1	1	1	2	3
2	2	2	2	3
3	3	3	3	3

$\rightarrow$	0	1	2	3
0	3	3	3	3
1	2	2	2	3
2	1	1	2	3
3	0	1	2	3

x	$\neg x$
0	3
1	2
2	1
3	0

x	$\sim x$
0	3
1	3
2	3
3	0

Table 7 Truth table of connective $\neg$ in $\mathbf{M4}'$.

x	$\neg x$
0	3
1	2
2	2
3	0

3.2.2 The $\mathbf{M4}_p$ *logic* **M4** (see [6]) is a 4-valued modal logic with truth values in the domain $D = \{0, 1, 2, 3\}$ where 2 and 3 are the designated values. The connectives $\wedge$, $\vee$, $\neg$, and $\rightarrow$ are defined according to the truth tables given in Table 6. We can see that conjunction and disjunction are defined in the usual way by the operators min and max. The implication $a \rightarrow b$ is defined as $\neg a \vee b$.

From **M4** logic, we obtain a paraconsistent logic, called $\mathbf{M4}_p$, when we modify the matrix of the negation connective as shown in Table 6, where the new negation is denoted by $\sim$.

3.2.3 The $\mathbf{M4}'$ *logic* The author of [6] also proposes to modify the matrix of negation of **M4** logic to obtain another paraconsistent logic that we call $\mathbf{M4}'$. In this case, we change the matrix of negation according to the truth table given in Table 7.

3.3 The logic $\mathbb{Z}$ $\mathbb{Z}$ is our main subject of interest, so we next present its definition and some of its properties.

The logic $\mathbb{Z}$ is introduced semantically by means of bivaluations which are functions from $\text{For}_{\mathbb{Z}}$ to $\{0, 1\}$, where $\text{For}_{\mathbb{Z}}$ is the set of formulas of $\mathbb{Z}$. A $\mathbb{Z}$-cosmos is any nonempty set $\mathcal{C}$ of bivaluations defined by the following condition: $v \in \mathcal{C}$ iff it obeys the classical conditions for $\vee, \wedge, \rightarrow$, and also the following condition for the $\neg$ (intended to be a paraconsistent negation), $v(\neg A) = 1$ iff $\exists u \in \mathcal{C}, u(A) = 0$. A formula A is $\mathbb{Z}$-valid iff, for any $\mathbb{Z}$-cosmos $\mathcal{C}$, $\forall v \in \mathcal{C}, v(A) = 1$.

For example, the formulas $A \vee \neg A$ and $\neg\neg A \to A$ given as axiom schemata of the logic $\mathbb{Z}$ are $\mathbb{Z}$-valid, since the condition that defines the paraconsistent negation guaranties that $v(A) = v(\neg A) = 0$ does not hold for any bivaluation; hence $v(A \vee \neg A) = 1$. Also, let us assume that we have a bivaluation v in a $\mathbb{Z}$-cosmos $\mathcal{C}$ such that $v(\neg\neg A) = 1$; then by the same property there exists a $u \in \mathcal{C}$ such that $u(\neg A) = 0$, and then $v(A) = 1$; hence $\neg\neg A \to A$ is $\mathbb{Z}$-valid.

Béziau offers also an axiomatization for $\mathbb{Z}$, the Hilbert system HZ (see [7, Definition 3.1]). The system HZ contains all of the axioms of C_ω plus the following three axiom schemata:

Pierce: $((a \to b) \to a) \to a$,
AZ2: $(a \wedge \neg b) \wedge \neg(a \wedge \neg b) \to (a \wedge \neg a)$,
AZ3: $\neg(a \wedge b) \to (\neg a \vee \neg b)$.

This is the only logic presented here that, in addition to modus ponens, has a second rule of inference called **RZ**; namely, from $a \to b$ one can derive $\neg(a \wedge \neg b)$.

In [7] it is proved that a formula $A \in \mathrm{For}_{\mathbb{Z}}$ is a theorem in the system **HZ** iff A is $\mathbb{Z}$-valid. Also the system **HZ** can be interpreted as an axiomatization of the modal logic $\mathbb{S}5$ that uses a negation operator as a primitive connective rather than a necessity or possibility operator. In fact the logic $\mathbb{Z}$ is translatable into $\mathbb{S}5$, and $\mathbb{S}5$ contains $\mathbb{Z}$ in the sense that $\mathbb{Z}$ is a reduct of $\mathbb{S}5$ (in the model-theoretic sense).

Definition 3.1 (see [7]) Let $*$ be the following translation function from the set of formulas of $\mathbb{Z}$ into the set of formulas of $\mathbb{S}5$: $\langle \mathrm{For}_{S5}; \wedge, \vee, \to, \sim, \Box \rangle$ ($\sim$ is the classical negation). We have the following:

$$a^* = a \quad \text{if } a \text{ is atomic},$$
$$(a \circ b)^* = a^* \circ b^*, \quad \circ \in \{\wedge, \vee, \to\},$$
$$(\neg a)^* = \sim\Box a^*.$$

For a family T of formulas in $\mathbb{Z}$, T^* is defined as $\{f^* | f \in T\}$.

Remark According to this translation,

$$(\neg\neg A)^* = \sim\Box\sim\Box A^* \quad \text{and} \quad \sim\Box\sim\Box A^* \leftrightarrow \Diamond\Box A^* \leftrightarrow \Box A^* \quad \text{(see [6])}.$$

Theorem 3.2 (see [7]) *Let T be a $\mathbb{Z}$-theory, and let α be a formula of $\mathbb{Z}$; then $T \vdash_{\mathbb{Z}} \alpha$ iff $T^* \vdash_{\mathbb{S}5} \alpha^*$.*

Let us consider the following two examples.

In order to prove that $\neg\neg X \to X$ is provable in $\mathbb{Z}$, it is enough to check that $\sim\Box\sim\Box X \to X$ is provable in logic $\mathbb{S}5$; this turns out to be an easy exercise.[4]

In a similar way, in order to verify that the formula

$$\big(\neg(x \vee y) \wedge \neg(y \to x)\big) \to \neg\big(\neg\big(x \to \big(\neg x \wedge \neg(\neg x)\big)\big)\big) \tag{1}$$

is not provable in $\mathbb{Z}$, it is enough to check that

$$\big(\sim\Box(x \vee y) \wedge \sim\Box(y \to x)\big) \to \sim\Box\big(\sim\Box\big(x \to \big(\sim\Box x \wedge \sim\Box(\sim\Box x)\big)\big)\big)$$

is not provable in modal $\mathbb{S}5$, which is also an easy exercise.

Since formula (1) will be used in a result presented in a later section, we call it Φ.

More recently Omori and Waragai in [23] prove that some axioms of **HZ** are not independent, and they then propose another axiomatization of $\mathbb{Z}$. They also discuss a new perspective on the relation between $\mathbb{S}5$ and classical propositional logic with the help of the new axiomatization of $\mathbb{Z}$ and conclude the paper by making a remark on the paraconsistency of $\mathbb{Z}$. Specifically, the system **HZ** seems to be of great interest, since it gives an axiomatization of $\mathbb{S}5$ using not the necessity operator or the possibility operator explicitly but a specific negation-like operator as its primitive connective. Therefore, they indicate that they might be able to reach a new point of view on the system of modal logic $\mathbb{S}5$. They show that the property of necessitation in the system **HZ** holds, give the definition of the bottom particle $\perp$, and give the proof of the replacement theorem for the negation in the system **HZ**. They also show an alternative axiomatization of $\mathbb{Z}$-logic given by: **Pos1–Pos8** of C_ω-logic, Pierce's axiom schema of $\mathbb{Z}$, and the two following negation axiom schemata:

$\boldsymbol{AZ2'}$: $\neg(a \wedge \neg b) \to (\neg b \to \neg a)$,
$\mathbf{C_\omega 2}$: $\neg\neg a \to a$.

In that axiomatization the authors include two rules of inference, modus ponens and **RZ**, and prove the next result, which they call rule **R3**:

$$\text{If } \vdash a \to b, \quad \text{then } \vdash \neg b \to \neg a.$$

We now present the first result of this paper as Theorem 3.4 (as noted above, all the proofs are in the appendix). Theorem 3 assures us that any theorem in $\mathbb{S}5$ can be mapped into a theorem in $\mathbb{Z}$, so it extends Theorem 3.2, by proving the opposite direction. To see this, we define a mapping *: $\mathrm{For}_{\mathbb{S}5} \longrightarrow \mathrm{For}_{\mathbb{Z}}$ such that for any $\alpha \in \mathrm{For}_{\mathbb{S}5}$, we have that $\vdash_{\mathbb{S}5} \alpha$ iff $\vdash_{\mathbb{Z}} \alpha^\star$. This mapping is defined as follows (the notation is taken from [17]):

$$\begin{aligned} a^\star &= a \quad \text{if } a \text{ is atomic,} \\ \perp^\star &= \neg p \wedge \neg\neg p \quad \text{for a fixed atom } p, \\ (a \circ b)^\star &= a^\star \circ b^\star \circ \in \{\wedge, \vee, \to\} \quad \text{for any formulas } a, b, \\ (\Box a)^\star &= \neg\neg(a^\star) \quad \text{for any formula } a. \end{aligned}$$

We have the following lemma to support the result.

Lemma 3.3 *For any formula* $\beta \in \mathrm{For}_{\mathbb{S}5}, \vdash_{\mathbb{S}5} (\beta^\star)^* \leftrightarrow \beta$.

Theorem 3.4 *Given the transformation* **:* $\mathrm{For}_{\mathbb{S}5} \longrightarrow \mathrm{For}_{\mathbb{Z}}$*, we have that* $\vdash_{\mathbb{S}5} \alpha$ *iff* $\vdash_{\mathbb{Z}} \alpha^\star$ *for any* $\alpha \in \mathrm{For}_{\mathbb{S}5}$.

3.4 Summarizing the properties of some paraconsistent logics In Table 8, we summarize the properties of the several logics we have explored in this section. Many of them can be found in the literature (see [7], [11], [12], [23] [28], [30], [33]), while others are verified in later sections.

4 Some General Results in Paraconsistent Logics

An interesting theoretical question that arises in the study of logics is whether a given logic satisfies the substitution property, also known as the *congruence relation* (see [34]). It is well known that there are several paraconsistent logics for which that property is not valid (see [10]). In this section we examine this property along with some other important results in the context of paraconsistent logics.

Definition 4.1 A logic X satisfies the substitution property: if $\vdash_X \alpha \leftrightarrow \beta$,[5] then $\vdash_X \Psi[\alpha/p] \leftrightarrow \Psi[\beta/p]$ for any formulas α, β, and Ψ and any atom p that appears

Table 8 Logics and their properties.

Formula	cije	ciore	ciorw	civw	cive	G'_3	CG'_3	*P-FOUR*	$M4_p$	$M4'$	$\mathbb{Z}$
$\neg\alpha \to (\neg\neg\alpha \to \beta)$	X	X	✓	✓	X	✓	✓	✓	✓	X	✓
$\neg\neg(\alpha \to \alpha)$	✓	✓	X	✓	✓	✓	✓	✓	X	✓	✓
$(\alpha \to \beta) \to \neg\neg(\alpha \to \beta)$	✓	✓	X	✓	✓	X	X	X	X	✓	X
$(\neg\alpha \to \neg\beta) \leftrightarrow (\neg\neg\beta \to \neg\neg\alpha)$	X	X	✓	✓	X	✓	✓	✓	✓	X	✓
$\neg\neg(\alpha \to \beta) \to (\neg\neg\alpha \to \neg\neg\beta)$	✓	✓	X	X	✓	✓	✓	✓	✓	X	✓
$\neg(\alpha \wedge \neg\alpha)$	✓	✓	X	X	X	✓	✓	✓	✓	✓	✓
$\neg\neg(\alpha \wedge \beta) \leftrightarrow (\neg\neg\alpha \wedge \neg\neg\beta)$	✓	✓	X	X	✓	✓	✓	✓	✓	✓	✓
$\neg\neg\alpha \leftrightarrow \alpha$	✓	✓	X	X	✓	X	X	X	X	X	X
$\neg(\alpha \wedge \beta) \to (\neg\alpha \vee \neg\beta)$	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
$(\neg\alpha \vee \neg\beta) \to \neg(\alpha \wedge \beta)$	✓	X	X	X	X	✓	✓	✓	✓	✓	✓
$\neg(\alpha \vee \beta) \to (\neg\alpha \wedge \neg\beta)$	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
$(\neg\alpha \wedge \neg\beta) \to \neg(\alpha \vee \beta)$	✓	X	X	X	X	✓	✓	X	✓	✓	X
Axioms of C_ω-logic	✓	✓	✓	✓	✓	✓	✓	✓	✓	X	✓
Necessitation	✓	✓	X	X	✓	✓	X	✓	X	✓	✓
$((\alpha \to \beta) \to \alpha) \to \alpha$ (Pierce)	✓	✓	✓	✓	✓	X	✓	✓	✓	✓	✓

in Ψ, where $\Psi[\alpha/p]$ denotes the resulting formula that is left after every occurrence of p is substituted by the formula α.

The next weak version of the substitution property is relevant for our work.

Definition 4.2 A logic X satisfies the weak substitution property: if $\vdash_X \alpha \leftrightarrow \beta$, then $\vdash_X \neg\alpha \leftrightarrow \neg\beta$.

Besides these two properties, there are some other properties that are relevant in the study of these logics. The next definition lists some of them.

Definition 4.3 We say that a logic X satisfies, respectively, the following:

1. *Double negation* if $\vdash_X \alpha \leftrightarrow \neg\neg\alpha$;
2. *Standard De Morgan* if $\vdash_X \neg(\alpha \wedge \beta) \leftrightarrow (\neg\alpha \vee \neg\beta)$ and $\vdash_X \neg(\alpha \vee \beta) \leftrightarrow (\neg\alpha \wedge \neg\beta)$;
3. *Distribution of* $\neg\neg$ *over* $\wedge$ if $\vdash_X \neg\neg(\alpha \wedge \beta) \leftrightarrow (\neg\neg\alpha \wedge \neg\neg\beta)$;
4. *Distribution of* $\neg\neg$ *over* $\to$ if $\vdash_X \neg\neg(\alpha \to \beta) \to (\neg\neg\alpha \to \neg\neg\beta)$;
5. $\neg\neg$-*Necessitation* if $\vdash_X \alpha$, then $\vdash_X \neg\neg\alpha$;
6. *Weak Explosion* if the following property holds: $\vdash_X \neg\alpha \to (\neg\neg\alpha \to \beta)$;
7. *Weak Contrapositive* if $\vdash_X (\neg\alpha \to \neg\beta) \leftrightarrow (\neg\neg\beta \to \neg\neg\alpha)$.

The formulas in Definition 4.3 are relevant for our work, since they will be used to define a family of logics that will help us to reconstruct logic $\mathbb{Z}$.

The next theorem ensures that, in a logic that has modus ponens and the $\neg\neg$-necessitation rule as its inference rules and that counts among its axioms some of the formulas that define the properties listed above, the $\neg\neg$-*necessitation* rule can be replaced by an extension of the axiomatic system with only modus ponens as an inference rule. This result will be useful later to present one of our main contributions.

Theorem 4.4 *Let Ł be a logic defined in terms of a family of axioms* Δ *and with modus ponens and the* $\neg\neg$*-necessitation rule as its inference rules. Let us assume that* $\{\neg\neg\alpha \to \alpha, \neg\neg(\alpha \to \beta) \to (\neg\neg\alpha \to \neg\neg\beta), (\neg\alpha \to \neg\beta) \leftrightarrow (\neg\neg\beta \to \neg\neg\alpha)\}$ *are theorems of Ł. Let NŁ be the logic defined by the family of axioms* $\{\neg\neg\alpha \to \alpha\} \cup \{\neg\neg A | A \in \Delta\}$ *and with modus ponens as its unique inference rule. Then Ł and NŁ have the same theorems.*

The proof consists in showing that if A is a theorem in NŁ, then $\neg\neg A$ is also a theorem in NŁ and is done by induction on the size of the proof of A.

Observe that, according to Theorem 4.4, if each axiom in the logic is replaced by its double negation, then three specific axiom schemata together with modus ponens as inference rule guarantee the $\neg\neg$-necessitation rule.

The following lemma says that we cannot have three suitable properties in a paraconsistent logic that AX-extends C_ω.

Lemma 4.5 *Let X be a paraconsistent logic that AX-extends C_ω. Then X does not satisfy double negation, standard De Morgan, and substitution (the congruence relation) at the same time.*

The next lemma says that, in any extension of C_ω, the properties of *double negation* and *weak contrapositive* are not consistent with paraconsistency.

Lemma 4.6 *If we add to C_ω the 2-axiom schemata $\alpha \leftrightarrow \neg\neg\alpha$ (double negation) and $(\neg\alpha \to \neg\beta) \leftrightarrow (\neg\neg\beta \to \neg\neg\alpha)$, the resulting logic is not paraconsistent.*

Remark 4.7 In [5] the author refers to a related result in paraconsistent logic; namely, if the principle of noncontradiction and the double negation are both valid, then the replacement theorem cannot hold.

We see now that any logic stronger than C_ω that satisfies the weak substitution property also satisfies the substitution property, and vice versa.

Lemma 4.8 *Let X be a logic that contains all of the axioms of C_ω as theorems. X satisfies the weak substitution property iff X satisfies the substitution property.*

The next theorem is the main part of the proof of Theorem 4.10; its proof is based on an answer-set encoding which includes the formula that defines the weak substitution property.

Theorem 4.9 (see [27]) *G'_3 is the only 3-valued logic, up to isomorphism, which extends C_ω and in which the weak substitution property is valid.*

As an immediate consequence we have the next result.

Theorem 4.10 *G'_3 is the only 3-valued logic, up to isomorphism, which extends C_ω and in which the substitution property is valid.*

The formula $(\neg\alpha \to \neg\beta) \leftrightarrow (\neg\neg\beta \to \neg\neg\alpha)$ that appears in Lemma 4.6 is valid in several of the logics shown in Table 8, in particular in $\mathbb{Z}$. We are interested in logics for which this formula is valid; in fact it will help us to rebuild $\mathbb{Z}$, and that is why we will call it $E1$.

Lemma 4.11 *If we add to C_ω any one of the two axiom schemata $\neg\alpha \to (\neg\neg\alpha \to \beta)$, $E1 = (\neg\alpha \to \neg\beta) \leftrightarrow (\neg\neg\beta \to \neg\neg\alpha)$, the resulting logics are the same.*

Double negation is a feature that shows up more often than weak explosion in the literature related to paraconsistent logics. However, there are paraconsistent logics such as Sette's logic and some of the logics shown in Table 8, where weak explosion holds (see [9]). In what follows, we are more interested in paraconsistent logics where the property of weak explosion holds (see [4]). According to Lemma 4.11, weak explosion is equivalent to axiom schema $E1$ for logics that extend C_ω. L_1 is defined as the result of adding $E1$ to C_ω.

5 Extending C_ω

In this section we build a new Hilbert-style axiomatization for $\mathbb{Z}$ that uses only modus ponens as inference rule. In the process, we define a family of logics that are defined in terms of important properties satisfied by some known paraconsistent logics. Each of these logics corresponds to a Hilbert-style axiomatization that extends C_ω.

The definition of the family of logics is progressive; each step will consist of adding a new axiom schema or property to the previous logic to define a new one. Then we present some properties of the new logic; in particular, we point out its relation with $\mathbb{Z}$. Finally, we indicate which logics in Table 8 satisfy the property used to define the new logic.

As before, proofs of the following results can be found in the appendix.

5.1 The logic L_1 Logic L_1 has four primitive logical connectives, namely, $\mathcal{GL} := \{\rightarrow, \wedge, \vee, \neg\}$. We add the following axiom schema to the axiom schemata of C_ω:

E1: $(\neg A \rightarrow \neg B) \leftrightarrow (\neg\neg B \rightarrow \neg\neg A)$ (*weak contrapositive*).

To see that L_1 is paraconsistent, observe that L_1 is sound with respect to G'_3, but $(\alpha \wedge \neg\alpha) \rightarrow \beta$ is not a tautology of G'_3. Also note that, according to Lemma 4.6, adding the double negation property to L_1 would result in a logic which is no longer paraconsistent.

In [12], the authors observe that the logic **ciore** (also called **LFI2**) and the modal logic $\mathbb{Z}$ plus the axiom schema $A \rightarrow \neg\neg A$ seem extremely close to each other. According to [23], $\mathbb{Z}$ satisfies the weak explosion principle; therefore, from Lemma 4.11, formula $E1$ is a theorem in $\mathbb{Z}$. Hence it follows from Lemma 4.6 that adding axiom schema $\alpha \rightarrow \neg\neg\alpha$ to logic $\mathbb{Z}$ results in a logic that is not paraconsistent.

By Lemma 4.11, L_1, and in general any extension of L_1, has the *bottom particle*; that is, the formula $\neg\alpha \wedge \neg\neg\alpha$ trivializes the logic. This means that $\neg\alpha \wedge \neg\neg\alpha \vdash \beta$. We define $\bot$ as some fixed formula of the form $\neg\alpha \wedge \neg\neg\alpha$. We have the new defined negation connective: $\sim A := (A \rightarrow \bot)$.

Lemma 5.1 *The following formula is a theorem in L_1: $\vdash \neg\beta \leftrightarrow \neg\neg\neg\beta$.*

Observe that L_1 restricted to the language $\mathcal{IL} := \{\rightarrow, \wedge, \vee, \bot\}$ defines a constructive logic. Recall that the formula $(\alpha \rightarrow \beta) \vee (\beta \rightarrow \alpha)$ is not a theorem in intuitionistic logic, the weakest constructive logic.

The next result points out an interesting property of L_1, which is that it can express intuitionistic logic.

Theorem 5.2 *Let α be any formula based on $\mathcal{IL}$. Then any theorem α in intuitionistic logic is a theorem in L_1. Moreover, none of the following formulas is a theorem in L_1: $(\alpha \rightarrow \beta) \vee (\beta \rightarrow \alpha)$, $\alpha \vee \sim\alpha$, $\sim\sim\alpha \rightarrow \alpha$, $\sim\alpha \vee \sim\sim\alpha$.*

Note that, according to the next lemma, $\sim$ is a strong negation.

Lemma 5.3 *In L_1, $\vdash \sim\alpha \rightarrow \neg\alpha$.*

A weak principle of disjunctive syllogism is valid in L_1, as the following lemma shows.

Lemma 5.4 *In L_1, $\neg\neg\alpha, \neg\alpha \vee \beta \vdash \beta$.*

Observe that although L_1 AX-extends C_ω, it does not satisfy any *double negation*, *standard De Morgan*, or *substitution* properties.

Lemma 5.5 *In L_1 none of the following properties hold: double negation, standard De Morgan, and substitution.*

The next lemma asserts that the formulas $\neg\neg(\alpha \to \beta) \to (\neg\neg\alpha \to \neg\neg\beta)$ and $(\alpha \to \beta) \to \neg\neg(\alpha \to \beta)$ destroy paraconsistency in any extension of L_1 that has them both as theorems. However, each formula is accepted in some paraconsistent logics. For example, among the logics exhibited in Table 8, CG'_3, *P-FOUR*, and $\mathbb{Z}$ have the first formula as a theorem, whereas civw, cive, and $\mathbf{M4}_p$ have the second one as a theorem. It is worth mentioning that civw is an example of an extension of L_1 for which the second formula is valid.

Lemma 5.6 *If we add to L_1 the axiom schemata $\neg\neg(\alpha \to \beta) \to (\neg\neg\alpha \to \neg\neg\beta)$ and $(\alpha \to \beta) \to \neg\neg(\alpha \to \beta)$, then the resulting logic is not paraconsistent.*

According to the translation in Definition 3.1 the formula $\neg\neg(\alpha \to \beta) \to (\neg\neg\alpha \to \neg\neg\beta)$ becomes the condition that defines normality in modal logic, $\Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta)$ (see [17]), hence the relevance of this formula in the study of paraconsistent logics.

Finally, note that $E1$ is accepted in some paraconsistent logics, such as ciorw, civw, G'_3, CG'_3, *P-FOUR*, $\mathbf{M4}_p$, and $\mathbb{Z}$ (see Table 8, [26], and [29]).

5.2 The logic L_2 Let us define L_2 as the logic resulting when the following axiom schema is added to L_1:

E2: $\neg\neg(\alpha \to \beta) \to (\neg\neg\alpha \to \neg\neg\beta)$.

Lemma 5.7 *If L_2 is extended by adding the $\neg\neg$-necessitation rule (i.e., if $\vdash \alpha$, then $\vdash \neg\neg\alpha$) or by adding the following two properties:*

(1) there exists a formula θ such that $\vdash \theta$ and $\vdash \neg\neg\theta$,
(2) if $\vdash \alpha \leftrightarrow \beta$, then $\vdash \neg\beta \leftrightarrow \neg\alpha$ (weak substitution),

then the resulting logics are the same.

Let us note that $E2$ is a theorem in $\mathbb{Z}$. In order to prove this statement we first prove the following weak version of disjunctive syllogism for logic $\mathbb{Z}$.

Proposition 5.8 *The formulas $((\neg\alpha \vee \beta) \wedge \neg\neg\alpha) \to \beta$ and $\neg(\neg\alpha \vee \neg\neg\beta) \leftrightarrow (\neg\neg\alpha \wedge \neg\beta)$ are theorems in $\mathbb{Z}$.*

As indicated before, the next result is well known in $\mathbb{S}5$; however, we offer a proof in $\mathbb{Z}$ in order to obtain more insight into the axiomatization system defined in [23].

Theorem 5.9 *$E2$ is a theorem in $\mathbb{Z}$.*

Finally, as Table 8 notes, some of the paraconsistent logics in which formula $E2$ as well as formula $E1$ are theorems are G'_3, CG'_3, *P-FOUR*, $\mathbf{M4}_p$, and $\mathbb{Z}$.

5.3 The logic L_3 L_3 extends L_2 by adding the following axiom schema:

$$\neg(\alpha \wedge \neg\alpha) \quad (\textit{principle of noncontradiction}).$$

This formula is one possible way of expressing the *principle of noncontradiction* (see [14]), intuitively read as saying that α and $\neg\alpha$ cannot be true at the same time.

We note that this formula is a theorem in $\mathbb{Z}$ as established in the next proposition.

Proposition 5.10 *The formula* $\neg(\alpha \wedge \neg\alpha)$ *is a theorem in* $\mathbb{Z}$.

At this point the $\neg\neg$-necessitation rule and the substitution properties do not exclude each other, and according to Lemmas 4.8 and 5.7, we have the following easy consequence.

Corollary 5.11 *If* L_3 *is extended by adding the* $\neg\neg$*-necessitation rule (i.e., if* $\vdash \alpha$, *then* $\vdash \neg\neg\alpha$*) or by adding the substitution property, then the resulting logics are the same.*

Eight of the logics presented in Table 8 satisfy the principle of noncontradiction; among them G'_3, CG'_3, *P-FOUR*, $\mathbf{M4}_p$, and $\mathbb{Z}$ also have formulas $E1$ and $E2$ as theorems.

5.4 The logic L_4 L_4 extends L_3 by adding the following axiom schema:

$$\neg\neg(\alpha \wedge \beta) \leftrightarrow (\neg\neg\alpha \wedge \neg\neg\beta) \quad (\textit{distributivity of double negation over } \wedge).$$

This formula is a well-known property in any normal modal logic when the double negation is interpreted as the operator $\Box$ (see [17]).

Next we consider a couple of formulas that are theorems in L_4 and will be used to show that the next logic we define satisfies the inference rule RZ of $\mathbb{Z}$.

Proposition 5.12 *In* L_4 *the formulas* $\neg\neg(\alpha \to \beta) \to \neg(\alpha \wedge \neg\beta)$ *and* $\neg(\alpha \wedge \neg\beta) \to (\neg\beta \to \neg\alpha)$ *are theorems.*

Theorem 5.13 *The formula* $\neg\neg(\alpha \wedge \beta) \leftrightarrow (\neg\neg\alpha \wedge \neg\neg\beta)$ *is a theorem in* $\mathbb{Z}$.

G'_3, CG'_3, *P-FOUR*, $\mathbf{M4}_p$, and $\mathbb{Z}$ satisfy the *distributivity of double negation* axiom schema as well as the formula that expresses the *principle of noncontradiction*; besides, all of them have $E1$, $E2$ as theorems (see Table 8).

5.5 The logic L_5 We define L_5 as logic L_4 together with the following inference rule:

$$\vdash \alpha \text{ then } \vdash \neg\neg\alpha \quad (\neg\neg\text{- } \textit{necessitation} \text{ rule}).$$

As an immediate consequence of the definition of L_5 we have the following result.

Theorem 5.14 *In* L_5 *the inference rule* RZ *of* $\mathbb{Z}$ *is valid; namely, from* $\alpha \to \beta$ *one can derive* $\neg(\alpha \wedge \neg\beta)$.

Corollary 5.15 *The substitution property is valid in* L_5. *Let* α, β, *and* ψ *be formulas, and let* p *be an atom; if* $\vdash_{L_5} \alpha \leftrightarrow \beta$, *then* $\vdash_{L_5} \psi[\alpha/p] \leftrightarrow \psi[\beta/p]$.

L_5 is sound with respect to G'_3; hence as Table 8 notes, Pierce's axiom schema is not valid in L_5.

Logics G'_3, *P-FOUR*, and $\mathbb{Z}$ satisfy the $\neg\neg$-*necessitation* inference rule and also have the formulas we have used to define $L1$, $L2$, $L3$, and $L4$ as theorems (again, see Table 8).

5.6 The logic L_6 Going one step further, we define L_6 by adding to L_5 Pierce's axiom schema:

$$\big((\alpha \to \beta) \to \alpha\big) \to \alpha \quad (\textit{Pierce axiom}).$$

The following theorem shows that we have reached $\mathbb{Z}$.

Theorem 5.16 L_6 *and* $\mathbb{Z}$ *are equivalent; that is, they have the same theorems.*

5.7 $\mathbb{Z}$ is not a maximal paraconsistent logic $\mathbb{Z}$ is not maximal paraconsistent. The logic (denoted $\mathbb{Z} + \Phi$) that results when adding Φ (formula (1) in Section 3.3) to $\mathbb{Z}$ is paraconsistent according to the following lemma whose proof is straightforward by using the truth tables of *P-FOUR*.

Lemma 5.17 *Every theorem of $\mathbb{Z} + \Phi$ is a tautology in P-FOUR, whereas $\alpha \wedge \neg\alpha \to \beta$ is not a tautology in P-FOUR. Therefore, $\mathbb{Z} + \Phi$ is a paraconsistent logic that extends $\mathbb{Z}$.*

5.8 One further axiomatization for $\mathbb{Z}$ With the results we have reached so far, we can propose a new axiomatic system for $\mathbb{Z}$ in which modus ponens is the only inference rule. Basically, we apply Theorem 4.4 to obtain the following proposition that gives an axiomatization of $\mathbb{Z}$ with modus ponens as its unique inference rule.

Theorem 5.18 *Let us denote by NL_6 the logic defined by the set of axiom schemata consisting of $\neg\neg\alpha \to \alpha$ and all the formulas of the form $\neg\neg A$, where A is an axiom of L_6. NL_6 has only one inference rule, which is modus ponens. Then NL_6 is equivalent to logic $\mathbb{Z}$; that is, they have the same theorems.*

6 Discussion: $\mathbb{Z}$ Compared to Some Multivalued Logics

At this point it is convenient to take a look at some of the differences and analogies between $\mathbb{Z}$ and some of the multivalued logics presented here. One of the main results presented in this work is the fact that $\mathbb{Z}$ is comparable with *P-FOUR*, a maximal paraconsistent logic whose set of tautologies contains properly the set of valid formulas of $\mathbb{Z}$. Logics G'_3, CG'_3, and $\mathbf{M4}_p$ are close to $\mathbb{Z}$, but in each of these logics all of the De Morgan laws are valid, whereas in $\mathbb{Z}$ one of them is not valid. The $\neg\neg$-necessitation rule, which under certain conditions (see Lemma 5.7) guaranties the substitution property, is not valid in either CG'_3 or $\mathbf{M4}_p$.

We observe also that from L_5 there are two ways to go in terms of extending L_5 to bigger paraconsistent logics. One is by using Pierce's formula as an axiom schema to obtain $\mathbb{Z}$, and the other one is by extending L_5 so as to guarantee the validity of all of the De Morgan laws, as it is done in the axiomatic version of G'_3. In this version of G'_3, the differences with $\mathbb{Z}$ are clear. Whereas Pierce's axiom schema appears in the definition of $\mathbb{Z}$, it is not a theorem in G'_3 (i.e., it is not a tautology in its 3-valued version; see Section 3). On the other hand, the De Morgan law $(\neg\alpha \wedge \neg\beta) \to \neg(\alpha \vee \beta)$ which is not valid in $\mathbb{Z}$ is one of the axiom schemata of G'_3; hence $\mathbb{Z}$ and G'_3 are not comparable. *P-FOUR* and CG'_3 are the result of going one step further in extending paraconsistent logics in these two directions: *P-FOUR* satisfies the substitution property, but CG'_3 does not, according to Lemmas 5.7 and 4.8 and Table 8. Also, *P-FOUR* satisfies the double negation of Pierce's axiom schema, while CG'_3 does not. Figure 1 illustrates the relations between some of these logics. The interested reader is invited to examine Figure 1, in which any two logics above L_5 that are not connected in the diagram are not comparable (see Table 8).

7 Conclusions and Future Work

In the present work we have presented a family of logics, interesting in their own right, in the process of reconstructing $\mathbb{Z}$ as an axiomatic system which requires only modus ponens as its inference rule. Some of these logics are presented in such a way that each is obtained from the previous one by adding a new axiom schema,

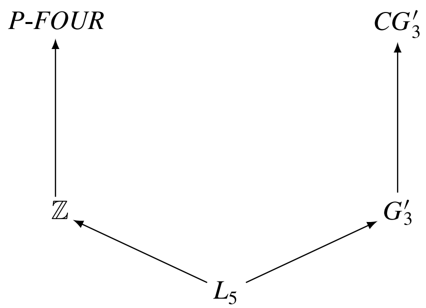

Figure 1 Relations between some of the logics considered.

which usually represents a desirable property in a paraconsistent logic. Along the way we noted that some pairs of desirable properties are not jointly compatible with the paraconsistency of some logics. We also noted some other properties that, under certain conditions, are equivalent in some paraconsistent logics. One of the most relevant properties one would expect a logic to have is the substitution property. Some of the logics presented in this work have this property, for example, the three-valued logics G'_3 and $\mathbb{Z}$, both of them stronger than C_ω. These two logics lack another desirable property, that of being maximal; that is, each of them can be extended to another paraconsistent logic. As Table 8 (see Section 3.4) shows, the logics G'_3, $\mathbb{Z}$, and $M4_p$ share many properties. Exploring in detail the analogies and differences existing among them, as well as exploring further properties of the logics defined in this work, is the object of future research. In particular, we would like to know which constructive logic corresponds to L_1 based on $\mathcal{IL}$.

Appendix

Proof of Lemma 3.3 The proof is straightforward, and it is done by induction on the length of β.

For an atomic formula, the proposition is immediate: $(\bot^\star)^* = {\sim}\Box p \wedge {\sim}\Box{\sim}\Box p$, which is equivalent to $\bot$ in $\mathbb{S}5$.

For the other cases the result follows by induction and using the relation $((\Box\beta)^\star)^* = {\sim}\Box{\sim}\Box(\beta^\star)^*$, which is equivalent to $\Box(\beta^\star)^*$. □

Proof of Theorem 3.4 Given $\alpha \in For_{\mathbb{S}5}$, if $\nvdash_{\mathbb{Z}} \alpha^\star$, then according to Theorem 3.2, $\nvdash_{\mathbb{S}5} (\alpha^\star)^*$. From $\vdash_{\mathbb{S}5} (\alpha^\star)^* \leftrightarrow \alpha$ (see Lemma 3.3), we conclude that $\nvdash_{\mathbb{S}5} \alpha$. □

Proof of Theorem 4.4 It is clear that the theorems in NŁ are theorems in Ł since the $\neg\neg$-necessitation rule guarantees that all axioms in NŁ are theorems in Ł. For the converse it is enough to prove that if A is a theorem in NŁ, then $\neg\neg A$ is also a theorem in NŁ.

We note that $\neg\neg\alpha \to \alpha$ and $\neg\neg(\neg\neg\alpha \to \alpha)$ are axiom schemata in NŁ.

Let us assume that A is a theorem in NŁ and prove by induction on the size of the proof of A that $\neg\neg A$ is also a theorem in NŁ.

First, we prove that $\neg\neg A$ is a theorem whenever A is an axiom of NŁ. From the fact that $C_\omega 2 \in \Delta$, the formula $\neg\neg\neg\alpha \to \neg\alpha$ is a theorem; hence if we apply $E1$ (which is also in Δ), we get that $\neg\neg\alpha \to \neg\neg\neg\neg\alpha$ is a theorem. If A is an axiom in NŁ of the form $\neg\neg\alpha$, then it follows that $\neg\neg A$ is a theorem by applying modus ponens to the last formula.

If A is the formula $\neg\neg\alpha \to \alpha$, then its double negation is also a theorem since we have included $\neg\neg C_\omega 2$ as an axiom schema.

Next, let us assume that the result is true for any theorem whose proof in NŁ consists of less than n steps, and let A be a theorem whose proof consists of n steps, where the last step is an application of modus ponens to two previous steps, say, β and $\beta \to A$. Then by hypothesis $\neg\neg\beta$ and $\neg\neg(\beta \to A)$ are theorems. By using axiom schema $E2$ (which is also in Δ), we conclude that $\neg\neg\beta \to \neg\neg A$ is a theorem. We reach the conclusion by applying modus ponens to this formula. □

Proof of Lemma 4.5 In what follows, by basic reasoning in X we will mean the application of simple results of C_ω, since we know that all theorems in C_ω are valid in X.

We know that $\vdash_{C_\omega} (\alpha \vee \neg\alpha) \leftrightarrow (\beta \vee \neg\beta)$. Thus, $\vdash_X (\alpha \vee \neg\alpha) \leftrightarrow (\beta \vee \neg\beta)$. By *substitution*, we can show that $\vdash_X \neg(\alpha \vee \neg\alpha) \leftrightarrow \neg(\beta \vee \neg\beta)$. By *standard De Morgan* and basic reasoning in X we obtain $\vdash_X (\neg\alpha \wedge \neg\neg\alpha) \leftrightarrow (\neg\beta \wedge \neg\neg\beta)$. By *substitution* and *double negation* we get $\vdash_X (\neg\alpha \wedge \alpha) \leftrightarrow (\neg\beta \wedge \beta)$. By basic reasoning in X we obtain $\vdash_X (\neg\alpha \wedge \alpha) \to \beta$. □

Proof of Lemma 4.6 Let us assume $\beta, \neg\beta$ as premises. We apply modus ponens to Pos1, $\neg\beta \to (\neg\alpha \to \neg\beta)$ to conclude that $\neg\alpha \to \neg\beta$. Now we apply modus ponens to the second axiom schema we are adding to C_ω to obtain $\neg\neg\beta \to \neg\neg\alpha$. By double negation we obtain from the premises $\neg\neg\beta$, and applying modus ponens to last formula we obtain $\neg\neg\alpha$. Now we apply double negation and modus ponens to obtain α. □

Proof of Lemma 4.8 One of the implications of this statement is straightforward; therefore, we prove that if X satisfies the weak substitution property, then it satisfies the substitution theorem. First, we observe that we will use some basic consequences from the axioms of positive logic; specifically, we have the following.

Remark A.1 We have $A \leftrightarrow B, C \leftrightarrow D \vdash (A \to C) \leftrightarrow (B \to D)$, also $A \leftrightarrow B, C \leftrightarrow D \vdash (A \wedge C) \leftrightarrow (B \wedge D)$ and $A \leftrightarrow B, C \leftrightarrow D \vdash (A \vee C) \leftrightarrow (B \vee D)$.

The proof is done by induction on the size of ψ.

Base case:

If $\psi = q$, q an atom, we have

$$\psi[\alpha/p] \Leftrightarrow \psi[\beta/p] = \begin{cases} q \leftrightarrow q & \text{if } p \neq q, \\ \alpha \leftrightarrow \beta & \text{if } p = q, \end{cases}$$

where $q \leftrightarrow q$ is a theorem since $\eta \to \eta$ is a theorem for any formula η and $\alpha \leftrightarrow \beta$ is a theorem by hypothesis.

Now we assume that the statement is true for the formulas $\varphi, \varphi_1, \varphi_2$.

- Case 1. $\psi = \neg\varphi$.

  We want to prove that $\neg\varphi[\alpha/p] \leftrightarrow \neg\varphi[\beta/p]$ is a theorem.

  Since by hypothesis $\varphi[\alpha/p] \leftrightarrow \varphi[\beta/p]$ is a theorem, we only need to apply the hypothesis to conclude that $\neg\varphi[\alpha/p] \leftrightarrow \neg\varphi[\beta/p]$ is a theorem.

- Case 2. $\psi = \varphi_1 \to \varphi_2$.
  We assume that $(\varphi_i[\alpha/p] \leftrightarrow \varphi_i[\beta/p])$ is a theorem for $i \in 1, 2$. We need to prove that the following formula is a theorem: $(\varphi_1[\alpha/p] \to \varphi_2[\alpha/p]) \leftrightarrow (\varphi_1[\beta/p] \to \varphi_2[\beta/p])$. This follows from the remark above.
- Case 3. $\psi = \varphi_1 \wedge \varphi_2$.
  Assume that $(\varphi_i[\alpha/p] \Leftrightarrow \varphi_i[\beta/p])$ is a theorem for $i \in 1, 2$. From the remark, it follows that $(\varphi_1[\alpha/p] \wedge \varphi_2[\alpha/p]) \leftrightarrow (\varphi_1[\beta/p] \wedge \varphi_2[\beta/p])$ is a theorem.
- Case 4. $\psi = \varphi_1 \vee \varphi_2$.
  Assume that $(\varphi_i[\alpha/p] \Leftrightarrow \varphi_i[\beta/p])$ is a theorem for $i \in 1, 2$. From the remark it follows that $(\varphi_1[\alpha/p] \vee \varphi_2[\alpha/p]) \leftrightarrow (\varphi_1[\beta/p] \vee \varphi_2[\beta/p])$ is a theorem. □

Proof of Theorem 4.10 The result is an immediate consequence of Lemma 4.8 and Theorem 4.9. □

Proof of Lemma 4.11 Assuming that we have the axioms of C_ω plus $\neg\alpha \to (\neg\neg\alpha \to \beta)$, we will prove $(\neg\neg\beta \to \neg\neg\alpha)$ from $(\neg\alpha \to \neg\beta)$.

From the hypothesis, $\neg\beta \to (\neg\neg\beta \to \neg\neg\alpha)$, and from $\neg\alpha \to \neg\beta$ we obtain $\neg\alpha \to (\neg\neg\beta \to \neg\neg\alpha)$. Also from Pos1, we have $\neg\neg\alpha \to (\neg\neg\beta \to \neg\neg\alpha)$.

Next we apply Pos8:

$$\big(\neg\alpha \to (\neg\neg\beta \to \neg\neg\alpha)\big) \\ \to \Big(\big(\neg\neg\alpha \to (\neg\neg\beta \to \neg\neg\alpha)\big) \to (\neg\alpha \vee \neg\neg\alpha) \to (\neg\neg\beta \to \neg\neg\alpha)\Big).$$

We apply two times modus ponens to this last formula to obtain $(\neg\alpha \vee \neg\neg\alpha) \to (\neg\neg\beta \to \neg\neg\alpha)$. Then we use $C_\omega 1$ and modus ponens again to get $\neg\neg\beta \to \neg\neg\alpha$.

Now, again from the same set of axioms, we will prove $(\neg\alpha \to \neg\beta)$ from $(\neg\neg\beta \to \neg\neg\alpha)$.

Pos1 gives us $\neg\beta \to (\neg\alpha \to \neg\beta)$, and from the axiom we are adding to C_ω, $\neg\neg\alpha \to (\neg\alpha \to \neg\beta)$ together with $\neg\neg\beta \to \neg\neg\alpha$ we obtain $\neg\neg\beta \to (\neg\alpha \to \neg\beta)$. As in the previous part, we apply Pos8 to the last two formulas to get $(\neg\beta \vee \neg\neg\beta) \to (\neg\alpha \to \neg\beta)$, and by using $C_\omega 1$ and modus ponens the result follows.

Conversely, we now assume the axioms of C_ω plus axiom schema $(\neg\alpha \to \neg\beta) \leftrightarrow (\neg\neg\beta \to \neg\neg\alpha)$. From $\neg\alpha$ we prove $\neg\neg\alpha \to \beta$.

By Pos1 we have $\neg\alpha \to (\neg\beta \to \neg\alpha)$. From $\neg\alpha$ and modus ponens we obtain $\neg\beta \to \neg\alpha$. From here, the use of $E1$, and modus ponens we obtain $\neg\neg\alpha \to \neg\neg\beta$, and by $C_\omega 2$ the result follows. □

Proof of Lemma 5.1 From $(\neg\neg\neg\neg\beta \to \neg\neg\beta) \leftrightarrow (\neg\beta \to \neg\neg\neg\beta)$ which is $E1$, we use $C_\omega 2$ and modus ponens to obtain that $(\neg\beta \to \neg\neg\neg\beta)$ is a theorem. □

Proof of Theorem 5.2 That a theorem in intuitionistic logic is a theorem in L_1 is direct. To see that $(\alpha \to \beta) \vee (\beta \to \alpha)$ is not a theorem in L_1 we proceed as follows. Construct the following lattice of 5 elements: $0, a, b, c, 1$, where $0 < a, 0 < b, a < c, b < c, c < 1$. The connectives $\vee$, $\wedge$, and $\to$ are interpreted as usual in the semantics of intuitionistic logic based on Heyting algebras (see Rutherford [32]). This means that $\vee$ corresponds to the least upper bound of their arguments and $\wedge$ corresponds to the greatest lower bound of their arguments; $y \to z$ corresponds to the greatest element x such that $glb(x, y) \leq z$. Now, define $\neg x$ as 1 if x is different from 1 and as 0 otherwise (see Table 9). Then one can easily show

Table 9 Table for $\neg$ and $\rightarrow$.

x	$\neg x$
0	1
a	1
b	1
c	1
1	0

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

Table 10 Matrices for the operators.

A	$\neg A$
0	1
1	0
2	1

$\rightarrow$	0	1	2
0	2	2	2
1	0	2	2
2	0	2	2

$\wedge$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	2

$\vee$	0	1	2
0	0	1	2
1	1	1	1
2	2	1	2

that L_1 is sound with respect to the lattice, namely, that every axiom evaluates to 1 and also that if $\alpha \rightarrow \beta$ and α evaluate both to 1, then β evaluates to 1 too. This way, every theorem in L_1 evaluates to 1. However, $(a \rightarrow b) \vee (b \rightarrow a)$ evaluates to c (not to 1). In the same way, it is possible to check that the other three formulas do not always evaluate to the designated value 1. □

Proof of Lemma 5.3 Note that $\neg\alpha, \alpha \rightarrow \bot \vdash \neg\alpha$ and $\alpha, \alpha \rightarrow \bot \vdash \neg\alpha$. Hence $\alpha \rightarrow \bot \vdash \neg\alpha$. Thus, $\vdash {\sim}\alpha \rightarrow \neg\alpha$. □

Proof of Lemma 5.4 This follows directly from the following two basic facts and Lemma 4.11:

$$\neg\alpha, \neg\neg\alpha, \neg\alpha \vee \beta \vdash \beta, \qquad \beta, \neg\neg\alpha, \neg\alpha \vee \beta \vdash \beta.$$ □

Proof of Lemma 5.5 For the first case, one can show that L_1 is sound with respect to G_3' but *double negation* does not hold in G_3'. For the second case, one can show that L_1 is sound with respect to *P-FOUR*, but one of the *standard De Morgan* properties does not hold in *P-FOUR*. For the last case, observe that $\vdash_{L_1} \alpha \leftrightarrow (\alpha \vee \alpha)$, but it is false that $\vdash_{L_1} (\neg\alpha) \leftrightarrow \neg(\alpha \vee \alpha)$. Table 10 provides the truth tables of a multivalued logic for which all axioms of L_1 are tautologies if 1 and 2 are taken as designated values, the implication connective preserves tautologies, but the formula $(\neg\alpha) \leftrightarrow \neg(\alpha \vee \alpha)$ does not always evaluate to 1 or 2. □

Proof of Lemma 5.6 Let us prove that in such a logic the formula $\beta \wedge \neg\beta \rightarrow \alpha$ is a theorem. Assuming the formula β and axiom schema Pos1, $\beta \rightarrow (\neg\alpha \rightarrow \beta)$, by modus ponens we arrive at $\neg\alpha \rightarrow \beta$. By one of the axioms in the hypothesis, $(\neg\alpha \rightarrow \beta) \rightarrow \neg\neg(\neg\alpha \rightarrow \beta)$, and by applying the other axiom in the hypothesis, we arrive at $(\neg\alpha \rightarrow \beta) \rightarrow (\neg\neg\neg\alpha \rightarrow \neg\neg\beta)$. Now by using axiom schema $E1$, we conclude that $(\neg\alpha \rightarrow \beta) \rightarrow (\neg\beta \rightarrow \neg\neg\alpha)$. Using modus ponens on this last expression and one of the formulas just obtained, we get $\neg\beta \rightarrow (\neg\neg\alpha)$. We conclude from all this that $(\beta \wedge \neg\beta) \rightarrow \neg\neg\alpha$. Now we just use axiom schema $C_\omega 2$ to conclude that the formula $(\beta \wedge \neg\beta) \rightarrow \alpha$ is a theorem. □

Proof of Lemma 5.7 Let us assume that we have an extension of L_2 which satisfies the following property: if $\vdash \alpha$, then $\vdash \neg\neg\alpha$. Then we have to prove two properties.

The first one is immediate from the hypothesis, since $\alpha \to \alpha$ is a theorem in positive logic.

As for the second property, we proceed as follows. Let us assume that $\vdash \alpha \to \beta$; then by hypothesis $\vdash \neg\neg(\alpha \to \beta)$, and by using axiom schema $E2$, $\vdash \neg\neg\alpha \to \neg\neg\beta$. Then an application of axiom schema $E1$ gives $\vdash \neg\beta \to \neg\alpha$. By symmetry, we obtain the desired result.

For the converse, let α be a theorem. Then from $\vdash \alpha$ and $\vdash \theta$ we conclude that $\vdash \alpha \leftrightarrow \theta$, and by hypothesis it follows that $\vdash \neg\alpha \leftrightarrow \neg\theta$. Next we apply axiom schema $E1$ to obtain $\vdash \neg\neg\theta \leftrightarrow \neg\neg\alpha$. And since $\neg\neg\theta$ is a theorem by hypothesis, we reach the conclusion $\vdash \neg\neg\alpha$. □

Proof of Proposition 5.8 We start with the next two theorems of $\mathbb{Z}$, $\neg\alpha \to (\neg\neg\alpha \to \beta)$ (weak explosion) and $\beta \to (\neg\neg\alpha \to \beta)$ (Pos1). Applying axiom schema Pos8 to these two formulas we get $\neg\alpha \vee \beta \to (\neg\neg\alpha \to \beta)$, from which the first formula follows.

As for the second formula, from Pos6, $\neg\alpha \to (\neg\alpha \vee \neg\neg\beta)$, we apply rule R3 to obtain $\neg(\neg\alpha \vee \neg\neg\beta) \to \neg\neg\alpha$. In a similar way, from $\neg\neg\beta \to (\neg\alpha \vee \neg\neg\beta)$, we obtain $\neg(\neg\alpha \vee \neg\neg\beta) \to \neg\neg\neg\beta$, and by using the formula $\neg\neg\neg\beta \to \neg\beta$, we get $\neg(\neg\alpha \vee \neg\neg\beta) \to \neg\beta$. So we obtain $\neg(\neg\alpha \vee \neg\neg\beta) \to (\neg\neg\alpha \wedge \neg\beta)$.

For the other implication observe that, from $(\neg\alpha \vee \neg\neg\beta) \wedge (\neg\neg\alpha \wedge \neg\beta)$ anything follows due to the weak explosion principle, so that we have $(\neg\alpha \vee \neg\neg\beta) \to ((\neg\neg\alpha \wedge \neg\beta) \to \neg(\neg\alpha \vee \neg\neg\beta))$ and by Pos1, $\neg(\neg\alpha \vee \neg\neg\beta) \to ((\neg\neg\alpha \wedge \neg\beta) \to \neg(\neg\alpha \vee \neg\neg\beta))$. Then by applying Pos8 to these two formulas and then using the formula $(\neg\alpha \vee \neg\neg\beta) \vee \neg(\neg\alpha \vee \neg\neg\beta)$ (axiom schema $C_\omega 1$) and modus ponens, we obtain $(\neg\neg\alpha \wedge \neg\beta) \to \neg(\neg\alpha \vee \neg\neg\beta)$. □

Proof of Theorem 5.9 According to Proposition 5.8,

$$\vdash \big((\neg\alpha \vee \beta) \wedge \neg\neg\alpha\big) \to \beta.$$

Then from $\vdash ((\neg\alpha \vee \beta) \wedge (\neg\neg\alpha \wedge \neg\beta)) \to ((\neg\alpha \vee \beta) \wedge \neg\neg\alpha)$ and transitivity,

$$\vdash \big((\neg\alpha \vee \beta) \wedge (\neg\neg\alpha \wedge \neg\beta)\big) \to \beta,$$

and using the following theorem,

$$\vdash \big((\neg\alpha \vee \beta) \wedge (\neg\neg\alpha \wedge \neg\beta)\big) \to \neg\beta,$$

we obtain

$$\vdash \big((\neg\alpha \vee \beta) \wedge (\neg\neg\alpha \wedge \neg\beta)\big) \to (\beta \wedge \neg\beta).$$

Now using Proposition 5.8,

$$\vdash \neg(\neg\alpha \vee \neg\neg\beta) \leftrightarrow (\neg\neg\alpha \wedge \neg\beta).$$

From the last two steps we have the following implication:

$$\vdash \big((\neg\alpha \vee \beta) \wedge \neg(\neg\alpha \vee \neg\neg\beta)\big) \to (\beta \wedge \neg\beta).$$

Now, according to rule $R3$,

$$\vdash \neg(\beta \wedge \neg\beta) \to \neg\big((\neg\alpha \vee \beta) \wedge \neg(\neg\alpha \vee \neg\neg\beta)\big).$$

Now by applying the rule RZ to the theorem $\beta \to \beta$, we get

$$\vdash \neg(\beta \wedge \neg\beta),$$

and by applying modus ponens to the last two steps, we get

$$\vdash \neg\big((\neg\alpha \vee \beta) \wedge \neg(\neg\alpha \vee \neg\neg\beta)\big).$$

Now by using rule $AZ2'$ and modus ponens, we get

$$\vdash \neg(\neg\alpha \vee \neg\neg\beta) \to \neg(\neg\alpha \vee \beta).$$

Again by Proposition 5.8,

$$\vdash (\neg\neg\alpha \wedge \neg\beta) \to \neg(\neg\alpha \vee \neg\neg\beta).$$

Now we apply transitivity to the last two steps and obtain

$$\vdash (\neg\neg\alpha \wedge \neg\beta) \to \neg(\neg\alpha \vee \beta).$$

From $\vdash \alpha \wedge (\alpha \to \beta) \to \neg\alpha \vee \beta$ and $\vdash \neg\alpha \wedge (\alpha \to \beta) \to \neg\alpha \vee \beta$, we get

$$\vdash (\alpha \to \beta) \to (\neg\alpha \vee \beta).$$

Now we apply rule $R3$:

$$\vdash \neg(\neg\alpha \vee \beta) \to \neg(\alpha \to \beta).$$

By transitivity applied to two of the last three steps, we get

$$\vdash (\neg\neg\alpha \wedge \neg\beta) \to \neg(\alpha \to \beta).$$

Equivalently

$$\vdash \neg\neg\alpha \to \big(\neg\beta \to \neg(\alpha \to \beta)\big),$$

and by using rule $R3$, we get

$$\vdash \neg\neg\alpha \to \big(\neg\neg(\alpha \to \beta) \to \neg\neg\beta\big).$$

Then we reach the conclusion,

$$\vdash \neg\neg(\alpha \to \beta) \to (\neg\neg\alpha \to \neg\neg\beta). \qquad \square$$

Proof of Proposition 5.10 It is an easy consequence of the fact that $\alpha \to \alpha$ is a theorem in $\mathbb{Z}$ and from rule RZ. □

Proof of Corollary 5.11 It follows immediately by putting $\theta = \neg(\alpha \wedge \neg\alpha)$ in Lemma 5.7, the fact that $\neg\alpha \leftrightarrow \neg\neg\neg\alpha$ is a theorem in L_3, and Lemma 4.8. □

Proof of Theorem 5.13 From $\vdash \alpha \wedge \beta \to \alpha$, it follows that $\vdash \neg\alpha \to \neg(\alpha \wedge \beta)$ by rule $R3$, and by applying $E1$ we get $\vdash \neg\neg(\alpha \wedge \beta) \to \neg\neg\alpha$, from which one of the implications follows by repeating the same argument.

For the other implication we use Pos8, $\neg\alpha \to (\neg\neg\alpha \to \neg\beta) \to (\neg\beta \to (\neg\neg\alpha \to \neg\beta) \to (\neg\alpha \vee \neg\beta) \to (\neg\neg\alpha \to \neg\beta))$, and using the fact that the formulas $\neg\alpha \to (\neg\neg\alpha \to \neg\beta)$ and $\neg\beta \to (\neg\neg\alpha \to \neg\beta)$ are theorems in $\mathbb{Z}$ together with modus ponens twice, we obtain $(\neg\alpha \vee \neg\beta) \to (\neg\neg\alpha \to \neg\beta)$. From here we get $\neg\neg\alpha \to ((\neg\alpha \vee \neg\beta) \to \neg\beta)$. Next we use the formula $\neg(\alpha \wedge \beta) \to (\neg\alpha \vee \neg\beta)$ which is an axiom schema in $\mathbb{Z}$ to get $\neg\neg\alpha \to (\neg(\alpha \wedge \beta) \to \neg\beta)$ by using basic reasoning. Now by using axiom schema $E1$, $\neg\neg\alpha \to (\neg\neg\beta \to \neg\neg(\alpha \wedge \beta))$, and this is equivalent to $(\neg\neg\alpha \wedge \neg\neg\beta) \to \neg\neg(\alpha \wedge \beta)$. □

Proof of Proposition 5.12 Let us prove the first formula.

Starting from $E2$,

$$\neg\neg(\alpha \to \beta) \to (\neg\neg\alpha \to \neg\neg\beta);$$

equivalently,

$$\neg\neg\alpha \to \big(\neg\neg(\alpha \to \beta) \to \neg\neg\beta\big).$$

Now we use axiom schema $E1$ to obtain

$$\neg\neg\alpha \to \big(\neg\beta \to \neg(\alpha \to \beta)\big),$$
$$\neg\neg\alpha \wedge \neg\beta \to \neg(\alpha \to \beta),$$
$$\neg\neg\alpha \wedge \neg\beta \to \neg\neg\neg(\alpha \to \beta).$$

Now we use the distributivity of the double negation in the conjunction and the fact that $\neg\neg\neg\beta \leftrightarrow \neg\beta$ to establish the next relation,

$$\neg\neg(\alpha \wedge \neg\beta) \leftrightarrow (\neg\neg\alpha \wedge \neg\beta).$$

Now we use transitivity in the last two steps to obtain

$$\neg\neg(\alpha \wedge \neg\beta) \to \neg\neg\neg(\alpha \to \beta).$$

Finally, we apply axiom schema $E1$ to get the desired formula,

$$\neg\neg(\alpha \to \beta) \to \neg(\alpha \wedge \neg\beta).$$

Let us prove the second formula; we start from the next two easy formulas,

$$\neg\beta, \neg\neg\alpha \vdash \neg\neg\alpha \qquad \text{and} \qquad \neg\beta, \neg\neg\alpha \vdash \neg\neg\neg\beta.$$

Hence $\neg\beta, \neg\neg\alpha \vdash \neg\neg\alpha \wedge \neg\neg\neg\beta$, and by using the distributivity of double negation over the conjunction, we get

$$\neg\beta, \neg\neg\alpha \vdash \neg\neg(\alpha \wedge \neg\beta).$$

Equivalently,

$$\neg\beta \vdash \neg\neg\alpha \to \neg\neg(\alpha \wedge \neg\beta),$$

and by using axiom schema $E1$, we obtain

$$\neg\beta \vdash \neg(\alpha \wedge \neg\beta) \to \neg\alpha,$$

from which it follows that

$$\neg\beta, \neg(\alpha \wedge \neg\beta) \vdash \neg\alpha,$$

which immediately gives the desired result. □

Proof of Theorem 5.14 If $\vdash \alpha \to \beta$, then $\vdash \neg\neg(\alpha \to \beta)$ according to the definition of L_5, and then by the previous theorem we conclude that $\vdash \neg(\alpha \wedge \neg\beta)$. □

Proof of Corollary 5.15 From $\vdash \alpha \to \beta$, it follows that $\vdash \neg\neg(\alpha \to \beta)$ by the $\neg\neg$-necessitation rule, and then by using the first part of Proposition 5.12 and modus ponens, we obtain $\vdash \neg(\alpha \wedge \neg\beta)$. Now by the second part of the same theorem and modus ponens, we obtain $\vdash (\neg\beta \to \neg\alpha)$. A symmetric argument shows that from $\vdash \beta \to \alpha$, one can derive $\vdash (\neg\alpha \to \neg\beta)$. Now we only have to appeal to Lemma 4.8 to finish the proof. □

Proof of Theorem 5.16 We prove first that all axioms of $\mathbb{Z}$ are theorems in L_6 and that rule RZ is valid in L_6. According to the axiomatic system given in [23], the only axioms of $\mathbb{Z}$ that are not included as axioms of C_ω are those represented by Pierce's axiom schema $(((\alpha \to \beta) \to \alpha) \to \alpha)$, which is an axiom schema of L_6 defined above, and axiom schema $AZ2'$, which is a theorem in L_6 as proved in the second part of Proposition 5.12. The inference rule RZ is valid in L_6 as shown by the previous theorem.

Conversely, every theorem in L_6 is a theorem in $\mathbb{Z}$, for we know that all axioms of C_ω are theorems in $\mathbb{Z}$ according to [23]. In [23] it is also proven that the principle of weak explosion is valid in $\mathbb{Z}$; therefore, according to Lemma 4.11 axiom schema $E1$ is valid in $\mathbb{Z}$. The $\neg\neg$-necessitation rule used to define L_5 is also valid in $\mathbb{Z}$ (see [23]). Finally, all the axioms that were added to L_1, L_2, L_3, and L_5 to define the next logic in the family are theorems in $\mathbb{Z}$ as we already verified after defining each of those logics. □

Proof of Theorem 5.18 It is a consequence of Theorems 5.16 and 4.4. □

Notes

1. "Minsky's frame paper" (see [22]) in its original form had an appendix titled "Criticism of the logistic approach."

2. We drop the subscript X in $\vdash_X$ when the given logic is understood from the context.

3. Note that this definition is not based on the terms "weaker" and "stronger" used to compare two logics as defined in Section 2.2.2.

4. This can be confirmed by using the Logics Workbench (LWB) at http://www.lwb.unibe.ch/

5. Here we use the notation $\vdash_X$ to indicate that the formula that follows it is a theorem or a tautology depending on how the logic is defined.

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