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More on d-Logics of Subspaces of the Rational Numbers

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Abstract We prove that each countable rooted **K4**-frame is a d-morphic image of a subspace of the space $\mathbb{Q}$ of rational numbers. From this we derive that each modal logic over **K4** axiomatizable by variable-free formulas is the d-logic of a subspace of $\mathbb{Q}$. It follows that subspaces of $\mathbb{Q}$ give rise to continuum many d-logics over **K4**, continuum many of which are neither finitely axiomatizable nor decidable. In addition, we exhibit several families of modal logics finitely axiomatizable by variable-free formulas over **K4** that d-define interesting classes of topological spaces. Each of these logics has the finite model property and is decidable. Finally, we introduce quasi-scattered and semi-scattered spaces as generalizations of scattered spaces, develop their basic properties, axiomatize their corresponding modal logics, and show that they also arise as the d-logics of some subspaces of $\mathbb{Q}$.

1 Introduction

The topological semantics for modal logic was developed in the pioneering work of McKinsey and Tarski [22], where they suggested two interpretations of modal diamond $\Diamond$, one as topological closure and another as topological derivative. In order to distinguish between these two semantics, we refer to interpreting $\Diamond$ as closure as the *c-semantics* and to interpreting $\Diamond$ as derivative as the *d-semantics*. Consequently, in c-semantics we will talk about c-definability, c-soundness, and c-completeness, while in d-semantics we will talk about d-definability, d-soundness, and d-completeness.

McKinsey and Tarski [22] showed that one of the best-known modal logics **S4** c-defines the class of all topological spaces, and that **S4** is the c-logic of any dense-in-itself separable metrizable space. On the other hand, Esakia [15] showed that

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$\mathbf{wK4} = \mathbf{K} + \Diamond\Diamond p \to (p \vee \Diamond p)$ d-defines the class of all topological spaces and that $\mathbf{K4} = \mathbf{K} + \Diamond\Diamond p \to \Diamond p$ d-defines the class of all T_d-spaces. He also showed that **wK4** is the d-logic of all topological spaces and that **K4** is the d-logic of all T_d-spaces.

Since the closure of a topological space is expressible by means of the derivative, namely, $\overline{A} = A \cup d(A)$, it follows that d-semantics is more expressive than c-semantics. In fact, d-semantics is strictly more expressive than c-semantics. Indeed, it follows from the McKinsey–Tarski theorem that the properties of being dense-in-itself, T_d, and T_0 are not c-definable. On the other hand, dense-in-itself topological spaces are d-defined by $\Diamond\top$ (see [22]), T_d-spaces are d-defined by $\Diamond\Diamond p \to \Diamond p$ (see [15]), and T_0-spaces are d-defined by $p \wedge \Diamond(q \wedge \Diamond p) \to \Diamond p \vee \Diamond(q \wedge \Diamond q)$ (see Bezhanishvili, Esakia, and Gabelaia [6]).

In [23] Shehtman utilized the technique developed by McKinsey and Tarski to show that $\mathbf{K4D} = \mathbf{K4} + \Diamond\top$ is the d-logic of any zero-dimensional dense-in-itself separable metrizable space. It follows that **K4D** is the d-logic of the space $\mathbb{Q}$ of rational numbers. An alternative purely geometric proof of this result can be found in Lucero-Bryan [21], where it is also shown that each of **K4**, $\mathbf{GL} = \mathbf{K} + \Box(\Box p \to p) \to \Box p$, and $\mathbf{GL}_n = \mathbf{GL} + \Box^n\bot$ can be obtained as the d-logic of some subspace of $\mathbb{Q}$. It was asked in [21] what other modal logics can be obtained as the d-logics of subspaces of $\mathbb{Q}$. In this paper we show that modal logics over **K4** axiomatizable by variable-free formulas and their intersections arise as the d-logics of subspaces of $\mathbb{Q}$. It follows that subspaces of $\mathbb{Q}$ give rise to continuum many d-logics over **K4**, continuum many of which are neither finitely axiomatizable nor decidable.

The paper is organized as follows. In Section 2 we recall the basics of Kripke semantics and d-semantics for extensions of **K4**. In Section 3 we generalize the geometric construction of [21] and show that each countable rooted **K4**-frame is a d-morphic image of a subspace of $\mathbb{Q}$. From this we derive that modal logics over **K4** axiomatizable by variable-free formulas and their intersections are the d-logics of subspaces of $\mathbb{Q}$. It follows that there are continuum many d-logics over **K4** arising from subspaces of $\mathbb{Q}$, continuum many of which are neither finitely axiomatizable nor decidable. In Section 4 we exhibit several families of modal logics finitely axiomatizable by variable-free formulas over **K4** that d-define interesting classes of topological spaces. Each of these logics has the finite model property (FMP) and is decidable. In Section 5 we discuss three generalizations of scattered spaces, thus arriving at the concepts of weakly scattered, quasi-scattered, and semi-scattered spaces. This leads to three more extensions of **K4**, each being a sublogic of **GL**. We call the first one weak **GL** and denote it by **wGL**, the second one quasi-**GL** and denote it by **qGL**, and the third one semi-**GL** and denote it by **sGL**. The logic **wGL** first appeared in Esakia [16] (see also Bezhanishvili, Esakia, and Gabelaia [5]) under the name **K4G**. The logics **qGL** and **sGL** appear to be new. We show that each of **wGL**, **qGL**, and **sGL** has the FMP, is decidable, and arises as the d-logic of a subspace of $\mathbb{Q}$.

2 Preliminaries

2.1 Kripke semantics We assume the reader's familiarity with Kripke semantics for modal logic. The proofs of the facts mentioned in this section can be found in Chagrov and Zakharyaschev [12]. We recall that a **K4**-*frame* is a pair $\mathfrak{F} = (W, R)$, where W is a nonempty set and R is a transitive binary relation on W. Let R^+ denote the reflexive closure of R; that is, $R^+ = R \cup \{(w, w) : w \in W\}$. Then R^+ is reflexive and transitive, and so $\mathfrak{F}^+ = (W, R^+)$ is an **S4**-frame.

Let $\mathfrak{F} = (W, R)$ be a **K4**-frame. We recall that $\mathfrak{F}$ is *rooted* if there exists $r \in W$, called a *root* of W, such that rRw for each $w \in W - \{r\}$. Let $w \in W$. Then w is *reflexive* if wRw and w is *irreflexive* otherwise. We let

$$C(w) = \{w\} \cup \{v \in W : wRv \text{ and } vRw\}$$

denote the *cluster generated by* w. We call $C \subseteq W$ a *cluster* if $C = C(w)$ for some $w \in W$. We also call a cluster C *proper* if it consists of more than one point, *simple* if it consists of a single reflexive point, and *degenerate* if it consists of a single irreflexive point. Since R is transitive, each proper cluster consists of reflexive points.

Let $A \subseteq W$. We call $w \in A$ a *maximal* point of A if wRv and $v \in A$ imply vRw. Let $\max(A)$ denote the set of maximal points of A. Let also $\operatorname{rmax}(A)$ denote the set of reflexive maximal points of A, and let $\operatorname{imax}(A) = \max(A) - \operatorname{rmax}(A)$ denote the set of irreflexive maximal points of A.

Definition 2.1 Let $\mathfrak{F} = (W, R)$ be a **K4**-frame. Define R^n recursively as follows:

$$\begin{aligned} wR^0v &\text{ iff } w = v,\\ wR^{n+1}v &\text{ iff } \exists u \in W \text{ with } wR^n u \text{ and } uRv,\\ &\text{ iff } \exists u \in W \text{ with } wRu \text{ and } uR^n v. \end{aligned}$$

For $A \subseteq W$, let $R^{-n}(A) = \{w \in W : \exists v \in A \text{ with } wR^n v\}$.

We recall that a **K4**-frame $\mathfrak{F} = (W, R)$ is *serial* if for each $w \in W$ there exists $v \in W$ with wRv. It is well known that $\mathfrak{F}$ is serial iff $\mathfrak{F} \models \Diamond\top$, and so **K4D** $=$ **K4** $+ \Diamond\top$ is the modal logic of serial **K4**-frames. In fact, **K4D** is the modal logic of finite serial **K4**-frames, and a finite **K4**-frame $\mathfrak{F} = (W, R)$ is serial iff $\max(W) = \operatorname{rmax}(W)$ (equivalently, $\operatorname{imax}(W) = \varnothing$).

We also recall that a **K4**-frame $\mathfrak{F} = (W, R)$ is *dually well founded* if $\operatorname{imax}(A) \neq \varnothing$ for each nonempty subset A of W, and that a **K4**-frame $\mathfrak{F}$ is dually well founded iff $\mathfrak{F} \models \mathbf{gl}$, where

$$\mathbf{gl} = \Box(\Box p \to p) \to \Box p.$$

Clearly a finite **K4**-frame is dually well founded iff it is irreflexive. In fact, the modal logic of all dually well-founded **K4**-frames is the same as the modal logic of all finite irreflexive **K4**-frames, and is the well-known *Gödel–Löb logic*

$$\mathbf{GL} = \mathbf{K} + \mathbf{gl} = \mathbf{K4} + \mathbf{gl}.$$

Let $\mathfrak{F} = (W, R)$ be a **K4**-frame. We call $A \subseteq W$ a *chain* if for each $w, v \in A$ we have wRv or vRw or $w = v$. We call $w \in A$ a *root* of the chain A if wRv for each $v \in A - \{w\}$. For $u, v \in W$, we say that u is *properly below* v and write $u\vec{R}v$ if uRv and $v\not{R}u$. We also say that the chain A is of *length* n if A contains at least one element from each cluster $C(w_i)$ for some $w_1\vec{R}\cdots\vec{R}w_n$ in $\mathfrak{F}$ and $A \subseteq \bigcup_{i=1}^n C(w_i)$. The *depth* of $w \in W$ is n if there is a chain with root w of length n and no other chain with root w is of greater length. Finally, we say that the *depth* of $\mathfrak{F}$ is n if there is $w \in W$ of depth n and for each $v \in W$, the depth of v is at most n.

For a formula φ, let $\Box^+\varphi = \varphi \wedge \Box\varphi$. Then $\Diamond^+\varphi = \varphi \vee \Diamond\varphi$. We also let $\Box^0\varphi = \varphi$ and $\Box^{n+1}\varphi = \Box\Box^n\varphi$. It is well known that $\mathbf{GL}_n = \mathbf{GL} + \Box^n\bot$ is the modal logic of finite irreflexive $\mathbf{K4}$-frames of depth $\leq n$, and that $\mathbf{GL} = \bigcap_\omega \mathbf{GL}_n$.

2.2 Topological semantics We assume the reader's familiarity with the basics of topological spaces. The proofs of the facts mentioned in this section can be found in Engelking [13]. Let X be a topological space. For $A \subseteq X$, let $\overline{A}$ denote the *closure* of A, and let $d(A)$ denote the *derivative* (the set of limit points) of A. We recall that

$$x \in \overline{A} \text{ iff } U \cap A \neq \varnothing \quad \text{for each open neighborhood } U \text{ of } x$$

and that

$$x \in d(A) \text{ iff } U \cap (A - \{x\}) \neq \varnothing \quad \text{for each open neighborhood } U \text{ of } x.$$

Moreover, the *interior* of A is defined as $\operatorname{int}(A) = X - \overline{X - A}$, and the *coderivative* of A is defined as $t(A) = X - d(X - A)$. Then

$$x \in \operatorname{int}(A) \text{ iff } U \subseteq A \quad \text{for some open neighborhood } U \text{ of } x,$$

and

$$x \in t(A) \text{ iff } (U - \{x\}) \subseteq A \quad \text{for some open neighborhood } U \text{ of } x.$$

We recall that $x \in X$ is an *isolated point* of X if $\{x\}$ is an open subset of X. Let $\operatorname{iso}(X)$ denote the set of isolated points of X. Then it is well known that $\operatorname{iso}(X) = X - d(X)$.

We call X *dense-in-itself* if $\operatorname{iso}(X) = \varnothing$, *scattered* if each nonempty subspace of X has an isolated point, and T_d if each point is an intersection of a closed and an open subset of X (i.e., each point of X is locally closed). It is well known that X is dense-in-itself iff $d(X) = X$, that X is scattered iff $d(A) \subseteq d(A - d(A))$ for each $A \subseteq X$, and that X is T_d iff $dd(A) \subseteq d(A)$ for each $A \subseteq X$.

For $A \subseteq X$ and an ordinal α, we define $d^\alpha(A)$ recursively as follows:

(1) $d^0(A) = A$,
(2) $d^{\alpha+1}(A) = d(d^\alpha(A))$,
(3) $d^\alpha(A) = \bigcap_{\beta<\alpha} d^\beta(A)$, for α a limit ordinal.

We call X *n-scattered* if $d^n(X) = \varnothing$.

Interpreting modal diamond as the derivative of a topological space, we obtain a semantics of modal logic, which we call the *d-semantics*. Consequently, we will talk about *d-definability*, *d-soundness*, and *d-completeness*, meaning definability, soundness, and completeness in d-semantics. We will also talk about the *d-logic* of a class K of topological spaces, meaning the set of all formulas valid under d-semantics in each member of K.

It is well known that $\mathbf{K4}$ is the d-logic of T_d-spaces, $\mathbf{K4D}$ is the d-logic of dense-in-itself T_d-spaces, $\mathbf{GL}$ is the d-logic of scattered spaces, and $\mathbf{GL}_n$ is the d-logic of n-scattered spaces. In fact, viewing the rational numbers $\mathbb{Q}$ and each ordinal as a topological space in the interval topology, $\mathbf{K4D}$ is the d-logic of $\mathbb{Q}$, $\mathbf{GL}$ is the d-logic of ω^ω, and $\mathbf{GL}_n$ is the d-logic of ω^n (see, e.g., van Benthem and Bezhanishvili [3, Section 3.1] and Bezhanishvili and Morandi [10, Theorem 3.5]).

One of the key tools in establishing these types of results is the concept of d-morphism. Let X be a topological space. We recall that $A \subseteq X$ is *dense-in-itself* if $A \subseteq d(A)$ and that A is *discrete* if $A \cap d(A) = \varnothing$. Also let $\mathfrak{F} = (W, R)$ be a $\mathbf{K4}$-frame. We recall that $U \subseteq W$ is an *upset* of $\mathfrak{F}$ if $w \in U$ and wRv imply that

$v \in U$. It is well known that the set of upsets of $\mathfrak{F}$ forms a topology τ_R on W, called the *Alexandroff topology*. Now let X be a T_d-space, and let $\mathfrak{F} = (W, R)$ be a **K4**-frame. We recall (see Bezhanishvili, Esakia, and Gabelaia [4, Definition 2.6]) that f is a *d-morphism* if it satisfies the following four conditions:

(1) $f : X \to (W, \tau_R)$ is continuous; that is, $f^{-1}(U)$ is open in X for each $U \in \tau_R$;
(2) $f : X \to (W, \tau_R)$ is open; that is, U open in X implies $f(U) \in \tau_R$;
(3) f is r-dense; that is, $f^{-1}(w)$ is dense-in-itself for each reflexive $w \in W$;
(4) f is i-discrete; that is, $f^{-1}(w)$ is discrete for each irreflexive $w \in W$.

It is well known (see [4, Corollary 2.9]) that if $f : X \to W$ is an onto d-morphism and $\mathfrak{F}$ refutes φ, then so does X. Having this result under our belt, it is easy to establish the d-completeness results mentioned above. For example, to show that **K4D** is the d-logic of $\mathbb{Q}$, observe that $\mathbb{Q}$ is a dense-in-itself T_d-space. Therefore, $\mathbb{Q} \models$ **K4D**. On the other hand, if **K4D** $\not\vdash \varphi$, then it is well known (see, e.g., Gabbay [17, Theorem 7]) that there is a valuation into the infinite binary tree $\mathcal{T}$ refuting φ. By [21, Lemma 3.14], there exists an onto d-morphism $f : \mathbb{Q} \to \mathcal{T}$. Thus, $\mathbb{Q} \not\models \varphi$, and so **K4D** is the d-logic of $\mathbb{Q}$. The other d-completeness results for **K4**, **GL**, and $\mathbf{GL}_n$ can be proved in a similar fashion (see, e.g., [21, Section 4]).

3 Main Results

In this section we generalize the geometric construction of [21] and prove that each countable rooted **K4**-frame is a d-morphic image of a subspace of $\mathbb{Q}$. From this we derive that modal logics over **K4** axiomatizable by variable-free formulas and their intersections are the d-logics of subspaces of $\mathbb{Q}$. It follows that there exist continuum many d-logics arising from subspaces of $\mathbb{Q}$, continuum many of which are neither finitely axiomatizable nor decidable.

We start by recalling the geometric construction of [21, Section 3], which provides a homeomorphic copy of $\mathbb{Q}$ in the lower half-plane. Let $\mathcal{C}$ denote the set of infinite closed bounded intervals of the real numbers $\mathbb{R}$. Let $L = \mathbb{R} \times (-\infty, 0]$ be the closed lower half-plane in $\mathbb{R}^2$, and let $L_0 = \mathbb{R} \times (-\infty, 0)$. Let $i : \mathbb{R} \to L$ be given by $i(x) = (x, 0)$. Clearly i is 1–1 and $L = L_0 \cup i(\mathbb{R})$. Let $\pi : L \to \mathbb{R}$ be the projection $\pi(x, y) = x$. For each $I \in \mathcal{C}$, let $\triangle_I$ be the right isosceles triangle in L whose hypotenuse coincides with $i(I)$ (see Figure 1).

There is a bijective correspondence between $\mathcal{C}$ and L_0 given by associating with I the only vertex of $\triangle_I$ in L_0. More formally, define $\alpha : L_0 \to \mathcal{C}$ and $\beta : \mathcal{C} \to L_0$ by

$$\alpha(x, y) = [x + y, x - y] \qquad \text{and} \qquad \beta([a, b]) = \left(\frac{a+b}{2}, \frac{a-b}{2}\right).$$

Then it is easy to check that α and β are well defined and that $\beta = \alpha^{-1}$.

Let Σ denote the set of finite strings over the nonzero integers. Clearly Σ is countable. We inject Σ into L_0 as follows. Let Λ denote the empty string. Also, for a string $\sigma = z_1 z_2 \cdots z_k$ and $z \in \mathbb{Z} - \{0\}$, let $\sigma.z$ denote the string $z_1 z_2 \cdots z_k z$. Define $h : \Sigma \to L_0$ recursively by setting $h(\Lambda) = (0, -1)$. For the recursive step, suppose that $h(\sigma) = (x, y) = p \in L_0$. Let $\alpha(p) = I$. Then $I = [x + y, x - y]$, and x is the midpoint of I. For $n \in \omega - \{0\}$, let

$$I^p_{-n} = \left[x + \frac{y}{2^{n-1}}, x + \frac{y}{2^n}\right] \qquad \text{and} \qquad I^p_n = \left[x - \frac{y}{2^n}, x - \frac{y}{2^{n-1}}\right].$$

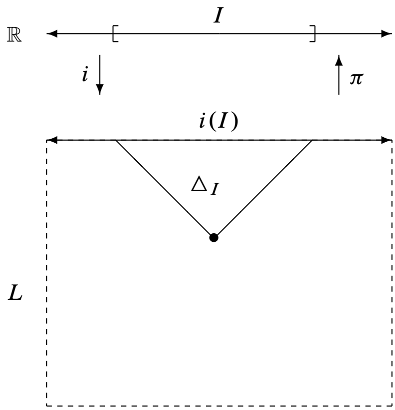

Figure 1

This leads to the following dissection of $I = \alpha(h(\sigma))$:

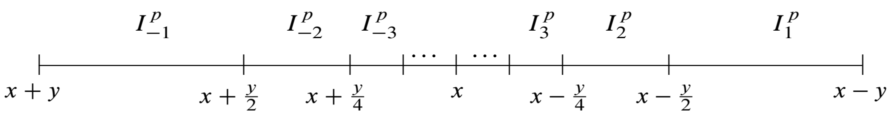

Figure 2

We set $h(\sigma.z) = \beta(I^p_z) = \beta(I^{h(\sigma)}_z)$. Figure 3 captures the recursive step.

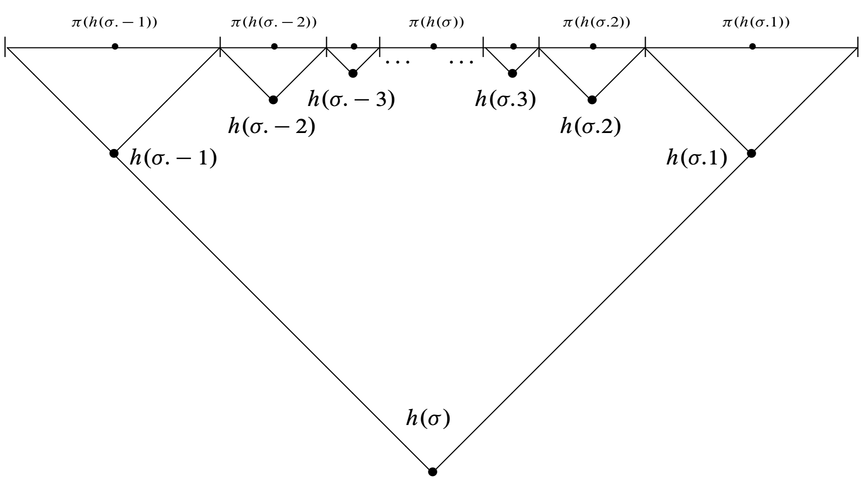

Figure 3

Define $<$ on Σ by

$$\sigma < \lambda \text{ iff } \pi\big(h(\sigma)\big) < \pi\big(h(\lambda)\big) \text{ in } \mathbb{R}.$$

By [21, Lemma 3.5], $<$ is a dense strict linear ordering on Σ which has no endpoints. By Cantor's theorem (see, e.g., Kuratowski and Mostowski [20, p. 217, Theorem 2]), $(\Sigma, <)$ is order-isomorphic to $\mathbb{Q}$ with the usual arithmetic order. Let τ be the interval topology on $(\Sigma, <)$. Then (Σ, τ) is homeomorphic to $\mathbb{Q}$. As a result, we obtain a "nice" copy of $\mathbb{Q}$ inside L_0.

Lemma 3.1 (Main Lemma) *Let $\mathfrak{F} = (W, R)$ be a countable rooted* **K4***-frame. Then $\mathfrak{F}$ is a d-morphic image of a subspace of $\mathbb{Q}$.*

Proof It is sufficient to show that $\mathfrak{F}$ is a d-morphic image of a subspace of Σ. We will define a partial function $f : \Sigma \to W$ so that, upon restricting f to those strings for which f is defined, we obtain an onto d-morphism. This will provide a d-morphism from a subspace X of Σ onto $\mathfrak{F}$. For $\sigma, \lambda \in \Sigma$, let $(\sigma, \lambda) = \{\mu \in \Sigma : \sigma < \mu < \lambda\}$. By [21, Lemma 3.11],

$$\big\{(\sigma.-n, \sigma.n) : n \in \omega - \{0\}\big\}$$

forms a local basis for each $\sigma \in \Sigma$. Let S denote the "initial segment" of relation on Σ. By [21, Lemma 3.12], for each $\sigma \in \Sigma$, the set $S(\sigma)$ is clopen in (Σ, τ).

Let r be a root of $\mathfrak{F}$. We set $f(\Lambda) = r$. For the recursive step, suppose that $f(\sigma) = w$ for $\sigma \in \Sigma$. If $R(w) = \varnothing$, then f is not defined on $S(\sigma) - \{\sigma\}$. If $R(w) \neq \varnothing$, we let $g_w : \omega - \{0\} \to R(w)$ be any sequence such that $(g_w)^{-1}(v)$ is infinite for each $v \in R(w)$. For $n \in \omega - \{0\}$, set $f(\sigma.n) = f(\sigma.-n) = g_w(n)$. Let $X = \{\sigma \in \Sigma : f(\sigma) \text{ is defined}\}$. Clearly $X \subseteq \Sigma$, and we equip X with the subspace topology. We claim that the function $f : X \to W$ is an onto d-morphism. That f is well defined follows from the construction. We show that f is onto. Let $w \in W$. If $w = r$, then $f(\Lambda) = r$. If $w \neq r$, then as r is a root of $\mathfrak{F}$, we have $w \in R(r)$. Therefore, $w = g_r(n)$ for some $n \in \omega - \{0\}$. Thus, $f(n) = f(\Lambda.n) = g_r(n) = w$. Consequently, f is onto.

Recall that R^+ denotes the reflexive closure of R. To see that f is continuous, it is sufficient to show that

$$f^{-1}\big(R^+(w)\big) = \bigcup_{wR^+f(\sigma)} S(\sigma) \cap X.$$

The $\subseteq$-inclusion is clear. For the other inclusion, we proceed by induction on $\lambda \in S(\sigma) \cap X$. Suppose that $\lambda \in S(\sigma) \cap X$ for some $\sigma \in \Sigma$ with $wR^+f(\sigma)$. If $\sigma = \lambda$, then obviously $wR^+f(\lambda)$, which establishes the base case. If not, then we show that $wR^+f(\lambda)$ implies $wR^+f(\lambda.z)$. We have

$$f(\lambda.z) = f(\lambda.|z|) = g_{f(\lambda)}(|z|) \in R\big(f(\lambda)\big),$$

which gives $wR^+f(\lambda)Rf(\lambda.z)$. As R is transitive, we have $wR^+f(\lambda.z)$. By induction, this gives $\lambda \in S(\sigma) \cap X$ implies $wR^+f(\lambda)$, and the $\supseteq$-inclusion follows. Now, since each $S(\sigma)$ is clopen in (Σ, τ), it follows that $f^{-1}(R^+(w))$ is open in X.

To see that f is open, we show that $f((\sigma.-n, \sigma.n) \cap X) = R^+(f(\sigma))$ for any $n \in \omega - \{0\}$ and $\sigma \in X$. By definition of f, we have that $f(\lambda)Rf(\lambda.z)$. As both S and R are transitive, it follows that $f(S(\sigma) \cap X) \subseteq R^+(f(\sigma))$. Since $(\sigma.-n, \sigma.n) \cap X \subseteq S(\sigma) \cap X$, we have $f((\sigma.-n, \sigma.n) \cap X) \subseteq R^+(f(\sigma))$. For the other inclusion, we first note that $f(\sigma) \in f((\sigma.-n, \sigma.n) \cap X)$. Let $w \in R(f(\sigma))$. Then there is $m > n$ such that $g_{f(\sigma)}(m) = w$. Therefore, $\sigma.m \in (\sigma.-n, \sigma.n)$. As $R(f(\sigma)) \neq \varnothing$, we have $\sigma.m \in X$. Thus,

$w = g_{f(\sigma)}(m) = f(\sigma.m) \in f((\sigma.-n, \sigma.n) \cap X)$, which shows that the other inclusion holds, and so the equality follows.

Let $w \in W$, and consider $\sigma \in f^{-1}(w)$. If w is reflexive, then $w \in R(w)$. Consider the basic open set $(\sigma.-n, \sigma.n)$ about σ. There is $m > n$ such that $g_w(m) = w$. Therefore, $\sigma.m \in [(\sigma.-n, \sigma.n) - \{\sigma\}] \cap X$ and $f(\sigma.m) = g_{f(\sigma)}(m) = g_w(m) = w$. Thus, $\sigma.m \in f^{-1}(w)$, showing that f is r-dense. Now assume that w is irreflexive. If w is maximal, we have $S(\sigma) \cap X = \{\sigma\}$, showing that σ is an isolated point of X. Suppose that w is not maximal. Let $\lambda \in (S(\sigma) - \{\sigma\}) \cap X$. Then $\lambda \in S(\sigma.z)$ for some nonzero integer z. Therefore, $f(\sigma.z)R^+ f(\lambda)$. If $f(\sigma.z)Rf(\lambda)$, the transitivity of R gives $f(\sigma)Rf(\lambda)$. If $f(\sigma.z) = f(\lambda)$, we have $f(\sigma)Rf(\lambda)$. In both cases we have $f(\lambda) \in R(f(\sigma))$. Therefore, $f(\lambda) \neq f(\sigma)$. This means that $(S(\sigma) \cap X) \cap f^{-1}(w) = \{\sigma\}$. It follows that f is i-discrete. Consequently, $f : X \to W$ is an onto d-morphism. □

In order to prove the Main Theorem of the paper, along with the Main Lemma, we also require the following lemma, which partially answers a question posed in [21, Section 6].

Lemma 3.2 *Let X be a T_d-space, let $\mathfrak{F} = (W, R)$ be a* **K4**-*frame, let $f : X \to W$ be an onto d-morphism, and let φ be a variable-free formula. For all $x \in X$, we have $x \models \varphi$ iff $f(x) \models \varphi$. Consequently, if $X \not\models \varphi$, then $\mathfrak{F} \not\models \varphi$.*

Proof First we show that for all $x \in X$, we have $x \models \varphi$ iff $f(x) \models \varphi$. The proof is by induction on the complexity of φ. That the base case holds is easy to see since $\varphi = \top$ or $\varphi = \bot$, and $\top$ is true and $\bot$ is false at every point of any model. Next suppose that for all $y \in X$, we have $y \models \psi$ iff $f(y) \models \psi$ for a variable-free formula ψ of lesser complexity than φ. Let $x \in X$. We have three cases to consider.

Case 1: $\varphi = \psi \vee \chi$, where both ψ and χ are variable-free. Then $x \models \varphi$ iff $x \models \psi$ or $x \models \chi$. By the inductive hypothesis, this happens iff $f(x) \models \psi$ or $f(x) \models \chi$, which happens iff $f(x) \models \varphi$.

Case 2: $\varphi = \neg\psi$, where ψ is variable-free. Then $x \models \varphi$ iff $x \not\models \psi$. By the (contrapositive of the) inductive hypothesis, this happens iff $f(x) \not\models \psi$, which happens iff $f(x) \models \varphi$.

Case 3: $\varphi = \Diamond\psi$, where ψ is variable-free. First suppose that $x \models \varphi$. Let $U = f^{-1}(R^+(f(x)))$. Since f is a d-morphism, U is an open neighborhood of x. Therefore, there exists $y \in U - \{x\}$ such that $y \models \psi$. By the inductive hypothesis, $f(y) \models \psi$. Clearly $f(y) \in R^+(f(x))$. If $f(x)$ is reflexive, then $R^+(f(x)) = R(f(x))$, so $f(y) \in R(f(x))$, and so $f(x) \models \varphi$. On the other hand, if $f(x)$ is irreflexive, then as f is i-discrete, there is an open neighborhood V of x such that $V \cap f^{-1}(f(x)) = \{x\}$. Therefore, $x \in U \cap V$, and so there exists $y \in (U \cap V) - \{x\}$ such that $y \models \psi$. The inductive hypothesis then gives $f(y) \models \psi$. Since $y \in U \cap V$, we have $f(y) \in R^+(f(x)) - \{f(x)\} = R(f(x))$. Thus, $f(x) \models \varphi$. Consequently, $x \models \varphi$ implies $f(x) \models \varphi$.

Next suppose that $x \not\models \varphi$. Then there is an open neighborhood U of x such that $y \not\models \psi$ for each $y \in U - \{x\}$. By the inductive hypothesis, $f(y) \not\models \psi$ for each $y \in U - \{x\}$. Since f is a d-morphism, $f(U)$ is an upset of $\mathfrak{F}$ containing $f(x)$. Therefore, $f(U - \{x\}) \supseteq R(f(x)) - \{f(x)\}$. Let $w \in R(f(x))$. If $w \neq f(x)$, we have $w = f(y)$ for some $y \in U - \{x\}$. Thus, $w \not\models \psi$. On the other hand, if $w = f(x)$, then $f(x)$ is reflexive. As f is r-dense, there is $y \in U - \{x\}$ with

$f(y) = f(x)$. Therefore, $w = f(y) \not\models \psi$. Thus, $f(x) \not\models \varphi$. Consequently, $x \not\models \varphi$ implies that $f(x) \not\models \varphi$, which completes the proof by induction.

Finally, if $X \not\models \varphi$, then there exists $x \in X$ such that $x \not\models \varphi$. By the above, $f(x) \not\models \varphi$. Thus, $\mathfrak{F} \not\models \varphi$. □

Theorem 3.3 (Main Theorem)

(1) Each modal logic over **K4** *axiomatizable by variable-free formulas is the d-logic of a subspace of* $\mathbb{Q}$.
(2) An arbitrary intersection of modal logics over **K4** *axiomatizable by variable-free formulas is the d-logic of a subspace of* $\mathbb{Q}$.

Proof (1) Let $L = \mathbf{K4} + \{\varphi_i : i \in I\}$, where each φ_i is variable-free. Since the standard translation (see, e.g., Blackburn, de Rijke, and Venema [11, Section 2.4]) of a variable-free formula produces a first-order condition on frames, L is complete with respect to a Δ-elementary class of Kripke frames. Therefore, by Gabbay and Shehtman [18, Proposition 5.4], L is complete with respect to a class of countable (rooted) Kripke frames. Let $\{\psi_n : n \in \omega\}$ be an enumeration of all nontheorems of L. For each ψ_n, there is a countable rooted L-frame $\mathfrak{F}_n$ refuting ψ_n. By the Main Lemma, there are a subspace X_n of $\mathbb{Q}$ and an onto d-morphism $f : X_n \to \mathfrak{F}_n$. Therefore, $X_n \not\models \psi$. Also, by Lemma 3.2, $X_n \models \{\varphi_i : i \in I\}$ (because $\mathfrak{F}_n \models \{\varphi_i : i \in I\}$). Let X be the disjoint union of $\{X_n : n \in \omega\}$. Then L is the d-logic of X. Clearly X is a subspace of a countable disjoint union of $\mathbb{Q}$, which is homeomorphic to $\mathbb{Q}$. Thus, X is homeomorphic to a subspace of $\mathbb{Q}$.

(2) Let $\{L_i : i \in I\}$ be a family of modal logics over **K4** axiomatizable by variable-free formulas. Let $L = \bigcap_{i \in I} L_i$, and let $\{\varphi_n : n \in \omega\}$ be an enumeration of nontheorems of L. For each $n \in \omega$ there is $i_n \in I$ such that $\varphi_n \notin L_{i_n}$. By (1), for each $n \in \omega$, there is a subspace X_n of $\mathbb{Q}$ whose d-logic is L_{i_n}. Take X to be the disjoint union of $\{X_n : n \in \omega\}$. Then X is homeomorphic to a subspace of $\mathbb{Q}$, and the d-logic of X is L. □

Corollary 3.4

(1) Subspaces of $\mathbb{Q}$ *give rise to continuum many d-logics over* **K4**.
(2) There exist continuum many d-logics of subspaces of $\mathbb{Q}$ *that are not finitely axiomatizable.*
(3) There exist continuum many d-logics of subspaces of $\mathbb{Q}$ *that are not decidable.*
(4) There exist d-logics of subspaces of $\mathbb{Q}$ *that do not have the FMP.*

Proof As follows from Gencer and de Jongh [19, Section 3], there exist continuum many modal logics over **K4** axiomatizable by variable-free formulas. Therefore, (1) follows from the Main Theorem. Since there are only countably many finitely axiomatizable (resp., decidable) modal logics, (2) and (3) follow from (1). Finally, by [12, Theorem 6.12], there is a modal logic over **K4** axiomatizable by variable-free formulas which does not have the FMP. Thus, (4) follows from the Main Theorem. □

In fact, there also exist continuum many d-logics of subspaces of $\mathbb{Q}$ that do not have the FMP (see [8]).

4 The Logics $\mathbf{K4\Gamma}_n$, $\mathbf{K4\Delta}_n$, $\mathbf{K4\Xi}_n$, $\mathbf{K4\Sigma}_n$, and $\mathbf{K4\Theta}_n$

In this section we consider five families of modal logics over **K4** axiomatized by variable-free formulas, and show that they d-define interesting classes of topological spaces.

Consider the following five families of formulas:

(1) $\gamma_n = \Box^n \bot$,
(2) $\delta_n = \Box^n \Diamond \top$,
(3) $\xi_n = \Diamond^n \Box \bot \to \Diamond \neg \Diamond^+ \Box \bot$,
(4) $\sigma_n = \gamma_{n+1} \to \gamma_n$,
(5) $\theta_n = \delta_{n+1} \to \delta_n$.

Definition 4.1 We let

(1) $\mathbf{K4\Gamma}_n = \mathbf{K4} + \gamma_n$,
(2) $\mathbf{K4\Delta}_n = \mathbf{K4} + \delta_n$,
(3) $\mathbf{K4\Xi}_n = \mathbf{K4} + \xi_n$,
(4) $\mathbf{K4\Sigma}_n = \mathbf{K4} + \sigma_n$,
(5) $\mathbf{K4\Theta}_n = \mathbf{K4} + \theta_n$.

Clearly $\gamma_0 = \bot$, and so $\mathbf{K4\Gamma}_0$ is the inconsistent logic. Moreover, since each of γ_n $(n \neq 0)$, δ_n, ξ_n, σ_n, and θ_n is variable-free, it follows from [12, Section 5.3] that each of $\mathbf{K4\Gamma}_n$ $(n \neq 0)$, $\mathbf{K4\Delta}_n$, $\mathbf{K4\Xi}_n$, $\mathbf{K4\Sigma}_n$, and $\mathbf{K4\Theta}_n$ has the FMP with respect to its Kripke semantics. Consequently, each of these modal logics is decidable.

Let $\mathfrak{F} = (W, R)$ be a **K4**-frame. We call $A \subseteq W$ *serial* if $A \subseteq R^{-1}(A)$. Note that if $A = \varnothing$, then A is serial.

Definition 4.2 Let $\mathfrak{F} = (W, R)$ be a finite **K4**-frame.

(1) We call $\mathfrak{F}$ *n-deep* if $R^{-n}(W) = \varnothing$.
(2) We call $\mathfrak{F}$ *n-top-deep* if $R^{-n}(\text{imax } W) = \varnothing$.
(3) We call $\mathfrak{F}$ *n-semi-top-deep* if $R^{-n}(\text{imax } W) \subseteq R^{-1}(\text{rmax } W)$.
(4) We call $\mathfrak{F}$ *n-serial* if $R^{-n}(W)$ is serial.
(5) We call $\mathfrak{F}$ *n-top-serial* if $R^{-n}(\text{imax } W)$ is serial.

Remark 4.3 We clearly have the following.

(1) $\mathfrak{F}$ is n-deep iff R is irreflexive and the depth of $\mathfrak{F}$ is $\leq n$.
(2) $\mathfrak{F}$ is n-top-deep iff $(R^+)^{-1}(\text{imax } W)$ is n-deep.
(3) $\mathfrak{F}$ is n-semi-top-deep iff $(R^+)^{-1}(\text{imax } W) - R^{-1}(\text{rmax } W)$ is n-deep.
(4) $\mathfrak{F}$ is n-serial iff $\max R^{-n}(W) = \text{rmax } R^{-n}(W)$.
(5) $\mathfrak{F}$ is n-top-serial iff $\max R^{-n}(\text{imax } W) = \text{rmax } R^{-n}(\text{imax } W)$.

Lemma 4.4 *Let $\mathfrak{F} = (W, R)$ be a finite* **K4**-*frame. Then*

(1) $\mathfrak{F} \models \gamma_n$ iff $\mathfrak{F}$ is n-deep;
(2) $\mathfrak{F} \models \delta_n$ iff $\mathfrak{F}$ is n-top-deep;
(3) $\mathfrak{F} \models \xi_n$ iff $\mathfrak{F}$ is n-semi-top-deep;
(4) $\mathfrak{F} \models \sigma_n$ iff $\mathfrak{F}$ is n-serial;
(5) $\mathfrak{F} \models \theta_n$ iff $\mathfrak{F}$ is n-top-serial.

Proof Let ν be a valuation into W.

(1) We have

$$\nu(\gamma_n) = \nu(\Box^n \bot) = \nu(\neg \Diamond^n \top) = W - R^{-n}(W).$$

Therefore, $\mathfrak{F} \models \gamma_n$ iff $W - R^{-n}(W) = W$ iff $R^{-n}(W) = \varnothing$ iff $\mathfrak{F}$ is n-deep.

(2) Since $\nu(\Box\bot) = \nu(\neg\Diamond\top) = W - R^{-1}(W) = \operatorname{imax}(W)$, we have

$$\nu(\delta_n) = \nu(\Box^n\Diamond\top) = \nu(\Box^n\neg\Box\bot) = \nu(\neg\Diamond^n\Box\bot) = W - R^{-n}(\operatorname{imax} W).$$

Therefore, $\mathfrak{F} \models \delta_n$ iff $W - R^{-n}(\operatorname{imax} W) = W$ iff $R^{-n}(\operatorname{imax} W) = \varnothing$ iff $\mathfrak{F}$ is n-top-deep.

(3) Since $\nu(\Diamond^n\Box\bot) = R^{-n}(\operatorname{imax} W)$ and $\nu(\Diamond^+\Box\bot) = (R^+)^{-1}(\operatorname{imax} W)$, it is obvious that $\mathfrak{F} \models \xi_n$ iff $R^{-n}(\operatorname{imax} W) \subseteq R^{-1}(W - (R^+)^{-1}(\operatorname{imax} W))$. We show that $R^{-1}(W - (R^+)^{-1}(\operatorname{imax} W)) = R^{-1}(\operatorname{rmax} W)$. Since $(R^+)^{-1}(\operatorname{imax} W) \cap \operatorname{rmax} W = \varnothing$, we have $\operatorname{rmax} W \subseteq W - (R^+)^{-1}(\operatorname{imax} W)$, and so $R^{-1}(\operatorname{rmax} W) \subseteq R^{-1}(W - (R^+)^{-1}(\operatorname{imax} W))$.

For the other inclusion, let $w \in R^{-1}(W - (R^+)^{-1}(\operatorname{imax} W))$. Then there exists $v \notin (R^+)^{-1}(\operatorname{imax} W)$ such that wRv. As $\mathfrak{F}$ is finite, there exists $u \in \max W$ such that vRu. Since R is transitive, we have wRu. Also, as $R^+(v) \cap \operatorname{imax} W = \varnothing$, we must have $u \in \operatorname{rmax} W$. Therefore, $w \in R^{-1}(\operatorname{rmax} W)$, showing that $R^{-1}(W - (R^+)^{-1}(\operatorname{imax} W)) \subseteq R^{-1}(\operatorname{rmax} W)$. As the desired equality holds, it follows that $\mathfrak{F} \models \xi_n$ iff $R^{-n}(\operatorname{imax} W) \subseteq R^{-1}(\operatorname{rmax} W)$, and hence $\mathfrak{F} \models \xi_n$ iff $\mathfrak{F}$ is n-semi-top-deep.

(4) We have

$$\begin{aligned}\nu(\sigma_n) &= \nu(\Box^{n+1}\bot \to \Box^n\bot) = \nu(\neg\Diamond^{n+1}\top \to \neg\Diamond^n\top)\\ &= (W - R^{-n-1}W) \to (W - R^{-n}W).\end{aligned}$$

Therefore, $\mathfrak{F} \models \sigma_n$ iff $W - R^{-n-1}(W) \subseteq W - R^{-n}(W)$ iff $R^{-n}(W) \subseteq R^{-n-1}(W)$. Thus, $\mathfrak{F} \models \sigma_n$ iff $R^{-n}(W)$ is serial, and so $\mathfrak{F} \models \sigma_n$ iff $\mathfrak{F}$ is n-serial.

(5) Since $\nu(\Box\bot) = \operatorname{imax}(W)$, we have

$$\begin{aligned}\nu(\theta_n) &= \nu(\Box^{n+1}\Diamond\top \to \Box^n\Diamond\top)\\ &= \nu(\Box^{n+1}\neg\Box\bot \to \Box^n\neg\Box\bot)\\ &= \nu(\neg\Diamond^{n+1}\Box\bot \to \neg\Diamond^n\Box\bot)\\ &= (W - R^{-n-1}\operatorname{imax} W) \to (W - R^{-n}\operatorname{imax} W).\end{aligned}$$

Therefore, $\mathfrak{F} \models \theta_n$ iff $W - R^{-n-1}(\operatorname{imax} W) \subseteq W - R^{-n}(\operatorname{imax} W)$ iff $R^{-n}(\operatorname{imax} W) \subseteq R^{-n-1}(\operatorname{imax} W)$. Thus, $\mathfrak{F} \models \theta_n$ iff $R^{-n}(\operatorname{imax} W)$ is serial, and so $\mathfrak{F} \models \theta_n$ iff $\mathfrak{F}$ is n-top-serial. $\Box$

In the next theorem we establish the containment relationships between the five families of logics we have introduced. It is captured by Figure 4. The full picture can

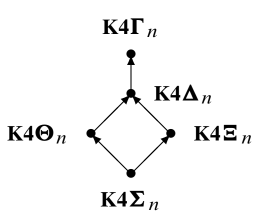

Figure 4

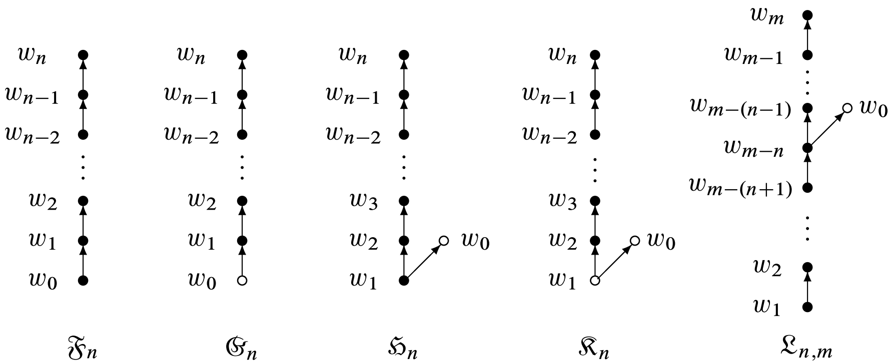

Figure 5

be found at the end of Section 5. Note that the arrows only indicate the inclusion relation, and so one should be careful when examining meets and joins in Figure 4. For example, $\mathbf{K4\Theta}_n \vee \mathbf{K4\Xi}_n \subset \mathbf{K4\Delta}_n$ and $\mathbf{K4\Sigma}_n \subset \mathbf{K4\Theta}_n \wedge \mathbf{K4\Xi}_n$.

Theorem 4.5

(1) $\mathbf{K4\Gamma}_n$ $(n \neq 0)$ *is the modal logic of finite n-deep* **K4***-frames;* $\mathbf{K4\Delta}_n$ *is the modal logic of finite n-top-deep* **K4***-frames;* $\mathbf{K4\Xi}_n$ *is the modal logic of finite n-semi-top-deep* **K4***-frames;* $\mathbf{K4\Theta}_n$ *is the modal logic of finite n-top-serial* **K4***-frames; and* $\mathbf{K4\Sigma}_n$ *is the modal logic of finite n-serial* **K4***-frames.*

(2) $\mathbf{K4\Gamma}_n = \mathbf{GL}_n$ *and* $\bigcap_\omega \mathbf{K4\Gamma}_n = \bigcap_\omega \mathbf{GL}_n = \mathbf{GL}$.

(3) $\mathbf{K4\Delta}_0 = \mathbf{K4D}$, $\mathbf{K4\Delta}_n \subset \mathbf{K4\Gamma}_n$ *for each* $n \neq 0$, *and* $\mathbf{K4\Gamma}_{n+m}$ *and* $\mathbf{K4\Delta}_n$ *are not comparable for* $m > 0$.

(4) $\mathbf{K4\Xi}_0 = \mathbf{K4D}$, $\mathbf{K4\Sigma}_n \subset \mathbf{K4\Xi}_n \subset \mathbf{K4\Delta}_n$, *and* $\mathbf{K4\Xi}_n$ *is not comparable with either* $\mathbf{K4\Gamma}_{n+m}$, $\mathbf{K4\Delta}_{n+m}$, *or* $\mathbf{K4\Theta}_m$ *for* $n, m > 0$.

(5) $\mathbf{K4\Sigma}_0 = \mathbf{K4\Theta}_0 = \mathbf{K4D}$, $\mathbf{K4\Sigma}_n \subset \mathbf{K4\Theta}_n \subset \mathbf{K4\Delta}_n$ *for each* $n > 0$, *and* $\mathbf{K4\Gamma}_{n+m}$ *and* $\mathbf{K4\Sigma}_n$ *are not comparable for* $m > 0$. *As a result, neither* $\mathbf{K4\Delta}_{n+m}$ *and* $\mathbf{K4\Sigma}_n$, $\mathbf{K4\Xi}_{n+m}$ *and* $\mathbf{K4\Sigma}_n$, *nor* $\mathbf{K4\Theta}_{n+m}$ *and* $\mathbf{K4\Sigma}_n$ *are comparable for* $n, m > 0$.

(6) $\mathbf{K4} = \bigcap_\omega \mathbf{K4\Sigma}_n = \bigcap_\omega \mathbf{K4\Theta}_n$.

Proof (1) Since each of the logics has the FMP, the result follows from Definition 4.2 and Lemma 4.4.

(2) As both $\mathbf{K4\Gamma}_n$ and $\mathbf{GL}_n$ are the modal logics of finite n-deep **K4**-frames, it follows that $\mathbf{GL}_n = \mathbf{K4\Gamma}_n$. Since $\mathbf{GL} = \bigcap_\omega \mathbf{GL}_n$, we have $\bigcap_\omega \mathbf{K4\Gamma}_n = \bigcap_\omega \mathbf{GL}_n = \mathbf{GL}$.

(3) Since $\delta_0 = \Diamond\top$, it is clear that $\mathbf{K4\Delta}_0 = \mathbf{K4D}$. We show that $\mathbf{K4\Delta}_n \subset \mathbf{K4\Gamma}_n$. Let $\mathfrak{F} \models \gamma_n$. Then $R^{-n}(W) = \varnothing$. As $R^{-n}(\operatorname{imax} W) \subseteq R^{-n}(W)$, it follows that $R^{-n}(\operatorname{imax} W) = \varnothing$. Therefore, $\mathfrak{F} \models \delta_n$, and so each n-deep frame is n-top-deep. Since $\mathbf{K4\Delta}_n$ is the modal logic of n-top-deep frames and $\mathbf{K4\Gamma}_n$ is the modal logic of n-deep frames, the containment follows. To see that the containment is strict, consider the frame $\mathfrak{H}_n$ shown in Figure 5. As usual, bullets denote irreflexive points and circles denote reflexive points. Clearly $\mathfrak{H}_n$ is n-top-deep, but not n-deep. Therefore, $\mathfrak{H}_n \models \delta_n$, but $\mathfrak{H}_n \not\models \gamma_n$, so $\mathbf{K4\Delta}_n \nvdash \gamma_n$, and so the containment is strict.

To see that $\mathbf{K4\Gamma}_{n+m}$ and $\mathbf{K4\Delta}_n$ are not comparable for $m > 0$, consider the frames $\mathfrak{F}_{n+m-1}$ and $\mathfrak{H}_n$ shown in Figure 5. Clearly the frame $\mathfrak{F}_{n+m-1}$ is a $\mathbf{K4\Gamma}_{n+m}$-frame that is not a $\mathbf{K4\Delta}_n$-frame. It is also clear that $\mathfrak{H}_n$ is a $\mathbf{K4\Delta}_n$-frame that is not a $\mathbf{K4\Gamma}_{n+1}$-frame. (In fact, $\mathfrak{H}_n$ is $(n-1)$-serial.) Thus, $\mathbf{K4\Gamma}_{n+m}$ and $\mathbf{K4\Delta}_n$ are incomparable.

(4) We start by showing that $\mathbf{K4\Xi}_0 = \mathbf{K4D}$. Let $\mathfrak{F} = (W, R)$ be a finite $\mathbf{K4}$-frame. By Lemma 4.4(3), $\mathfrak{F} \models \xi_0$ iff $R^{-0}(\operatorname{imax} W) \subseteq R^{-1}(\operatorname{rmax} W)$ iff $\operatorname{imax} W \subseteq R^{-1}(\operatorname{rmax} W)$ iff $\operatorname{imax} W = \varnothing$ iff $\mathfrak{F}$ is serial. Therefore, $\mathbf{K4\Xi}_0 = \mathbf{K4D}$.

Second, we show that $\mathbf{K4\Sigma}_n \subset \mathbf{K4\Xi}_n$ for $n > 0$. Let $\mathfrak{F} = (W, R)$ be a $\mathbf{K4\Xi}_n$-frame, and let $w \in R^{-n}(W)$. Then $wR^n v$ for some $v \in W$. If $v \notin \operatorname{imax} W$, then $v \in W - \operatorname{imax} W = R^{-1}(W)$. Therefore, vRu for some $u \in W$. Thus, $wR^{n+1}u$, and so $w \in R^{-n-1}(W)$. On the other hand, if $v \in \operatorname{imax} W$, then $w \in R^{-n}(\operatorname{imax} W) \subseteq R^{-1}(\operatorname{rmax} W)$. Therefore, wRu for some $u \in \operatorname{rmax} W$. As $u \in \operatorname{rmax} W$, we have that u is reflexive, and so $uR^n u$. Thus, $wR^{n+1}u$, and so $w \in R^{-n-1}(W)$. In either case we obtain $R^{-n}(W) \subseteq R^{-n-1}(W)$, so $\mathfrak{F}$ is n-serial, and so $\mathfrak{F}$ is a $\mathbf{K4\Sigma}_n$-frame. Consequently, $\mathbf{K4\Sigma}_n \subseteq \mathbf{K4\Xi}_n$. To see that the containment is strict, consider the frame $\mathfrak{G}_n$ in Figure 5. Clearly $\mathfrak{G}_n$ is n-serial, and $R^{-n}(\operatorname{imax} W) = \{w_0\} \not\subseteq \varnothing = R^{-1}(\operatorname{rmax} W)$. Thus, $\mathfrak{G}_n$ is a $\mathbf{K4\Sigma}_n$-frame which is not a $\mathbf{K4\Xi}_n$-frame, and so the containment is strict.

Third, we show that $\mathbf{K4\Xi}_n \subset \mathbf{K4\Delta}_n$ for $n > 0$. Let $\mathfrak{F} = (W, R)$ be a $\mathbf{K4\Delta}_n$-frame. Then $R^{-n}(\operatorname{imax} W) = \varnothing \subseteq R^{-1}(\operatorname{rmax} W)$. Therefore, $\mathfrak{F}$ is a $\mathbf{K4\Xi}_n$-frame, and so $\mathbf{K4\Xi}_n \subseteq \mathbf{K4\Delta}_n$. That the inclusion is strict can be seen by considering the frame $\mathfrak{H}_{n+1}$ in Figure 5, which is clearly not n-top-deep, but it is n-semi-top-deep, because

$$R^{-n}(\operatorname{imax} W) = \{w_1\} \subseteq \{w_1, w_0\} = R^{-1}(\operatorname{rmax} W).$$

Next we show that $\mathbf{K4\Xi}_n$ is not comparable with either $\mathbf{K4\Gamma}_{n+m}$ or $\mathbf{K4\Delta}_{n+m}$ for $n, m > 0$. The frame $\mathfrak{K}_2$ in Figure 5 is n-semi-top-deep, but it is not $(n+m)$-top-deep. Therefore, $\mathbf{K4\Delta}_{n+m} \not\subseteq \mathbf{K4\Xi}_n$, and so $\mathbf{K4\Gamma}_{n+m} \not\subseteq \mathbf{K4\Xi}_n$. Also, the frame $\mathfrak{F}_{n+m-1}$ is $(n+m)$-deep, but it is not n-semi-top-deep. Thus, $\mathbf{K4\Xi}_n \not\subseteq \mathbf{K4\Gamma}_{n+m}$, and so $\mathbf{K4\Xi}_n \not\subseteq \mathbf{K4\Delta}_{n+m}$. Consequently, $\mathbf{K4\Xi}_n$ is not comparable with either $\mathbf{K4\Gamma}_{n+m}$ or $\mathbf{K4\Delta}_{n+m}$ for $n, m > 0$.

Last, we show that $\mathbf{K4\Xi}_n$ is not comparable with $\mathbf{K4\Theta}_m$ for $n, m > 0$. Since $\mathfrak{G}_1$ in Figure 5 is not n-semi-top-deep, but is m-top-serial, we have $\mathbf{K4\Xi}_n \not\subseteq \mathbf{K4\Theta}_m$. To show that $\mathbf{K4\Theta}_m \not\subseteq \mathbf{K4\Xi}_n$, we consider $n \geq m$ and $n < m$ separately. If $n \geq m$, then the frame $\mathfrak{H}_{n+1}$ in Figure 5 is a $\mathbf{K4\Xi}_n$-frame which is not a $\mathbf{K4\Theta}_m$-frame. If $n < m$, then the frame $\mathfrak{L}_{n,m+1}$ in Figure 5 is a $\mathbf{K4\Xi}_n$-frame which is not a $\mathbf{K4\Theta}_m$-frame because $w_1 \in R^{-m}(\operatorname{imax} W)$, but $w_1 \notin R^{-m-1}(\operatorname{imax} W)$. Consequently, $\mathbf{K4\Theta}_m \not\subseteq \mathbf{K4\Xi}_n$, and so $\mathbf{K4\Xi}_n$ and $\mathbf{K4\Theta}_m$ are incomparable for $n, m > 0$.

(5) First, we show that $\mathbf{K4\Sigma}_0 = \mathbf{K4D}$. For a $\mathbf{K4}$-frame $\mathfrak{F}$, we have

$$\begin{aligned} \mathfrak{F} \models \sigma_0 \text{ iff } \mathfrak{F} \models \gamma_1 \to \gamma_0 \text{ iff } \mathfrak{F} \models \Box\bot \to \bot \text{ iff } \mathfrak{F} \models \neg\Box\bot \\ \text{iff } \mathfrak{F} \models \Diamond\neg\bot \text{ iff } \mathfrak{F} \models \Diamond\top. \end{aligned}$$

Thus, a finite $\mathbf{K4}$-frame $\mathfrak{F}$ is 0-serial iff it is serial, and so $\mathbf{K4\Sigma}_0 = \mathbf{K4D}$.

Second, we show that $\mathbf{K4\Theta}_0 = \mathbf{K4D}$. For a finite $\mathbf{K4}$-frame $\mathfrak{F} = (W, R)$, we have

$$\begin{aligned}\mathfrak{F} \models \theta_0 \text{ iff } \mathfrak{F} \models \delta_1 \to \delta_0 \text{ iff } \mathfrak{F} \models \Box\Diamond\top \to \Diamond\top \text{ iff } \mathfrak{F} \models \Box\bot \to \Diamond\Box\bot \\ \text{iff } \operatorname{imax} W \subseteq R^{-1}(\operatorname{imax} W) \text{ iff } \operatorname{imax} W = \varnothing \text{ iff } \mathfrak{F} \text{ is serial} \\ \text{iff } \mathfrak{F} \models \Diamond\top.\end{aligned}$$

Therefore, $\mathfrak{F}$ is 0-top-serial iff it is serial, and so $\mathbf{K4\Theta}_0 = \mathbf{K4D}$.

Third, we show that $\mathbf{K4\Sigma}_n \subset \mathbf{K4\Theta}_n$. Let $\mathfrak{F} = (W, R)$ be a finite $\mathbf{K4}$-frame. If $\mathfrak{F} \models \theta_n$, then $\mathfrak{F}$ is n-top-serial, and so $R^{-n}(\operatorname{imax} W) \subseteq R^{-n-1}(\operatorname{imax} W)$. Therefore,

$$\begin{aligned}R^{-n}(W) &= R^{-n}\big(R^{-1}(W) \cup \operatorname{imax} W\big) \\ &= R^{-n}\big(R^{-1}(W)\big) \cup R^{-n}(\operatorname{imax} W) \\ &\subseteq R^{-n-1}(W) \cup R^{-n-1}(\operatorname{imax} W) \\ &= R^{-n-1}(W).\end{aligned}$$

Thus, $\mathfrak{F}$ is n-serial, and so $\mathfrak{F} \models \sigma_n$. This shows that each finite $\mathbf{K4\Theta}_n$-frame is a $\mathbf{K4\Sigma}_n$-frame, and the containment follows. For $n > 0$, we show that the containment is strict. The frame $\mathfrak{H}_{n+1}$ is n-serial, but it is not n-top-serial as $R^{-n}(\operatorname{imax} W) = \{w_1\}$, while $R^{-n-1}(\operatorname{imax} W) = \varnothing$. Therefore, $\mathfrak{H}_{n+1} \models \sigma_n$, but $\mathfrak{H}_{n+1} \not\models \theta_n$, so $\mathbf{K4\Sigma}_n \nvdash \theta_n$, and so the containment is strict.

Fourth, we show that $\mathbf{K4\Theta}_n \subset \mathbf{K4\Delta}_n$. Let $\mathfrak{F} = (W, R)$ be a finite $\mathbf{K4}$-frame. If $\mathfrak{F} \models \delta_n$, then $R^{-n}(\operatorname{imax} W) = \varnothing \subseteq R^{-n-1}(\operatorname{imax} W)$, showing that $\mathfrak{F} \models \theta_n$. Therefore, each finite n-top-deep frame is n-top-serial, and so $\mathbf{K4\Theta}_n \subseteq \mathbf{K4\Delta}_n$. We show that the containment is strict. Consider the frame $\mathfrak{G}_n$ shown in Figure 5. Obviously $\mathfrak{G}_n$ is n-top-serial because $R^{-n}(W) = \{w_0\} = R^{-n-1}(W)$, but it is not n-top-deep. Thus, $\mathbf{K4\Theta}_n \subset \mathbf{K4\Delta}_n$.

Next we show that $\mathbf{K4\Gamma}_{n+m}$ and $\mathbf{K4\Sigma}_n$ are not comparable for $m > 0$. Since the frame $\mathfrak{G}_n$ in Figure 5 is a $\mathbf{K4\Sigma}_n$-frame, but is not a $\mathbf{K4\Gamma}_{m+n}$-frame, we have $\mathbf{K4\Gamma}_{m+n} \nsubseteq \mathbf{K4\Sigma}_n$. The frame $\mathfrak{F}_{n+m-1}$ in Figure 5 is $(n+m)$-deep, but it is not n-serial. Thus, $\mathbf{K4\Sigma}_n \nsubseteq \mathbf{K4\Gamma}_{n+m}$.

It follows that $\mathbf{K4\Sigma}_n$ is properly contained in $\mathbf{K4\Xi}_n$, $\mathbf{K4\Theta}_n$, $\mathbf{K4\Delta}_n$, and $\mathbf{K4\Gamma}_n$ for each $n > 0$, but neither $\mathbf{K4\Gamma}_{n+m}$, $\mathbf{K4\Delta}_{n+m}$, $\mathbf{K4\Xi}_{n+m}$, nor $\mathbf{K4\Theta}_{n+m}$ is comparable with $\mathbf{K4\Sigma}_n$ for $m > 0$.

(6) Obviously $\mathbf{K4} \subseteq \mathbf{K4\Theta}_n$ for each $n \in \omega$. Let $\mathbf{K4} \nvdash \varphi$. As $\mathbf{K4}$ has the FMP with respect to its Kripke semantics, there is a finite rooted $\mathbf{K4}$-frame $\mathfrak{F}$ refuting φ. Let n be the depth of $\mathfrak{F}$. Either $R^{-n}(\operatorname{imax} W) = \varnothing$ or $R^{-n}(\operatorname{imax} W) \neq \varnothing$. If $R^{-n}(\operatorname{imax} W) = \varnothing$, then $\mathfrak{F}$ is n-top-deep and hence n-top-serial. If $R^{-n}(\operatorname{imax} W) \neq \varnothing$, then as $\mathfrak{F}$ has depth n, $R^{-n}(\operatorname{imax} W)$ contains a nondegenerate cluster of $\mathfrak{F}$. Therefore, $R^{-n-1}(\operatorname{imax} W) = R^{-n}(\operatorname{imax} W)$, and so $\mathfrak{F}$ is again n-top-serial. Thus, in either case $\mathfrak{F}$ is a $\mathbf{K4\Theta}_n$-frame refuting φ, and so $\mathbf{K4\Theta}_n \nvdash \varphi$. We conclude that $\bigcap_{n\in\omega} \mathbf{K4\Theta}_n \nvdash \varphi$. Consequently, $\mathbf{K4} = \bigcap_\omega \mathbf{K4\Theta}_n$. Now observe that $\mathbf{K4} \subseteq \bigcap_\omega \mathbf{K4\Sigma}_n \subseteq \bigcap_\omega \mathbf{K4\Theta}_n = \mathbf{K4}$. $\square$

Definition 4.6 Let X be a T_d-space.

(1) We call X *n-scattered* if $d^n(X) = \varnothing$.
(2) We call X *n-quasi-scattered* if $d^n(\operatorname{iso} X) = \varnothing$.
(3) We call X *n-semi-scattered* if $d^n(\operatorname{iso} X) \subseteq d(X - \overline{\operatorname{iso} X})$.

(4) We call X *n-dense-in-itself* if $d^n(X)$ is dense-in-itself.
(5) We call X *n-strongly-dense-in-itself* if $d^n(\text{iso}\, X)$ is dense-in-itself.

Lemma 4.7 *Let X be a T_d-space. Then*

(1) $X \models \gamma_n$ iff X is n-scattered;
(2) $X \models \delta_n$ iff X is n-quasi-scattered;
(3) $X \models \xi_n$ iff X is n-semi-scattered;
(4) $X \models \sigma_n$ iff X is n-dense-in-itself;
(5) $X \models \theta_n$ iff X is n-strongly-dense-in-itself.

Proof Let ν be a valuation into X.

(1) We have

$$\nu(\gamma_n) = \nu(\Box^n \bot) = \nu(\neg\Diamond^n \top) = X - d^n(X).$$

Therefore, $X \models \gamma_n$ iff $X - d^n(X) = X$ iff $d^n(X) = \varnothing$ iff X is n-scattered.

(2) Since $\nu(\Box\bot) = \nu(\neg\Diamond\top) = X - d(X) = \text{iso}(X)$, we have

$$\nu(\delta_n) = \nu(\Box^n \Diamond \top) = \nu(\Box^n \neg\Box\bot) = \nu(\neg\Diamond^n \Box\bot) = X - d^n(\text{iso}\, X).$$

Therefore, $X \models \delta_n$ iff $X - d^n(\text{iso}\, X) = X$ iff $d^n(\text{iso}\, X) = \varnothing$ iff X is n-quasi-scattered.

(3) Since $\nu(\Box\bot) = \text{iso}\, X$, $\nu(\Diamond^n \Box\bot) = d^n(\text{iso}\, X)$, and $\nu(\Diamond^+ \Box\bot) = \overline{\text{iso}\, X}$, we have $X \models \xi_n$ iff $d^n(\text{iso}\, X) \subseteq d(X - \overline{\text{iso}\, X})$ iff X is n-semi-scattered.

(4) We have

$$\begin{aligned}\nu(\sigma_n) &= \nu(\Box^{n+1}\bot \to \Box^n \bot) = \nu(\neg\Diamond^{n+1}\top \to \neg\Diamond^n \top) \\ &= (X - d^{n+1}X) \to (X - d^n X).\end{aligned}$$

Therefore, $X \models \sigma_n$ iff $X - d^{n+1}(X) \subseteq X - d^n(X)$ iff $d^n(X) \subseteq d^{n+1}(X)$. Thus, $X \models \sigma_n$ iff $d^n(X)$ is dense-in-itself, and so $X \models \sigma_n$ iff X is n-dense-in-itself.

(5) Since $\nu(\Box\bot) = \text{iso}(X)$, we have

$$\begin{aligned}\nu(\theta_n) &= \nu(\Box^{n+1}\Diamond\top \to \Box^n \Diamond\top) \\ &= \nu(\Box^{n+1}\neg\Box\bot \to \Box^n\neg\Box\bot) \\ &= \nu(\neg\Diamond^{n+1}\Box\bot \to \neg\Diamond^n \Box\bot) \\ &= \big(X - d^{n+1}(\text{iso}\, X)\big) \to \big(X - d^n(\text{iso}\, X)\big).\end{aligned}$$

Therefore, $X \models \theta_n$ iff $X - d^{n+1}(\text{iso}\, X) \subseteq X - d^n(\text{iso}\, X)$ iff $d^n(\text{iso}\, X) \subseteq d^{n+1}(\text{iso}\, X)$. Thus, $X \models \theta_n$ iff $d^n(\text{iso}\, X)$ is dense-in-itself, and so $X \models \theta_n$ iff X is n-strongly-dense-in-itself. $\square$

Theorem 4.8

(1) $\mathbf{K4\Gamma}_n$ ($n \neq 0$) d-defines and is the d-logic of the class of n-scattered spaces. In fact, $\mathbf{K4\Gamma}_n$ ($n \neq 0$) is the d-logic of an n-scattered subspace of $\mathbb{Q}$.
(2) $\mathbf{K4\Delta}_n$ d-defines and is the d-logic of the class of n-quasi-scattered spaces. In fact, $\mathbf{K4\Delta}_n$ is the d-logic of an n-quasi-scattered subspace of $\mathbb{Q}$.
(3) $\mathbf{K4\Xi}_n$ d-defines and is the d-logic of the class of n-semi-scattered spaces. In fact, $\mathbf{K4\Xi}_n$ is the d-logic of an n-semi-scattered subspace of $\mathbb{Q}$.
(4) $\mathbf{K4\Sigma}_n$ d-defines and is the d-logic of the class of n-dense-in-itself spaces. In fact, $\mathbf{K4\Sigma}_n$ is the d-logic of an n-dense-in-itself subspace of $\mathbb{Q}$.

(5) $\mathbf{K4\Theta}_n$ *d-defines and is the d-logic of the class of n-strongly-dense-in-itself spaces. In fact,* $\mathbf{K4\Theta}_n$ *is the d-logic of an n-strongly-dense-in-itself subspace of* $\mathbb{Q}$.

Proof The proof follows from the Main Theorem and Lemma 4.7. □

5 The Logics wGL, qGL, and sGL

In this final section we provide three natural generalizations of the concept of scattered space. The first leads to the concept of weakly scattered space, the second to the concept of quasi-scattered space, and the third to the concept of semi-scattered space. These three classes of spaces give rise to the modal logics weak-**GL** (denoted **wGL**), quasi-**GL** (denoted **qGL**), and semi-**GL** (denoted **sGL**). The logic **wGL** has already appeared in the literature under the name **K4G** (see [16]; see also [5]). The logics **qGL** and **sGL** appear to be new.

We show that **wGL** d-defines the class of weakly scattered spaces, **qGL** d-defines the class of quasi-scattered spaces, and **sGL** d-defines the class of semi-scattered spaces. We also show that each of **wGL**, **qGL**, and **sGL** has the FMP with respect to its Kripke semantics and is decidable. In addition, we show that each of these three logics arises as the d-logic of a subspace of $\mathbb{Q}$, that $\mathbf{GL} = \mathbf{wGL} \vee \mathbf{qGL} = \mathbf{wGL} \vee \mathbf{sGL}$, that $\mathbf{qGL} = \bigcap_\omega \mathbf{K4\Delta}_n$, and that $\mathbf{sGL} = \bigcap_\omega \mathbf{K4\Xi}_n$. Here and below $\vee$ denotes the join in the lattice of normal extensions of **K4**.

5.1 wGL

Let X be a topological space. We recall that $A \subseteq X$ is *dense* in X if $\overline{A} = X$. If X is scattered, then for each nonempty subspace Y of X, the set $\mathrm{iso}(Y)$ of isolated points of Y is nonempty. But in fact $\mathrm{iso}(Y)$ is dense in Y. In particular, $\mathrm{iso}(X)$ is dense in X.

Definition 5.1 We call a T_d-space X *weakly scattered* if $\mathrm{iso}(X)$ is dense in X.

Clearly each scattered space is weakly scattered. An example of a weakly scattered space which is not scattered is the Stone–Čech compactification $\beta(\omega)$ of ω. Indeed, $\mathrm{iso}(\beta(\omega)) = \omega$ is dense in $\beta(\omega)$, but $\omega^* = \beta(\omega) - \omega$ is a nonempty dense-in-itself subspace of $\beta(\omega)$.

Definition 5.2 Let $\mathbf{wgl} = \Diamond^+\Box\bot$, and let $\mathbf{wGL} = \mathbf{K4} + \mathbf{wgl}$. We call **wGL** *weak-**GL***.

We recall that $\mathbf{K4G} = \mathbf{K4} + (\neg\Box\bot \to \neg\Box\neg\Box\bot)$. If we read $\Box$ as "it is provable in a theory T," then $\neg\Box\bot \to \neg\Box\neg\Box\bot$ reads as "if T is consistent, then T cannot prove its own consistency," thus providing the modal version of Gödel's second incompleteness theorem (see [16]).

Lemma 5.3 $\mathbf{K4G} = \mathbf{wGL}$.

Proof We have

$$\mathbf{K4} \vdash \Diamond^+\Box\bot \leftrightarrow \Box\bot \vee \Diamond\Box\bot \leftrightarrow \neg\neg\Box\bot \vee \Diamond\Box\bot \leftrightarrow \neg\Box\bot \to \Diamond\Box\bot$$
$$\leftrightarrow \neg\Box\bot \to \neg\Box\neg\Box\bot.$$

Therefore, $\mathbf{K4G} = \mathbf{K4} + (\neg\Box\bot \to \neg\Box\neg\Box\bot) = \mathbf{K4} + \Diamond^+\Box\bot = \mathbf{wGL}$. □

Theorem 5.4

(1) $\mathbf{wGL}$ *has the FMP with respect to its Kripke semantics and hence is decidable.*
(2) For each finite $\mathbf{K4}$*-frame* $\mathfrak{F} = (W, R)$*, we have* $\mathfrak{F} \models \mathbf{wgl}$ *iff* $\max W = \operatorname{imax} W$.
(3) For each T_d*-space* X*, we have* $X \models \mathbf{wgl}$ *iff* X *is weakly scattered.*
(4) $\mathbf{wGL}$ *d-defines the class of weakly scattered spaces.*
(5) $\mathbf{wGL}$ *is the d-logic of weakly scattered spaces. In fact,* $\mathbf{wGL}$ *is the d-logic of a weakly scattered subspace of* $\mathbb{Q}$.
(6) $\mathbf{wGL} \subset \mathbf{GL}$*. In particular,* $\mathbf{wGL} \not\vdash \mathbf{gl}$.
(7) $\mathbf{wGL} \vee \mathbf{K4\Sigma}_n = \mathbf{wGL} \vee \mathbf{K4\Theta}_n$.
(8) $\mathbf{wGL} \vee \mathbf{K4\Delta}_n = \mathbf{K4\Gamma}_n$.

Proof (1) As $\mathbf{wGL}$ is an extension of $\mathbf{K4}$ by a variable-free formula, by [12, Section 5.3], $\mathbf{wGL}$ has the FMP with respect to its Kripke semantics (see also [5, Proposition 2.1.4]). As $\mathbf{wGL}$ is finitely axiomatizable and has the FMP, it is decidable.

(2) Let $\mathfrak{F} = (W, R)$ be a finite $\mathbf{K4}$-frame. Recall that R^+ denotes the reflexive closure of R. For each valuation ν into $\mathfrak{F}$, we have

$$\nu(\mathbf{wgl}) = \nu(\Diamond^+\Box\bot) = (R^+)^{-1}(W - R^{-1}W) = (R^+)^{-1}(\operatorname{imax} W).$$

Therefore, $\mathfrak{F} \models \mathbf{wgl}$ iff $\mathfrak{F} \models \Diamond^+\Box\bot$ iff $\nu(\Diamond^+\Box\bot) = W$ iff $(R^+)^{-1}(\operatorname{imax} W) = W$ iff $\operatorname{rmax} W = \varnothing$ iff $\max W = \operatorname{imax} W$ (see also [5, Section 2]).

(3) Let X be a T_d-space. For each valuation ν into X, we have

$$\nu(\mathbf{wgl}) = \nu(\Diamond^+\Box\bot) = \overline{X - d(X)} = \overline{\operatorname{iso}(X)}.$$

Therefore, $X \models \mathbf{wgl}$ iff $X \models \Diamond^+\Box\bot$ iff $\nu(\Diamond^+\Box\bot) = X$ iff $\overline{\operatorname{iso}(X)} = X$ iff X is weakly scattered.

(4) This is an immediate consequence of (3).

(5) Apply (4) and the Main Theorem.

(6) Clearly the class of finite $\mathbf{wGL}$-frames contains the class of finite $\mathbf{GL}$-frames. Since both logics have the FMP with respect to their Kripke semantics, it follows that $\mathbf{wGL} \subseteq \mathbf{GL}$. A simple example of a finite $\mathbf{wGL}$-frame which is not a $\mathbf{GL}$-frame is the frame $\mathfrak{G}_1$ shown in Figure 5. Therefore, $\mathbf{wGL} \not\vdash \mathbf{gl}$, and so $\mathbf{wGL} \subset \mathbf{GL}$.

(7) Since $\mathbf{K4\Sigma}_n \subseteq \mathbf{K4\Theta}_n$, we have $\mathbf{wGL} \vee \mathbf{K4\Sigma}_n \subseteq \mathbf{wGL} \vee \mathbf{K4\Theta}_n$. For the converse inclusion, first we show that $\mathbf{wGL} \vdash \Diamond\top \leftrightarrow \Diamond\Box\bot$. Since $\mathbf{wGL} \vdash \Diamond^+\Box\bot$, we have $\mathbf{wGL} \vdash \Diamond\top \leftrightarrow \Diamond\Diamond^+\Box\bot$, so $\mathbf{wGL} \vdash \Diamond\top \leftrightarrow \Diamond(\Box\bot \vee \Diamond\Box\bot)$, and so $\mathbf{wGL} \vdash \Diamond\top \leftrightarrow (\Diamond\Box\bot \vee \Diamond^2\Box\bot)$. As $\mathbf{wGL} \vdash \Diamond^2\varphi \rightarrow \Diamond\varphi$, we obtain $\mathbf{wGL} \vdash \Diamond\top \leftrightarrow \Diamond\Box\bot$. Next we show that $\mathbf{wGL} \vee \mathbf{K4\Sigma}_n \vdash \theta_n$. We clearly have $\mathbf{K4} \vdash \theta_n \leftrightarrow (\Diamond^n\Box\bot \rightarrow \Diamond^{n+1}\Box\bot)$ and $\mathbf{K4} \vdash \sigma_n \leftrightarrow (\Diamond^n\top \rightarrow \Diamond^{n+1}\top)$. Now $\mathbf{wGL} \vdash \Diamond^n\Box\bot \leftrightarrow \Diamond^{n-1}\Diamond\Box\bot \leftrightarrow \Diamond^{n-1}\Diamond\top \leftrightarrow \Diamond^n\top$. Since $\mathbf{wGL} \vee \mathbf{K4\Sigma}_n \vdash \Diamond^n\top \rightarrow \Diamond^{n+1}\top$, we have $\mathbf{wGL} \vee \mathbf{K4\Sigma}_n \vdash \Diamond^n\Box\bot \rightarrow \Diamond^{n+1}\Box\bot$. Therefore, $\mathbf{wGL} \vee \mathbf{K4\Sigma}_n \vdash \theta_n$, and so $\mathbf{wGL} \vee \mathbf{K4\Theta}_n \subseteq \mathbf{wGL} \vee \mathbf{K4\Sigma}_n$. Thus, $\mathbf{wGL} \vee \mathbf{K4\Sigma}_n = \mathbf{wGL} \vee \mathbf{K4\Theta}_n$.

(8) We clearly have $\mathbf{wGL}, \mathbf{K4\Delta}_n \subseteq \mathbf{K4\Gamma}_n$. Therefore, $\mathbf{wGL} \vee \mathbf{K4\Delta}_n \subseteq \mathbf{K4\Gamma}_n$. We show that $\mathbf{wGL} \vee \mathbf{K4\Delta}_n \vdash \gamma_n$. We have $\mathbf{K4} \vdash \gamma_n \leftrightarrow \neg\Diamond^n\top$ and $\mathbf{K4} \vdash \delta_n \leftrightarrow \neg\Diamond^n\Box\bot$. Since $\mathbf{wGL} \vdash \Diamond^n\top \leftrightarrow \Diamond^n\Box\bot$, we have $\mathbf{wGL} \vdash \neg\Diamond^n\Box\bot \leftrightarrow \neg\Diamond^n\top$. As $\mathbf{wGL} \vee \mathbf{K4\Delta}_n \vdash \neg\Diamond^n\Box\bot$, it follows that $\mathbf{wGL} \vee \mathbf{K4\Delta}_n \vdash \neg\Diamond^n\top$. Therefore, $\mathbf{wGL} \vee \mathbf{K4\Delta}_n \vdash \gamma_n$, and so $\mathbf{K4\Gamma}_n \subseteq \mathbf{wGL} \vee \mathbf{K4\Delta}_n$. Thus, $\mathbf{wGL} \vee \mathbf{K4\Delta}_n = \mathbf{K4\Gamma}_n$. $\square$

5.2 qGL and sGL Let X be a scattered space. Then $\overline{\mathrm{iso}(X)} = X$, and so $\overline{\mathrm{iso}(X)}$ is a scattered space. But in general $\overline{\mathrm{iso}(X)}$ is a proper subspace of X. In fact, often $\overline{\mathrm{iso}(X)}$ is even empty. So it may happen that $\overline{\mathrm{iso}(X)}$ is scattered without X being scattered. Similarly, $\mathrm{int}(\overline{\mathrm{iso}(X)})$ is scattered whenever $\overline{\mathrm{iso}(X)}$ is scattered, but it may happen that $\mathrm{int}(\overline{\mathrm{iso}(X)})$ is scattered without $\overline{\mathrm{iso}(X)}$ being scattered.

Definition 5.5 Let X be a T_d-space.

(1) We call X *quasi-scattered* if $\overline{\mathrm{iso}(X)}$ is a scattered space.
(2) We call X *semi-scattered* if $\mathrm{int}(\overline{\mathrm{iso}(X)})$ is a scattered space.

Clearly each quasi-scattered space is semi-scattered, but the converse may not be true in general. For the next lemma, we recall that if Y is a subspace of X and $A \subseteq Y$, then $d_Y(A) = d(A) \cap Y$. In particular, if Y is closed in X, then $d_Y(A) = d(A)$.

Lemma 5.6 *Let X be a T_d-space.*

(1) The following conditions are equivalent.
  (a) X is quasi-scattered.
  (b) Each nonempty subspace of $\overline{\mathrm{iso}(X)}$ contains an isolated point.
  (c) There is an ordinal α such that $d^\alpha(\overline{\mathrm{iso}(X)}) = \varnothing$.
  (d) $d(A \cap \overline{\mathrm{iso}(X)}) \subseteq d\big((A \cap \overline{\mathrm{iso}(X)}) - d(A \cap \overline{\mathrm{iso}(X)})\big)$ for each $A \subseteq X$.

(2) The following conditions are equivalent.
  (a) X is semi-scattered.
  (b) Each nonempty subspace of $\mathrm{int}(\overline{\mathrm{iso}(X)})$ contains an isolated point.
  (c) There is an ordinal α such that $d^\alpha\big(\mathrm{int}(\overline{\mathrm{iso}(X)})\big) \cap \mathrm{int}(\overline{\mathrm{iso}(X)}) = \varnothing$.
  (d) We have $d\big(A \cap \mathrm{int}(\overline{\mathrm{iso}(X)})\big) \cap \mathrm{int}(\overline{\mathrm{iso}(X)}) \subseteq d\big((A \cap \mathrm{int}(\overline{\mathrm{iso}(X)})) - d(A \cap \mathrm{int}(\overline{\mathrm{iso}(X)}))\big)$ for each $A \subseteq X$.

Proof It follows from Definition 5.5 and [10, Proposition 2.2] that for both (1) and (2) the first three conditions (a), (b), and (c) are equivalent. To see that (1.a)$\Longleftrightarrow$(1.d), recall from Esakia [14] (see also Bezhanishvili, Mines, and Morandi [9, Theorem 2.11]) that $\overline{\mathrm{iso}(X)}$ is scattered iff $d(B) \subseteq d(B - d(B))$ for each $B \subseteq \overline{\mathrm{iso}(X)}$, which is clearly equivalent to

$$d\big(A \cap \overline{\mathrm{iso}(X)}\big) \subseteq d\big((A \cap \overline{\mathrm{iso}(X)}) - d(A \cap \overline{\mathrm{iso}(X)})\big)$$

for each $A \subseteq X$. To see that (2.a)$\Longleftrightarrow$(2.d), in the proof of (1.a)$\Longleftrightarrow$(1.d), replace $\overline{\mathrm{iso}(X)}$ by $\mathrm{int}(\overline{\mathrm{iso}(X)})$, replace d by $d_{\mathrm{int}(\overline{\mathrm{iso}(X)})}$, and observe that

$$d_{\mathrm{int}(\overline{\mathrm{iso}(X)})}\big(A \cap \mathrm{int}(\overline{\mathrm{iso}(X)})\big) \subseteq d_{\mathrm{int}(\overline{\mathrm{iso}(X)})}\big((A \cap \mathrm{int}(\overline{\mathrm{iso}(X)})) - d_{\mathrm{int}(\overline{\mathrm{iso}(X)})}(A \cap \mathrm{int}(\overline{\mathrm{iso}(X)}))\big)$$

iff

$$d\big(A \cap \mathrm{int}(\overline{\mathrm{iso}(X)})\big) \cap \mathrm{int}(\overline{\mathrm{iso}(X)}) \subseteq d\big((A \cap \mathrm{int}(\overline{\mathrm{iso}(X)})) - d(A \cap \mathrm{int}(\overline{\mathrm{iso}(X)}))\big).$$

□

Definition 5.7 Let

$$\mathbf{qgl} = \Box\big(\Box(p \vee \Box^+\Diamond\top) \to (p \vee \Box^+\Diamond\top)\big) \to \Box(p \vee \Box^+\Diamond\top),$$

and let

$$\mathbf{sgl} = \Box\big(\Box(p \vee \Diamond^{+}\Box^{+}\Diamond\top) \to (p \vee \Diamond^{+}\Box^{+}\Diamond\top)\big)$$
$$\to \Box(p \vee \Diamond^{+}\Box^{+}\Diamond\top) \vee \Diamond^{+}\Box^{+}\Diamond\top.$$

Note that **qgl** is obtained from **gl** by substituting $p \vee \Box^{+}\Diamond\top$ for p. Therefore, **gl** $\vdash$ **qgl**. Also, substituting $p \vee \Diamond^{+}\Box^{+}\Diamond\top$ for p, we obtain **gl** $\vdash \Box(\Box(p \vee \Diamond^{+}\Box^{+}\Diamond\top) \to (p \vee \Diamond^{+}\Box^{+}\Diamond\top)) \to \Box(p \vee \Diamond^{+}\Box^{+}\Diamond\top) \vdash$ **sgl**. On the other hand, as we will see shortly, **qgl** $\nvdash$ **gl** and **sgl** $\nvdash$ **qgl**.

Lemma 5.8 *Let X be a T_d-space.*

*(1) X is quasi-scattered iff $X \models$ **qgl**.*
*(2) X is semi-scattered iff $X \models$ **sgl**.*

Proof (1) Observe that by substituting $\neg p$ for p in **qgl**, we obtain that **qgl** is equivalent to

$$\Diamond(p \wedge \Diamond^{+}\Box\bot) \to \Diamond\big((p \wedge \Diamond^{+}\Box\bot) \wedge \neg\Diamond(p \wedge \Diamond^{+}\Box\bot)\big).$$

Now as $\nu(\Diamond^{+}\Box\bot) = \overline{\mathrm{iso}(X)}$, we obtain

$$\begin{aligned} X \models \mathbf{qgl} \text{ iff } & X \models \Diamond(p \wedge \Diamond^{+}\Box\bot) \to \Diamond\big((p \wedge \Diamond^{+}\Box\bot) \wedge \neg\Diamond(p \wedge \Diamond^{+}\Box\bot)\big) \\ & \text{iff } d\left(A \cap \overline{\mathrm{iso}(X)}\right) \subseteq d\big((A \cap \overline{\mathrm{iso}(X)}) - d(A \cap \overline{\mathrm{iso}(X)})\big) \\ & \quad \text{for each } A \subseteq X \\ & \text{iff } X \text{ is a quasi-scattered space.} \end{aligned}$$

(2) Similarly, by substituting $\neg p$ for p in **sgl**, we obtain that **sgl** is equivalent to

$$\Diamond(p \wedge \Box^{+}\Diamond^{+}\Box\bot) \wedge \Box^{+}\Diamond^{+}\Box\bot \to \Diamond\big((p \wedge \Box^{+}\Diamond^{+}\Box\bot) \wedge \neg\Diamond(p \wedge \Box^{+}\Diamond^{+}\Box\bot)\big).$$

Now as $\nu(\Box^{+}\Diamond^{+}\Box\bot) = \mathrm{int}(\overline{\mathrm{iso}(X)})$, we obtain

$$\begin{aligned} X \models \mathbf{sgl} \text{ iff } & X \models \Diamond(p \wedge \Box^{+}\Diamond^{+}\Box\bot) \wedge \Box^{+}\Diamond^{+}\Box\bot \\ & \quad \to \Diamond\big((p \wedge \Box^{+}\Diamond^{+}\Box\bot) \wedge \neg\Diamond(p \wedge \Box^{+}\Diamond^{+}\Box\bot)\big) \\ & \text{iff } d\left(A \cap \mathrm{int}(\overline{\mathrm{iso}(X)})\right) \cap \mathrm{int}\left(\overline{\mathrm{iso}(X)}\right) \subseteq d\big((A \cap \mathrm{int}(\overline{\mathrm{iso}(X)})) \\ & \quad - d(A \cap \mathrm{int}(\overline{\mathrm{iso}(X)}))\big) \text{ for each } A \subseteq X \\ & \text{iff } X \text{ is a semi-scattered space.} \end{aligned}$$

□

Definition 5.9

(1) Let **qGL** = **K4** + **qgl**. We call **qGL** *quasi*-**GL**.
(2) Let **sGL** = **K4** + **sgl**. We call **sGL** *semi*-**GL**.

As an immediate consequence of Lemma 5.8, we obtain the following.

Corollary 5.10

(1) **qGL** *d-defines the class of quasi-scattered spaces.*
(2) **sGL** *d-defines the class of semi-scattered spaces.*

Next we characterize Kripke frames for **qGL** and **sGL**.

Lemma 5.11 *Let $\mathfrak{F} = (W, R)$ be a* **K4***-frame.*

*(1) $\mathfrak{F} \models$ **qgl** iff $(R^{+})^{-1}(\mathrm{imax}\, W)$ is dually well founded.*
*(2) $\mathfrak{F} \models$ **sgl** iff $(R^{+})^{-1}(\mathrm{imax}\, W) - R^{-1}(\mathrm{rmax}\, W)$ is dually well founded.*

Proof Let ν be a valuation into $\mathfrak{F}$.

(1) We have $\nu(\Diamond^+\Box\bot) = (R^+)^{-1}(\mathrm{imax}\, W)$. Therefore,

$$\begin{aligned}\mathfrak{F} \models \mathbf{qgl} \text{ iff } \mathfrak{F} &\models \Diamond(p \wedge \Diamond^+\Box\bot) \to \Diamond\big((p \wedge \Diamond^+\Box\bot) \wedge \neg\Diamond(p \wedge \Diamond^+\Box\bot)\big)\\ &\text{iff } (R^+)^{-1}(\mathrm{imax}\, W) \models \Diamond p \to \Diamond(p \wedge \neg\Diamond p)\\ &\text{iff } (R^+)^{-1}(\mathrm{imax}\, W) \models \mathbf{gl}\\ &\text{iff } (R^+)^{-1}(\mathrm{imax}\, W) \text{ is dually well founded.}\end{aligned}$$

(2) We have $\nu(\Box^+\Diamond^+\Box\bot) = W - (R^+)^{-1}(W - (R^+)^{-1}(\mathrm{imax}\, W))$. Observe that

$$\begin{aligned}w \in W &- (R^+)^{-1}\big(W - (R^+)^{-1}(\mathrm{imax}\, W)\big)\\ &\text{iff } R^+(w) \subseteq (R^+)^{-1}(\mathrm{imax}\, W)\\ &\text{iff } w \in (R^+)^{-1}(\mathrm{imax}\, W) - R^{-1}(\mathrm{rmax}\, W).\end{aligned}$$

Therefore, $W - (R^+)^{-1}(W - (R^+)^{-1}(\mathrm{imax}\, W)) = (R^+)^{-1}(\mathrm{imax}\, W) - R^{-1}(\mathrm{rmax}\, W)$, and so $\nu(\Box^+\Diamond^+\Box\bot) = (R^+)^{-1}(\mathrm{imax}\, W) - R^{-1}(\mathrm{rmax}\, W)$. Thus,

$$\begin{aligned}\mathfrak{F} \models \mathbf{sgl} \text{ iff } \mathfrak{F} &\models \Diamond(p \wedge \Box^+\Diamond^+\Box\bot) \wedge \Box^+\Diamond^+\Box\bot\\ &\quad \to \Diamond\big((p \wedge \Box^+\Diamond^+\Box\bot) \wedge \neg\Diamond(p \wedge \Box^+\Diamond^+\Box\bot)\big)\\ &\text{iff } (R^+)^{-1}(\mathrm{imax}\, W) - R^{-1}(\mathrm{rmax}\, W) \models \Diamond p \to \Diamond(p \wedge \neg\Diamond p)\\ &\text{iff } (R^+)^{-1}(\mathrm{imax}\, W) - R^{-1}(\mathrm{rmax}\, W) \models \mathbf{gl}\\ &\text{iff } (R^+)^{-1}(\mathrm{imax}\, W) - R^{-1}(\mathrm{rmax}\, W) \text{ is dually well founded.}\end{aligned}$$

$\Box$

Corollary 5.12 *Let $\mathfrak{F} = (W, R)$ be a finite* **K4***-frame.*

*(1) $\mathfrak{F} \models$ **qgl** iff $(R^+)^{-1}(\mathrm{imax}\, W)$ is irreflexive.*
*(2) $\mathfrak{F} \models$ **sgl** iff $(R^+)^{-1}(\mathrm{imax}\, W) - R^{-1}(\mathrm{rmax}\, W)$ is irreflexive.*

Our next goal is to show that both **qGL** and **sGL** have the FMP with respect to their Kripke semantics. For this it is sufficient to show that both **qGL** and **sGL** are cofinal subframe logics and then apply [12, Theorem 11.20]. Let $\mathfrak{F} = (W, R)$ be a **K4**-frame. For a nonempty subset S of W, let R_S be the restriction of R to S. Then $\mathfrak{S} = (S, R_S)$ is also a **K4**-frame, called a *subframe* of $\mathfrak{F}$ (see [12, pp. 28, 65]); if in addition $R(S) \subseteq (R^+)^{-1}(S)$, then $\mathfrak{S}$ is called a *cofinal subframe* of $\mathfrak{F}$ (see [12, p. 292]). As follows from [12, Theorem 11.21], a logic L over **K4** is a subframe logic iff it is complete with respect to a class of **K4**-frames closed under subframes, and L is a cofinal subframe logic iff it is complete with respect to a class of **K4**-frames closed under cofinal subframes. It is a relatively easy consequence of Lemma 5.11 that the classes of all **qGL**-frames and of all **sGL**-frames are closed under cofinal subframes. However, proving that both **qGL** and **sGL** are complete is more of a challenge. This forces us to work with descriptive **K4**-frames instead.

Let X be a topological space. We recall that a subset U of X is *clopen* if U is closed and open, that X is *zero-dimensional* if clopen subsets of X form a basis, and that X is a *Stone space* if X is a zero-dimensional compact Hausdorff space. We also recall that a pair $\mathfrak{X} = (X, R)$ is a *descriptive* **K4***-frame* if X is a Stone space, (X, R) is a **K4**-frame, $R(x)$ is closed for each $x \in X$, and $R^{-1}(U)$ is clopen for each clopen U of X (for an equivalent definition of descriptive frames that does not use topology see [12, Chapter 8]).

Let $\mathfrak{X} = (X, R)$ be a descriptive **K4**-frame. As follows from Abashidze [1, Section 4] (see also Beklemishev, Bezhanishvili, and Icard [2, Section 2.2]), $\mathfrak{X} \models \mathbf{gl}$ iff $\max U = \operatorname{imax} U$ for each clopen U of X.

Lemma 5.13 *Let $\mathfrak{X} = (X, R)$ be a descriptive* **K4***-frame.*

(1) $\mathfrak{X} \models \mathbf{qgl}$ iff $\max U = \operatorname{imax} U$ for each clopen U of the subspace $(R^+)^{-1}(\operatorname{imax} X)$.

(2) $\mathfrak{X} \models \mathbf{sgl}$ iff $\max U = \operatorname{imax} U$ for each clopen U of the subspace $(R^+)^{-1}(\operatorname{imax} X) - R^{-1}(\operatorname{rmax} X)$.

Proof (1) We have $\mathfrak{X} \models \mathbf{qgl}$ iff $(R^+)^{-1}(\operatorname{imax} X) \models \mathbf{gl}$ iff $\max U = \operatorname{imax} U$ for each clopen U of the subspace $(R^+)^{-1}(\operatorname{imax} X)$.

(2) We have $\mathfrak{X} \models \mathbf{sgl}$ iff $(R^+)^{-1}(\operatorname{imax} X) - R^{-1}(\operatorname{rmax} X) \models \mathbf{gl}$ iff $\max U = \operatorname{imax} U$ for each clopen U of the subspace $(R^+)^{-1}(\operatorname{imax} X) - R^{-1}(\operatorname{rmax} X)$. □

Let $\mathfrak{X} = (X, R)$ be a descriptive **K4**-frame, and let S be a nonempty clopen subset of X. Then S is a Stone space in the subspace topology. Let R_S be the restriction of R to S. It is well known that $\mathfrak{S} = (S, R_S)$ is also a descriptive **K4**-frame. We call $\mathfrak{S} = (S, R_S)$ a *subframe* of $\mathfrak{X}$. Also, we call $\mathfrak{S}$ a *cofinal subframe* of $\mathfrak{X}$ if $R(S) \subseteq (R^+)^{-1}(S)$.

Lemma 5.14 *Let $\mathfrak{X} = (X, R)$ be a descriptive* **K4***-frame, and let $\mathfrak{S} = (S, S_R)$ be a cofinal subframe of $\mathfrak{X}$. Then $\operatorname{imax} S \subseteq \operatorname{imax} X$.*

Proof Let $x \in \operatorname{imax}(S)$. We recall that for each $x \in X$, either $x \in \operatorname{imax}(X)$ or there exists $y \in \max(X)$ such that xRy. Suppose that $x \notin \operatorname{imax}(X)$. Then there exists $y \in \max(X)$ such that xRy. Since S is cofinal and $y \in R(S)$, there is $z \in S$ such that yR^+z. Therefore, $xRyRz$ or $xRy = z$. As R is transitive, in either case we have xRz. Thus, $x \notin \operatorname{imax}(S)$, a contradiction. Consequently, $x \in \operatorname{imax}(X)$, and so $\operatorname{imax}(S) \subseteq \operatorname{imax}(X)$. □

Lemma 5.15

(1) A cofinal subframe of a descriptive **qGL***-frame is also a descriptive* **qGL***-frame.*

(2) A cofinal subframe of a descriptive **sGL***-frame is also a descriptive* **sGL***-frame.*

Proof (1) Suppose that $\mathfrak{X} = (X, R)$ is a descriptive **qGL**-frame and that $\mathfrak{S} = (S, R_S)$ is a cofinal subframe of $\mathfrak{X}$. Let U be a clopen subset of $(R^+)^{-1}(\operatorname{imax} S)$. By Lemma 5.14, $\operatorname{imax}(S) \subseteq \operatorname{imax}(X)$, and so $U \subseteq (R^+)^{-1}(\operatorname{imax} X)$. Since $\mathfrak{X}$ is a descriptive **qGL**-frame, by Lemma 5.13(1), $\max U = \operatorname{imax} U$. Therefore, $\mathfrak{S}$ is a descriptive **qGL**-frame.

(2) Suppose that $\mathfrak{X} = (X, R)$ is a descriptive **sGL**-frame and that $\mathfrak{S} = (S, R_S)$ is a cofinal subframe of $\mathfrak{X}$. Let U be a clopen subset of $((R^+)^{-1}(\operatorname{imax} S) - R^{-1}(\operatorname{rmax} S)) \cap S$. We show that $((R^+)^{-1}(\operatorname{imax} S) - R^{-1}(\operatorname{rmax} S)) \cap S \subseteq (R^+)^{-1}(\operatorname{imax} X) - R^{-1}(\operatorname{rmax} X)$. Let $x \in ((R^+)^{-1}(\operatorname{imax} S) - R^{-1}(\operatorname{rmax} S)) \cap S$. Since S is a cofinal subframe of X, we have $R^+(x) \subseteq (R^+)^{-1}(\operatorname{imax} S)$. Therefore, by Lemma 5.14, $R^+(x) \subseteq (R^+)^{-1}(\operatorname{imax} X)$. Thus, $x \in (R^+)^{-1}(\operatorname{imax} X) - R^{-1}(\operatorname{rmax} X)$, and so $((R^+)^{-1}(\operatorname{imax} S) - R^{-1}(\operatorname{rmax} S)) \cap S \subseteq (R^+)^{-1}(\operatorname{imax} X) - R^{-1}(\operatorname{rmax} X)$. Consequently, $U \subseteq (R^+)^{-1}(\operatorname{imax} X) - R^{-1}(\operatorname{rmax} X)$. As $\mathfrak{X}$ is a descriptive **sGL**-frame, by Lemma 5.13(2), $\max U = \operatorname{imax} U$. It follows that $\mathfrak{S}$ is a descriptive **sGL**-frame. □

Theorem 5.16

(1) **qGL** *is a cofinal subframe logic over* **K4**. *Consequently,* **qGL** *has the FMP with respect to its Kripke semantics, and is decidable.*

(2) **sGL** *is a cofinal subframe logic. Consequently,* **sGL** *has the FMP with respect to its Kripke semantics, and is decidable.*

Proof It follows from the characterization of subframe and cofinal subframe logics (see Wolter [25, Sections 2.2, 2.3]; see also Wolter [24, Section 3]) and the duality between modal algebras and descriptive frames that a modal logic L is a subframe logic iff the class of descriptive frames of L is closed under subframes and that L is a cofinal subframe logic iff the class of descriptive frames of L is closed under cofinal subframes. Therefore, by Lemma 5.15, both **qGL** and **sGL** are cofinal subframe logics over **K4**. That both **qGL** and **sGL** have the FMP with respect to their Kripke semantics now follows from [12, Theorem 11.20] (for a different proof see Bezhanishvili, Ghilardi, and Jibladze [7]). As a result, since both **qGL** and **sGL** are finitely axiomatizable, both **qGL** and **sGL** are decidable. □

On the other hand, we show that neither **qGL** nor **sGL** is a subframe logic. Let $\mathfrak{F} = (W, R)$ be the **K4**-frame shown in Figure 6, where w_0 and w_2 are reflexive points and w_1 is an irreflexive point. It is easy to see that $\mathfrak{F}$ is both a **qGL**-frame and an **sGL**-frame. Let $S = \{w_0, w_1\}$. Then $\mathfrak{S} = (S, R_S)$ is a subframe of $\mathfrak{F}$ which is neither a **qGL**-frame nor an **sGL**-frame. Consequently, neither **qGL** nor **sGL** is a subframe logic. This is in contrast with **GL**, which is well known to be a subframe logic. (Note that **wGL** is also a cofinal subframe logic that is not a subframe logic.)

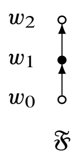

Figure 6

Theorem 5.17

(1) $\mathbf{K4} \subset \mathbf{sGL} \subset \mathbf{qGL} \subset \mathbf{GL}$.

(2) $\mathbf{qGL} = \bigcap_\omega \mathbf{K4\Delta}_n$ *and* $\mathbf{sGL} = \bigcap_\omega \mathbf{K4\Xi}_n$.

(3) **qGL** *is the d-logic of quasi-scattered spaces. In fact,* **qGL** *is the d-logic of a quasi-scattered subspace of* $\mathbb{Q}$.

(4) **sGL** *is the d-logic of semi-scattered spaces. In fact,* **sGL** *is the d-logic of a semi-scattered subspace of* $\mathbb{Q}$.

(5) *Neither* **qGL** *nor* **sGL** *is comparable with either* **wGL**, $\mathbf{K4\Sigma}_n$, *or* $\mathbf{K4\Theta}_n$ *for* $n > 0$. *Also,* **GL** *is not comparable with either* $\mathbf{K4\Sigma}_n$ *or* $\mathbf{K4\Theta}_n$ *for* $n > 0$.

(6) $\mathbf{GL} = \mathbf{wGL} \vee \mathbf{sGL} = \mathbf{wGL} \vee \mathbf{qGL}$.

(7) $\mathbf{K4\Delta}_n = \mathbf{qGL} \vee \mathbf{K4\Theta}_n$.

(8) $\mathbf{wGL} \vee \mathbf{K4\Xi}_n = \mathbf{wGL} \vee \mathbf{K4\Delta}_n = \mathbf{K4\Gamma}_n$, $\mathbf{sGL} \vee \mathbf{K4\Sigma}_n = \mathbf{K4\Xi}_n$, *and* $\mathbf{qGL} \vee \mathbf{K4\Sigma}_n = \mathbf{qGL} \vee \mathbf{K4\Xi}_n$.

Proof (1) Obviously $\mathbf{K4} \subseteq \mathbf{sGL}$. The frame $\mathfrak{G}_1$ in Figure 5 is a $\mathbf{K4}$-frame which is not an $\mathbf{sGL}$-frame. Therefore, the containment is strict. Let $\mathfrak{F} = (W, R)$ be a finite $\mathbf{qGL}$-frame. Then $(R^+)^{-1}(\operatorname{imax} W)$ is irreflexive. Thus, $(R^+)^{-1}(\operatorname{imax} W) - R^{-1}(\operatorname{rmax} W)$ is also irreflexive. Consequently, $\mathfrak{F}$ is a $\mathbf{sGL}$-frame, and so $\mathbf{sGL} \subseteq \mathbf{qGL}$. The containment is strict because the frame $\mathfrak{K}_2$ in Figure 5 is an $\mathbf{sGL}$-frame which is not a $\mathbf{qGL}$-frame. Clearly each finite $\mathbf{GL}$-frame is a $\mathbf{qGL}$-frame. Therefore, $\mathbf{qGL} \subseteq \mathbf{GL}$. The containment is strict because any finite reflexive $\mathbf{K4}$-frame is a $\mathbf{qGL}$-frame which is not a $\mathbf{GL}$-frame.

(2) First we show that $\mathbf{qGL} = \bigcap_\omega \mathbf{K4\Delta}_n$. For each $n \in \omega$, it is obvious that a finite $\mathbf{K4\Delta}_n$-frame is a $\mathbf{qGL}$-frame. Consequently, $\mathbf{qGL} \subseteq \bigcap_\omega \mathbf{K4\Delta}_n$. Conversely, if $\mathbf{qGL} \nvdash \varphi$, then by Theorem 5.16(1), there exists a finite $\mathbf{qGL}$-frame $\mathfrak{F}$ refuting φ. Clearly $\mathfrak{F}$ is n-top-deep for some $n \in \omega$. Therefore, $\mathbf{K4\Delta}_n \nvdash \varphi$, and so $\mathbf{qGL} = \bigcap_\omega \mathbf{K4\Delta}_n$.

Next we show that $\mathbf{sGL} = \bigcap_\omega \mathbf{K4\Xi}_n$. Let $\mathfrak{F} = (W, R)$ be a finite $\mathbf{K4\Xi}_n$-frame. Then $R^{-n}(\operatorname{imax} W) \subseteq R^{-1}(\operatorname{rmax} W)$. If $(R^+)^{-1}(\operatorname{imax} W) - R^{-1}(\operatorname{rmax} W)$ is not irreflexive, then there exists $w \in (R^+)^{-1}(\operatorname{imax} W) - R^{-1}(\operatorname{rmax} W)$ such that wRw. Since $w \in (R^+)^{-1}(\operatorname{imax} W)$ and wRw, we have $w \in R^{-n}(\operatorname{imax} W)$. But then $w \in R^{-1}(\operatorname{rmax} W)$, a contradiction. Therefore, $(R^+)^{-1}(\operatorname{imax} W) - R^{-1}(\operatorname{rmax} W)$ is irreflexive, and so $\mathfrak{F} \models \mathbf{sgl}$. Consequently, $\mathbf{sGL} \subseteq \mathbf{K4\Xi}_n$ for each $n \geq 0$, and so $\mathbf{sGL} \subseteq \bigcap_\omega \mathbf{K4\Xi}_n$. For the converse, suppose that $\mathbf{sGL} \nvdash \varphi$. By Theorem 5.16(2), there exists a finite $\mathbf{sGL}$-frame $\mathfrak{F} = (W, R)$ refuting φ. As $(R^+)^{-1}(\operatorname{imax} W) - R^{-1}(\operatorname{rmax} W)$ is irreflexive, there is $n \in \omega$ such that $R^{-n}(\operatorname{imax} W) \subseteq R^{-1}(\operatorname{rmax} W)$. Thus, $\mathfrak{F}$ is a $\mathbf{K4\Xi}_n$-frame, and so $\mathbf{K4\Xi}_n \nvdash \varphi$. Consequently, $\mathbf{sGL} = \bigcap_\omega \mathbf{K4\Xi}_n$.

(3) Apply Lemma 5.8(1), (2), and part (2) of the Main Theorem.

(4) Apply Lemma 5.8(2), (2), and part (2) of the Main Theorem.

(5) The frame $\mathfrak{G}_1$ in Figure 5 refutes $\mathbf{sgl}$, but validates $\mathbf{wgl}$, θ_n, and σ_n for $n > 0$. Therefore, $\mathbf{sGL} \nsubseteq \mathbf{wGL}, \mathbf{K4\Sigma}_n, \mathbf{K4\Theta}_n$ for $n > 0$. Consequently, $\mathbf{qGL} \nsubseteq \mathbf{wGL}, \mathbf{K4\Sigma}_n, \mathbf{K4\Theta}_n$ and $\mathbf{GL} \nsubseteq \mathbf{K4\Sigma}_n, \mathbf{K4\Theta}_n$ for $n > 0$. The frame $\mathfrak{H}_2$ in Figure 5 is a $\mathbf{qGL}$-frame which is not a $\mathbf{wGL}$-frame. Therefore, $\mathbf{wGL} \nsubseteq \mathbf{qGL}$, and so $\mathbf{wGL} \nsubseteq \mathbf{sGL}$. The frame $\mathfrak{F}_n$ in Figure 5 is a $\mathbf{GL}$-frame which is not n-serial. It follows that $\mathbf{K4\Sigma}_n \nsubseteq \mathbf{GL}$. Thus, we also have $\mathbf{K4\Sigma}_n \nsubseteq \mathbf{qGL}, \mathbf{sGL}$ and $\mathbf{K4\Theta}_n \nsubseteq \mathbf{GL}, \mathbf{qGL}, \mathbf{sGL}$. Consequently, neither $\mathbf{qGL}$ nor $\mathbf{sGL}$ is comparable with either $\mathbf{wGL}$, $\mathbf{K4\Sigma}_n$, or $\mathbf{K4\Theta}_n$ for $n > 0$. Also, $\mathbf{GL}$ is not comparable with either $\mathbf{K4\Sigma}_n$ or $\mathbf{K4\Theta}_n$ for $n > 0$.

(6) Since $\mathbf{wGL} \subset \mathbf{GL}$ and $\mathbf{sGL} \subset \mathbf{qGL} \subset \mathbf{GL}$, it follows that $\mathbf{wGL} \vee \mathbf{sGL} \subseteq \mathbf{wGL} \vee \mathbf{qGL} \subseteq \mathbf{GL}$. Conversely, as $\mathbf{wGL} \vdash \Diamond^+\Box\bot$, we also have $\mathbf{wGL} \vdash \Box^+\Diamond^+\Box\bot$. Therefore, since

$$
\begin{aligned}
\mathbf{sGL} \vdash \Diamond\,(p \wedge \Box^+\Diamond^+\Box\bot) \wedge \Box^+\Diamond^+\Box\bot \\
\to \Diamond\big((p \wedge \Box^+\Diamond^+\Box\bot) \wedge \neg\Diamond(p \wedge \Box^+\Diamond^+\Box\bot)\big),
\end{aligned}
$$

we have

$$\mathbf{wGL} \vee \mathbf{sGL} \vdash \Diamond p \to \Diamond(p \wedge \neg\Diamond p).$$

Thus, $\mathbf{wGL} \vee \mathbf{sGL} \vdash \mathbf{gl}$, so $\mathbf{GL} \subseteq \mathbf{wGL} \vee \mathbf{sGL}$, and so $\mathbf{GL} = \mathbf{wGL} \vee \mathbf{sGL} = \mathbf{wGL} \vee \mathbf{qGL}$.

(7) Since $\mathbf{qGL}, \mathbf{K4\Theta}_n \subseteq \mathbf{K4\Delta}_n$, it follows that $\mathbf{qGL} \vee \mathbf{K4\Theta}_n \subseteq \mathbf{K4\Delta}_n$. Conversely, we show that $\mathbf{qGL} \vee \mathbf{K4\Theta}_n \vdash \delta_n$. First note that θ_n is equivalent

to $\Diamond^n\Box\bot \to \Diamond^{n+1}\Box\bot$, and if $\mathbf{qGL} \vdash p \to \Diamond^+\Box\bot$, then $\mathbf{qGL} \vdash \Diamond p \to \Diamond(p \wedge \neg\Diamond p)$. Next, as $\mathbf{K4} \vdash \Diamond^n\Box\bot \to \Diamond^+\Box\bot$, we have

$$\mathbf{qGL} \vdash \Diamond(\Diamond^n\Box\bot) \to \Diamond\big(\Diamond^n\Box\bot \wedge \neg\Diamond(\Diamond^n\Box\bot)\big).$$

Therefore,

$$\mathbf{qGL} \vdash \Diamond^{n+1}\Box\bot \to \Diamond(\Diamond^n\Box\bot \wedge \neg\Diamond^{n+1}\Box\bot).$$

Since $\mathbf{K4\Theta}_n \vdash \Diamond^n\Box\bot \to \Diamond^{n+1}\Box\bot$, we obtain

$$\begin{aligned}\mathbf{qGL} \vee \mathbf{K4\Theta}_n &\vdash \Diamond^{n+1}\Box\bot \to \Diamond(\Diamond^{n+1}\Box\bot \wedge \neg\Diamond^{n+1}\Box\bot)\\ &\vdash \Diamond^{n+1}\Box\bot \to \Diamond\bot.\end{aligned}$$

As $\mathbf{K4} \vdash \Diamond\bot \to \bot$, we have

$$\begin{aligned}\mathbf{qGL} \vee \mathbf{K4\Theta}_n &\vdash \Diamond^{n+1}\Box\bot \to \bot\\ &\vdash \neg\Diamond^{n+1}\Box\bot\\ &\vdash \Box^{n+1}\Diamond\top\\ &\vdash \delta_{n+1}.\end{aligned}$$

But $\mathbf{K4\Theta}_n \vdash \delta_{n+1} \to \delta_n$. Therefore, $\mathbf{qGL} \vee \mathbf{K4\Theta}_n \vdash \delta_n$. Thus, $\mathbf{K4\Delta}_n \subseteq \mathbf{qGL} \vee \mathbf{K4\Theta}_n$, and so $\mathbf{K4\Delta}_n = \mathbf{qGL} \vee \mathbf{K4\Theta}_n$.

(8) First we show that $\mathbf{wGL} \vee \mathbf{K4\Xi}_n = \mathbf{wGL} \vee \mathbf{K4\Delta}_n = \mathbf{K4\Gamma}_n$. Since $\mathbf{K4\Xi}_n \subseteq \mathbf{K4\Delta}_n$, we have $\mathbf{wGL} \vee \mathbf{K4\Xi}_n \subseteq \mathbf{wGL} \vee \mathbf{K4\Delta}_n$. Conversely, let $\mathfrak{F} = (W, R)$ be a finite n-semi-top-deep $\mathbf{wGL}$-frame. Then $R^{-n}(\operatorname{imax} W) \subseteq R^{-1}(\operatorname{rmax} W) = R^{-1}(\varnothing) = \varnothing$. Therefore, $\mathfrak{F}$ is a $\mathbf{K4\Delta}_n$-frame. Thus, $\mathbf{wGL} \vee \mathbf{K4\Delta}_n \subseteq \mathbf{wGL} \vee \mathbf{K4\Xi}_n$, and so the equality holds. Now apply Theorem 5.4(8), by which $\mathbf{wGL} \vee \mathbf{K4\Delta}_n = \mathbf{K4\Gamma}_n$.

Next we show that $\mathbf{sGL} \vee \mathbf{K4\Sigma}_n = \mathbf{K4\Xi}_n$. Clearly $\mathbf{sGL} \vee \mathbf{K4\Sigma}_n \subseteq \mathbf{K4\Xi}_n$ since both $\mathbf{sGL}$ and $\mathbf{K4\Sigma}_n$ are contained in $\mathbf{K4\Xi}_n$. Suppose that $\mathfrak{F} = (W, R)$ is a finite n-serial $\mathbf{sGL}$-frame. Let $w \in R^{-n}(\operatorname{imax} W)$. Then $w \in (R^+)^{-1}(\operatorname{imax} W)$. If $w \notin R^{-1}(\operatorname{rmax} W)$, then $w \in (R^+)^{-1}(\operatorname{imax} W) - R^{-1}(\operatorname{rmax} W)$. Therefore, $R^{-n}(\operatorname{imax} W) \cap ((R^+)^{-1}(\operatorname{imax} W) - R^{-1}(\operatorname{rmax} W)) \neq \varnothing$. On the other hand, as $(R^+)^{-1}(\operatorname{imax} W) - R^{-1}(\operatorname{rmax} W)$ is irreflexive and $\mathfrak{F}$ is n-serial, we have $R^{-n}(\operatorname{imax} W) \cap ((R^+)^{-1}(\operatorname{imax} W) - R^{-1}(\operatorname{rmax} W)) = \varnothing$. The obtained contradiction proves that $w \in R^{-1}(\operatorname{rmax} W)$, and so $\mathfrak{F} \vDash \xi_n$. It follows that $\mathbf{K4\Xi}_n \subseteq \mathbf{sGL} \vee \mathbf{K4\Sigma}_n$; hence the equality.

Lastly, $\mathbf{qGL} \vee \mathbf{K4\Xi}_n = \mathbf{qGL} \vee (\mathbf{sGL} \vee \mathbf{K4\Sigma}_n) = (\mathbf{qGL} \vee \mathbf{sGL}) \vee \mathbf{K4\Sigma}_n = \mathbf{qGL} \vee \mathbf{K4\Sigma}_n$. $\Box$

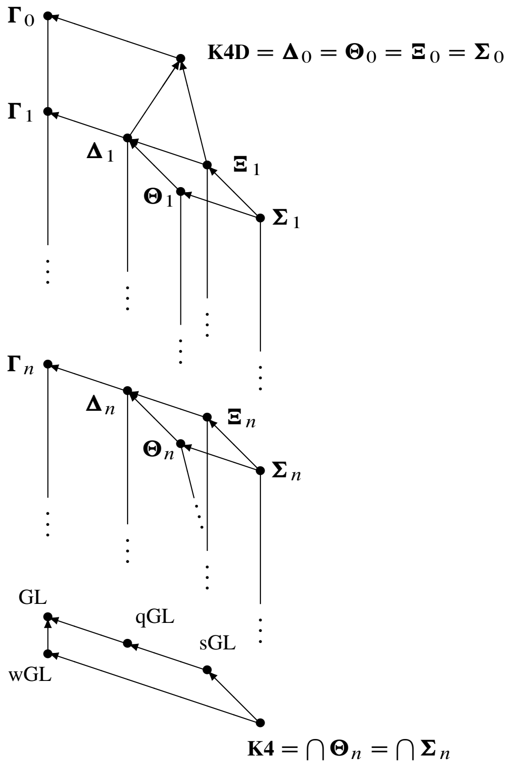

Figure 7

We conclude with Figure 7 depicting the inclusion relationships between the logics introduced in Sections 4 and 5. Each logic of the form $\mathbf{K4}*$ is labeled by $*$ for readability purposes. Also, the arrows indicate only the inclusion relation, and as such, one should be careful when examining meets and joins in Figure 7. For example, $\mathbf{K4\Theta}_n \vee \mathbf{K4\Xi}_n \neq \mathbf{K4\Delta}_n$ and $\mathbf{K4\Theta}_n \wedge \mathbf{K4\Xi}_n \neq \mathbf{K4\Sigma}_n$, while $\mathbf{wGL} \vee \mathbf{sGL} = \mathbf{wGL} \vee \mathbf{qGL} = \mathbf{GL}$.

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