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The K-Degrees, Low for K Degrees, and Weakly Low for K Sets

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Abstract We call A *weakly low for* K if there is a c such that $K^A(\sigma) \geq K(\sigma) - c$ for infinitely many σ; in other words, there are infinitely many strings that A does not help compress. We prove that A is weakly low for K if and only if Chaitin's Ω is A-random. This has consequences in the K-degrees and the low for K (i.e., low for random) degrees. Furthermore, we prove that the initial segment prefix-free complexity of 2-random reals is infinitely often maximal. This had previously been proved for plain Kolmogorov complexity.

1 Introduction

If $A \in 2^\omega$ is 1-random, then there is a connection between the degree of randomness of A, the prefix-free (Kolmogorov) complexity of initial segments of A, and the (lack of) power of A as an oracle. We explore some aspects of this connection. See Section 2 for a brief introduction to effective randomness.

We say that $A \in 2^\omega$ is *weakly low for* K if $(\exists c)(\exists^\infty n)\ K(n) \leq K^A(n) + c$. Making use of $\leq^+$ to indicate a suppressed additive constant, we can write this as $(\exists^\infty n)\ K(n) \leq^+ K^A(n)$. Nies, Stephan, and Terwijn [22] call A *low for* Ω if Chaitin's Ω is A-random. In Section 3, we show that A is weakly low for K if and only if it is low for Ω. This result is analogous to a celebrated result of Nies. Call $A \in 2^\omega$ *low for* K if $K(\sigma) \leq^+ K^A(\sigma)$ and *low for* 1*-random* if every 1-random is A-random. Nies [21] proved that these two notions—each stating that A is useless as an oracle in a specific context—are equivalent.

The equivalence of weakly low for K and low for Ω has a variety of consequences. In Section 4, we use it to prove that the initial segment prefix-free complexity of 2-random reals is infinitely often maximal. This had previously been proved for plain Kolmogorov complexity [22; 16].

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Section 5 looks at consequences in the *LR*/*LK*-degrees. Nies partially relativized the notions of low for K and low for 1-random to introduce two ways of comparing the power of oracles in the context of effective randomness. He defined $X \leq_{LK} Y$ to mean that $K^Y(\sigma) \leq^+ K^X(\sigma)$, and $X \leq_{LR} Y$ to mean that every Y-random is X-random. These partial orders induce the *low for K degrees* and the *low for random degrees*, respectively. They turn out to be the same. It is clear that $X \leq_{LK} Y$ implies $X \leq_{LR} Y$; Kjos-Hanssen, Miller, and Solomon [12] proved the converse. Note that this extends the result of Nies, since X is low for K if and only if $X \leq_{LK} \varnothing$ and X is low for 1-random if and only if $X \leq_{LR} \varnothing$.

We prove that if $X \leq_{LR} Y$ and Y is low for Ω, then $X \leq_T Y'$. Thus, if Y is low for Ω, it has countably many predecessors in the *LR*-degrees; the converse is open. It also follows that if X and Y are 2-random relative to each other, then they form a minimal pair in the *LR*-degrees.

In Section 6, we consider the K-degrees. Downey, Hirschfeldt, and LaForte [7; 8] defined $X \leq_K Y$ to mean that $K(X \upharpoonright n) \leq^+ K(Y \upharpoonright n)$. In other words, Y has higher initial segment prefix-free complexity than X, up to a constant. The induced partial order is called the *K-degrees*. If higher complexity implies more randomness, then one can interpret $X \leq_K Y$ as saying that Y is more random than X.

We prove that if X is 1-random, then prefix-free complexity relative to X can be expressed in terms of the prefix-free complexity of initial segments of X. In particular, $K^X(\sigma) =^+ \min_{s\in\omega} K(X \upharpoonright \langle\sigma, s\rangle) - \langle\sigma, s\rangle$. This implies that if X is 1-random and $X \leq_K Y$, then $Y \leq_{LK} X$. Note that this result is not new; it follows from the corresponding result for the *LR*-degrees [19] and the equivalence between $\leq_{LR}$ and $\leq_{LK}$. As a corollary to the work of Section 5, the cones above 2-random reals in the K-degrees are countable. This is not true for all 1-random reals.

2 Preliminaries

We assume that the reader is familiar with basic computability (recursion) theory, as would be found in Part I of Soare [25]. We give a quick introduction to effective randomness, touching on the definitions and results needed in this paper. For a more thorough introduction, see Li and Vitányi [14], Nies [20], or the upcoming monograph of Downey and Hirschfeldt [6].

By *strings* we refer to elements of $2^{<\omega}$. We identify strings with natural numbers using an effective bijection; for concreteness, identify $\sigma \in 2^{<\omega}$ with $n \in \omega$ if 1σ is the binary expansion of $n+1$. We call elements of 2^ω *reals* and abuse notation by conflating $X \in 2^\omega$ with the element of $[0, 1]$ that has binary expansion $0.X$. The nonuniqueness of binary expansion will not be an issue below.

For $\sigma \in 2^{<\omega}$, let $[\sigma] = \{X \in 2^\omega : \sigma \prec X\}$, that is, the set of reals extending σ. If $S \subseteq 2^\omega$ is a c.e. set, then $\bigcup_{\sigma\in S}[\sigma]$ is called a Σ^0_1 class. The complement of a Σ^0_1 class is called a Π^0_1 class. A *Martin-Löf test* is a uniform sequence $\{V_n\}_{n\in\omega}$ of Σ^0_1 classes such that $\mu(V_n) \leq 2^{-n}$, where μ is the standard Lebesgue measure on 2^ω. A real $X \in 2^\omega$ is said to *pass* a Martin-Löf test $\{V_n\}_{n\in\omega}$ if $X \notin \bigcap_{n\in\omega} V_n$. We say that $X \in 2^\omega$ is 1-*random* (or *Martin-Löf random*) if it passes all Martin-Löf tests. There is a *universal* Martin-Löf test, that is, a single test $\{U_n\}_{n\in\omega}$ that is passed only by the 1-random reals.

Define A-*randomness* by relativizing Martin-Löf's definition to an oracle $A \in 2^\omega$. An essential tool in understanding relativized randomness is van Lambalgen's theorem [27]: $X \oplus Y$ is 1-random if and only if X is 1-random and Y is X-random. Note that by applying van Lambalgen's theorem twice, we can show that if X and Y are both 1-random, then X is Y-random if and only if Y is X-random. We call X n-*random* if it is $\varnothing^{(n-1)}$-random.

There is an important connection between the randomness of real numbers and the complexity, or information content, of finite binary strings. A set $D \subseteq 2^{<\omega}$ is *prefix-free* if no element of D is a proper prefix of another element. A *prefix-free machine* is a partial computable function $M: 2^{<\omega} \to 2^{<\omega}$ with prefix-free domain. If M is a prefix-free machine, then let

$$K_M(\sigma) = \min\{|\tau|: M(\tau) = \sigma\},$$

so $K_M(\sigma)$ is the length of the shortest M-description of σ. There is a *universal prefix-free machine* $U: 2^{<\omega} \to 2^{<\omega}$ that is optimal for prefix-free machines: if M is any such machine, then $K_U(\sigma) \leq^+ K_M(\sigma)$. We write $K(\sigma)$ for $K_U(\sigma)$ and call it the *prefix-free* (*Kolmogorov*) *complexity* of $\sigma \in 2^{<\omega}$. Plain Kolmogorov complexity C is defined in the same way as prefix-free complexity except without restricting the domains of machines. It is well known that the 1-random reals can be characterized in terms of the prefix-free complexity of their initial segments. Schnorr proved that $X \in 2^\omega$ is 1-random if and only if $K(X \upharpoonright n) \geq^+ n$.

Since U has prefix-free domain, $\sum_{\sigma\in 2^{<\omega}} 2^{-K(\sigma)} \leq \sum_{\tau\in \operatorname{dom} U} 2^{-|\tau|} \leq 1$; this is called *Kraft's inequality*.[1] It has a useful effective converse. A *Kraft-Chaitin set* is a c.e. set $W \subseteq \omega \times 2^{<\omega}$ such that $\sum_{\langle d,\sigma\rangle\in W} 2^{-d} \leq 1$. Given such a set, the Kraft-Chaitin theorem says that there is a prefix-free machine M such that $\langle d, \sigma\rangle \in W$ implies that $K_M(\sigma) \leq d$. Thus, $K(\sigma) \leq^+ d$ for all $\langle d, \sigma\rangle \in W$. Closely related to the Kraft-Chaitin theorem is the fact that K is an optimal information content measure. A function $\widehat{K}: \omega \to \mathbb{R}\cup\{\infty\}$ is an *information content measure* if $\sum_{n\in\omega} 2^{-\widehat{K}(n)} \leq 1$ and $\{\langle k, n\rangle: \widehat{K}(n) \leq k\}$ is computably enumerable. Not only is K an information content measure (when viewed as a function of ω), but it is not hard to see that if $\widehat{K}$ is another information content measure, then $W = \{\langle k+1, n\rangle: \widehat{K}(n) < k\}$ is a Kraft-Chaitin set; hence $K(n) \leq^+ \widehat{K}(n)$.

We write U^A and K^A for the relativizations of the universal prefix-free machine and prefix-free complexity, respectively, to an oracle $A \in 2^\omega$. The results mentioned above remain true in their relativized forms. In particular, $X \in 2^\omega$ is A-random if and only if $K^A(X \upharpoonright n) \geq^+ n$. The following result relates K^A to unrelativized prefix-free complexity when $A \in 2^\omega$ is 1-random (see also Lemma 6.1).

Ample Excess Lemma (Miller and Yu [19]) Let $A \in 2^\omega$ be 1-random.

1. $\sum_{n\in\omega} 2^{n-K(A\upharpoonright n)} < \infty$.
2. $K^A(n) \leq^+ K(A \upharpoonright n) - n$.

Note that (1) implies (2) by applying the Kraft-Chaitin theorem relativized to A.

Chaitin proved that $\Omega = \sum_{\tau\in\operatorname{dom} U} 2^{-|\tau|}$ is 1-random. It is easy to see that Ω is a *c.e. real*, meaning that there is a computable, nondecreasing sequence of rational numbers $\{\Omega_s\}_{s\in\omega}$ such that $\Omega = \lim \Omega_s$. It follows from Calude, Hertling, Khoussainov, and Wang [4] and Kučera and Slaman [13] that every 1-random c.e. real is Ω for the right choice of universal machine. Chaitin showed that $\Omega \equiv_T \varnothing'$ (this also follows from Arslanov's completeness criterion). It is easy to see, given what we know, that the 2-random reals are exactly the 1-random, low for Ω reals.

Proposition 2.1 (Nies, Stephan, and Terwijn [22]) *Let $A \in 2^\omega$ be 1-random. Then A is 2-random if and only if it is low for Ω.*

Proof By definition, A is 2-random if and only if A is 1-random relative to $\varnothing'$. Since $\varnothing' \equiv_T \Omega$, this is equivalent to A being Ω-random. By van Lambalgen's theorem, A is Ω-random if and only if Ω is A-random, in other words, exactly when A is low for Ω. □

3 Weakly Low for K Is the Same As Low for Ω

We show that being weakly low for K is equivalent to being low for Ω. An interesting alternate proof of the harder direction, that low for Ω implies weakly low for K (Theorem 3.3), has recently been found by Bienvenu. A *Solovay function* is a computable $f\colon \omega \to \omega$ such that $K(n) \leq^+ f(n)$ (which is equivalent to $\sum_{n\in\omega} 2^{-f(n)}$ converging) and $(\exists^\infty n)\ f(n) \leq^+ K(n)$. Bienvenu and Downey [3] proved that a computable function f is a Solovay function if and only if $\sum_{n\in\omega} 2^{-f(n)}$ is finite and 1-random. To see how this implies Theorem 3.3, let f be a computable function such that $\sum_{n\in\omega} 2^{-f(n)} = \Omega$. Then f is a Solovay function, so $K(n) \leq^+ f(n)$. If A is not weakly low for K, then $\lim_{n\to\infty} K(n) - K^A(n) = \infty$; hence $\lim_{n\to\infty} f(n) - K^A(n) = \infty$. Therefore, f is not a Solovay function relative to A. Relativizing the result of Bienvenu and Downey, Ω is not A-random; that is, A is not low for Ω.

Theorem 3.1 *If A is weakly low for K, then A is low for Ω.*

Proof We show the contrapositive. First, we define two families of c.e. sets $\{W_\sigma\}_{\sigma\in 2^{<\omega}}$ and $\{D_\sigma\}_{\sigma\in 2^{<\omega}}$. Fix $\sigma \in 2^{<\omega}$. Search for the least stage $s \in \omega$ such that $\sigma \prec \Omega_s$, in other words, such that σ appears to be a prefix of Ω. If no such stage is found, then W_σ and D_σ will be empty. Now, for any $\tau \in 2^{<\omega}$ such that $U(\tau)\downarrow$ after stage s, enumerate $\langle |\tau|, U(\tau)\rangle$ into D_σ. Also enumerate $\langle |\tau|, U(\tau)\rangle$ into W_σ as long as it preserves the condition that $\sum_{\langle d,n\rangle\in W_\sigma} 2^{-d} \leq 2^{-|\sigma|}$. Note that if $K_s(n) \neq K(n)$, then $\langle K(n), n\rangle \in D_\sigma$.

We claim that if $\sigma \prec \Omega$, then $W_\sigma = D_\sigma$. It follows from our definition that $\sum_{\langle d,n\rangle\in D_\sigma} 2^{-d} \leq \Omega - \Omega_s$. Observe that if $\sigma \prec \Omega$, then $\Omega - \Omega_s \leq 2^{-|\sigma|}$. In this case, $\sum_{\langle d,n\rangle\in D_\sigma} 2^{-d} \leq 2^{-|\sigma|}$, so $W_\sigma = D_\sigma$. The idea is that we have used an approximation of Ω to efficiently approximate all but finitely many values of $K(n)$.

Consider the A-c.e. set $W = \{\langle d + |\tau| - |\sigma|, n\rangle : U^A(\tau) = \sigma \text{ and } \langle d, n\rangle \in W_\sigma\}$. By the construction of $\{W_\sigma\}_{\sigma\in 2^{<\omega}}$ and Kraft's inequality,

$$\sum_{\langle e,n\rangle\in W} 2^{-e} = \sum_{U^A(\tau)\downarrow=\sigma}\ \sum_{\langle d,n\rangle\in W_\sigma} 2^{-d-|\tau|+|\sigma|} \leq \sum_{U^A(\tau)\downarrow} 2^{-|\tau|} \leq 1.$$

This proves that W is a Kraft-Chaitin set relative to A. Therefore, there is a constant $k \in \omega$ such that if $\langle e, n\rangle \in W$, then $K^A(n) \leq e + k$.

Now, assume that Ω is not A-random. For any $c \in \omega$, there are $\tau, \sigma \in 2^\omega$ such that $U^A(\tau) = \sigma$, $|\sigma| - |\tau| \geq c$, and $\sigma \prec \Omega$. Let $s \in \omega$ be the least stage such that $\sigma \prec \Omega_s$ (which must exist because Ω is not a dyadic rational). There is an $N \in \omega$ such that if $n \geq N$, then $K_s(n) \neq K(n)$ (by the usual conventions on stages, $N = s + 1$ is sufficient). For all $n \geq N$, we have $\langle K(n), n\rangle \in W_\sigma$; hence $\langle K(n) + |\tau| - |\sigma|, n\rangle \in W$.

But this means that $K^A(n) \leq K(n) + |\tau| - |\sigma| + k \leq K(n) + k - c$, for all but finitely many n. But c was arbitrary, so A is not weakly low for K. $\square$

For the other direction we use the fact that Ω is essentially interchangeable with any other 1-random c.e. real. This follows from work on *Solovay reducibility*. Write $X \leq_S Y$ to mean that there is a $c \in \omega$ and a partial computable function $f : \mathbb{Q} \to \mathbb{Q}$ such that if $q < Y$ is rational, then $0 \leq X - f(q) \leq c(Y - q)$ [26]. In other words, good approximations of Y from the left give us good approximations of X from the left. Kučera and Slaman [13] showed that if X is a 1-random c.e. real, then $X \equiv_S \Omega$.

When he introduced the reducibility, Solovay [26] proved that $X \leq_S Y$ implies $X \leq_K Y$. Relativizing the proof to an oracle A, we see that $X \leq_S Y$ implies that $K^A(X \upharpoonright n) \leq^+ K^A(Y \upharpoonright n)$, from which it follows that A-randomness is closed upward in the Solovay degrees. Together with the result of Kučera and Slaman, if X is a 1-random c.e. real, then X is random relative to A if and only if A is low for Ω.

We also need a simple lemma.

Lemma 3.2 *Let A be an oracle. If V is any $\Sigma^0_1[A]$ class, then there is a $\Sigma^0_1[A]$ class $\widehat{V}$ such that*

1. $\mu(\widehat{V}) \leq 3\mu(V)$,
2. *if X is an endpoint of an open interval in V, then $X \in \widehat{V}$.*

Furthermore, an index for $\widehat{V}$ can be found uniformly from an index from V and is independent of A.

Proof Let $\widehat{V} = \bigcup\{(a - \varepsilon, a + 2\varepsilon) : [a, a + \varepsilon] \subseteq V\}$. It is easy to check that $\widehat{V}$ has the required properties. $\square$

Theorem 3.3 *If A is low for Ω, then A is weakly low for K.*

Proof Assume that A is not weakly low for K. Let $S \subseteq 2^\omega$ be a Σ^0_1 class such that $\mu(S) \leq 1/2$ and $2^\omega \setminus S$ contains only 1-random reals. For example, we could take $S = U_1$, where $\{U_n\}_{n\in\omega}$ is a universal Martin-Löf test. Let $X = \inf(2^\omega \setminus S)$. Note that X is a 1-random c.e. real; from the discussion above, if we prove that X is not A-random, then A is not low for Ω.

For each n, we define a $\Sigma^0_1[A]$ class V_n such that $\mu(V_n) \leq 2^{-n-2}$. It will not be the case that $X \in V_n$; in fact, we will have $V_n \subseteq S$. On the other hand, it *will* always be true that X is an endpoint of an open interval in V_n. We claim that this is sufficient. By the previous lemma, we can form a computable sequence $\{\widehat{V_n}\}_{n\in\omega}$ of $\Sigma^0_1[A]$ classes such that $X \in \bigcap_{n\in\omega} \widehat{V_n}$ and $\mu(\widehat{V_n}) \leq 3\mu(V_n) \leq 3 \cdot 2^{-n-2} < 2^{-n}$. Therefore, X is covered by a Martin-Löf test relative to A, so X is not A-random.

We turn to the definition of $\{V_n\}_{n\in\omega}$. Assume that $S = \bigcup_{s\in\omega}[\sigma_s]$, where $\{\sigma_s\}_{s\in\omega}$ is a prefix-free computable sequence of strings. Fix $n \in \omega$. If $m = |\sigma_s|$, put $[\sigma_s]$ into V_n as long as σ_s is among the first $2^{m-K^A(m)-n-2}$ strings of length m in $\{\sigma_s\}_{s\in\omega}$. In other words, V_n is built from the same sequence that defines S but with the restriction that strings of length m can contribute at most $2^{-K^A(m)-n-2}$ to its measure. Note that the stage-wise approximations to $2^{m-K^A(m)-n-2}$ approach it from below, so V_n is $\Sigma^0_1[A]$. Also note that

$$\mu(V_n) \leq \sum_{m\in\omega} 2^{-K^A(m)-n-2} = 2^{-n-2} \sum_{m\in\omega} 2^{-K^A(m)} \leq 2^{-n-2},$$

where the last step uses Kraft's inequality.

Next we prove that there is a $\upsilon \in \omega$ such that if $|\sigma_s| \geq \upsilon$, then $[\sigma_s] \subseteq V_n$. Let $J(m) = |\{s \in \omega \colon |\sigma_s| = m\}|$. We claim that $I(m) = m - \log(J(m))$ is an information content measure. Clearly, I is computable from above. Note that $2^{-I(m)} = J(m)2^{-m}$ is exactly the contribution to the measure of S made by the strings in $\{\sigma_s\}_{s\in\omega}$ of length m. Since $\{\sigma_s\}_{s\in\omega}$ is prefix-free, $\sum_{m\in\omega} 2^{-I(m)} = \mu(S) \leq 1/2$. This shows that I is an information content measure, so there is c such that $(\forall m)\ K(m) \leq I(m) + c$. Because A is not weakly low for K, there is a υ large enough that $K^A(m) \leq K(m) - c - n - 2$, for all $m \geq \upsilon$. For such m,

$$2^{m-K^A(m)-n-2} \geq 2^{m-K(m)+c} \geq 2^{m-I(m)} = 2^{\log(J(m))} = J(m).$$

Therefore, $[\sigma_s]$ is put into V_n as long as $|\sigma_s| \geq \upsilon$.

We can now show that X is an endpoint of an open interval in V_n. This is because X is not a binary rational and thus not an endpoint of $[\sigma_s]$, for any s. Since there are only finitely many strings in $\{\sigma_s\}_{s\in\omega}$ of length less than υ, there is an ε small enough such that $(X-\varepsilon, X)$ is disjoint from all corresponding intervals. But $(X-\varepsilon, X) \subseteq S$, so $(X - \varepsilon, X) \subseteq V_n$. This completes the proof. □

It has been shown that every nonempty Π^0_1 class has a low for Ω member [9; 23], giving us a weakly low for K basis theorem.

4 All 2-Random Reals Maximize K Infinitely Often

While it is impossible for every initial segment of a real to have maximal complexity (with respect to either C or K), almost every real infinitely often achieves maximal initial segment complexity up to a constant. Call $A \in 2^\omega$ *infinitely often (i.o.) K maximizing* if $(\exists^\infty n)\ K(A \upharpoonright n) \geq^+ n + K(n)$. Similarly, A is *i.o. C maximizing* if $(\exists^\infty n)\ C(A \upharpoonright n) \geq^+ n$.[2] The right side of each inequality represents the maximal possible complexity for a string of length n.

Solovay [26] proved that i.o. K maximizing implies i.o. C maximizing. In fact, he proved that strings with (essentially) maximal K complexity must have (essentially) maximal C complexity. Solovay also proved that almost all reals are i.o. K maximizing. Yu, Ding, and Downey [28] analyzed his argument to prove that 3-randomness is sufficient to imply that a real is i.o. K maximizing. In the other direction, Martin-Löf [15] showed that every i.o. C maximizing real is 1-random, while Schnorr [24] refuted the converse. Nies, Stephan, and Terwijn [22] showed that i.o. C maximizing implies 2-randomness. They also showed the converse, as did Miller [16], classifying the i.o. C maximizing reals. Putting these facts together,

$$\text{3-random} \implies \text{i.o. } K \text{ maximizing} \implies \text{i.o. } C \text{ maximizing} \iff \text{2-random}.$$

We resolve the status of i.o. K maximizing, answering a question in [17].

Theorem 4.1 *A is 2-random if and only if it is infinitely often K maximizing.*

Proof Assume that A is 2-random. Then A is low for Ω by Proposition 2.1, hence weakly low for K by Theorem 3.3. By the ample excess lemma, $K^A(n) \leq^+ K(A \upharpoonright n) - n$. Rearranging, we have $K(A \upharpoonright n) \geq^+ n + K^A(n)$. Because A is weakly low for K, there are infinitely many n such that $K^A(n) \geq^+ K(n)$. Note that $K(A \upharpoonright n) \geq^+ n + K(n)$ for these n, so A is infinitely often K maximizing. □

It is interesting to note that Solovay [26] proved that strings with maximal C-complexity need not have maximal K-complexity, so the equivalence of i.o. C and K maximizing is not true on the level of strings.

5 Applications to the *LR*/*LK*-Degrees

The work of Section 3 has consequences in the *LR*/*LK*-degrees.

Theorem 5.1 *If $X \leq_{LR} Y$ and Y is low for Ω, then $X \leq_T Y'$.*

Proof We have $X \leq_{LK} Y$, so

$$K^Y(X \upharpoonright n) \leq^+ K^X(X \upharpoonright n) \leq^+ K^X(n) \leq^+ K(n), \tag{1}$$

for all $n \in \omega$. Because Y is low for Ω, it is weakly low for K, so there is a $c \in \omega$ such that $S = \{n \in \omega \colon K(n) \leq K^Y(n) + c\}$ is infinite. Together with (1), there is a $d \in \omega$ such that if $n \in S$, then $K^Y(X \upharpoonright n) \leq K^Y(n) + d$. By relativizing Chaitin's counting theorem [5, Lemma I3], there is an $e \in \omega$ such that

$$|\{\sigma \in 2^n \colon K^Y(\sigma) \leq K^Y(n) + d\}| \leq e,$$

for all $n \in \omega$. Let $T \subseteq 2^{<\omega}$ be the tree defined by

$$\sigma \in T \text{ iff } (\forall n < |\sigma|)[\, n \in S \rightarrow K^Y(\sigma \upharpoonright n) \leq K^Y(n) + d\,].$$

Note that $|[T]| \leq e$ and $X \in [T]$, where $[T]$ denotes the set of infinite paths through T. Also note that $S \leq_T Y'$ and so $T \leq_T Y'$. But every isolated infinite path through a tree is computable from the tree; hence $X \leq_T Y'$. □

Corollary 5.2 *If Y is low for Ω, then it has countably many predecessors in the LR-degrees.*

It is possible that the converse holds.

Open Question If Y is not low for Ω, must it have continuum many predecessors in the *LR*-degrees?

Not all reals have countably many *LR*-predecessors. Barmpalias, Lewis, and Soskova [2] proved that if $Y \in 2^\omega$ is non-GL_2 (i.e., $Y'' \nleq_T (Y \oplus \varnothing')'$), then it has continuum many predecessors in the *LR*-degrees. Furthermore, Barmpalias [1] has answered the question positively for Δ^0_2 reals. Note that the *LR*-predecessors of a real form a Borel set, so if Y has uncountably many *LR*-predecessors, then it has continuum many. In fact, Y has continuum many *LR-degrees* below it, because each *LR*-degree is countable (if $X \equiv_{LK} Y$, then $X' \equiv_T Y'$ [21]).

The next proof uses some basic facts about 2-randomness. By definition, $X \in 2^\omega$ is 2-random if and only if X is $\varnothing'$-random. This is equivalent to X being Ω-random, since $\Omega \equiv_T \varnothing'$. Hence by van Lambalgen's theorem, $X \oplus \Omega$ is 1-random. Second, Kautz [11] proved that every 2-random X is GL_1; in other words, $X' \leq_T X \oplus \varnothing'$. Finally, we say that Y is 2-random relative to X if it is X'-random. Note that almost every pair of reals are two random relative to each other.

Corollary 5.3 *If $X, Y \in 2^\omega$ are 2-random relative to each other, then they form a minimal pair in the LR-degrees.*

Proof Since X is 2-random relative to Y, it must be 2-random, so $X' \equiv_T X \oplus \varnothing' \equiv_T X \oplus \Omega$ and $X \oplus \Omega$ is 1-random. We know that Y is X'-random, hence $X \oplus \Omega$-random. Applying van Lambalgen's theorem, $X \oplus \Omega$ is Y-random. So if we assume that $A \leq_{LR} Y$, then $X \oplus \Omega$ is A-random.

Now assume that $A \leq_{LR} X$. By Theorem 5.1 and the fact that X is low for Ω, we have $A \leq_T X' \equiv_T X \oplus \Omega$. So we have proved that A is computed by an A-random real. This means that A is a *base for* 1*-randomness*, which Hirschfeldt, Nies, and Stephan proved to be equivalent to A being low for 1-randomness [10]. In other words, $A \equiv_{LR} \varnothing$. This shows that X and Y are a minimal pair in the LR-degrees. □

6 Remarks on the K-Degrees

Recall that $X \leq_K Y$ means that $K(X \upharpoonright n) \leq^+ K(Y \upharpoonright n)$. Miller and Yu [19] proved that if $X \in 2^\omega$ is 1-random and $X \leq_K Y$, then $Y \leq_{LR} X$, which is equivalent to $Y \leq_{LK} X$. Below we give a more direct proof that $X \leq_K Y$ implies $Y \leq_{LK} X$ on the 1-randoms. We use the following lemma.

Bounding Lemma (Miller and Yu [18]) If $\sum_{n\in\omega} 2^{-g(n)} < \infty$ and $g \leq_T X$ with use n, then $K(X \upharpoonright n) \leq^+ n + g(n)$.

It turns out that the initial segment complexity of X codes the behavior of K^X in a fairly simple way. Fix a pairing function, that is, an effective bijection $\langle\cdot,\cdot\rangle\colon \omega^2 \to \omega$. We may assume that $\langle n, m\rangle$ is greater than or equal to both n and m. We also apply the pairing function to strings, having identified them with natural numbers.

Lemma 6.1 *If X is* 1*-random, then* $K^X(\sigma) =^+ \min_{s\in\omega} K(X \upharpoonright \langle\sigma, s\rangle) - \langle\sigma, s\rangle$.

Proof By the ample excess lemma, there is a $c \in \omega$ with $\sum_{n\in\omega} 2^{n-K(X\upharpoonright n)} \leq 2^c$. Let $W = \{\langle K(X \upharpoonright \langle\sigma, s\rangle) - \langle\sigma, s\rangle + c + k + 1, \sigma\rangle : \sigma \in 2^{<\omega} \text{ and } s, k \in \omega\}$. Note that W is X-c.e. (which is the purpose of k). Also,

$$\sum_{\langle d,\sigma\rangle\in W} 2^{-d} = \sum_{n\in\omega}\sum_{k\in\omega} 2^{-K(X\upharpoonright n)+n-c-k-1} = \sum_{n\in\omega} 2^{n-K(X\upharpoonright n)-c} \leq 1,$$

so W is a Kraft-Chaitin set relative to X. This implies that

$$K^X(\sigma) \leq^+ \min_{s\in\omega} K(X \upharpoonright \langle\sigma, s\rangle) - \langle\sigma, s\rangle.$$

For the other direction, let

$$g(\langle\sigma, s\rangle) = \begin{cases} K_s^X(\sigma) & \text{if } s = 0 \text{ or } K_s^X(\sigma) < K_{s-1}^X(\sigma), \\ \langle\sigma, s\rangle & \text{otherwise.} \end{cases}$$

Here, $K_s^X(\sigma)$ is the stage s approximation of $K^X(\sigma)$, which by standard conventions on the use function can only depend on $X \upharpoonright s \preccurlyeq X \upharpoonright \langle\sigma, s\rangle$.[3] Therefore, $g \leq_T X$ with use n. Next we want to bound $\sum_{n\in\omega} 2^{-g(n)}$. Note that $\sigma \in 2^{<\omega}$ contributes less than $\sum_{k\in\omega} 2^{-K^X(\sigma)-k} + \sum_{s\in\omega} 2^{-\langle\sigma,s\rangle} = 2\cdot 2^{-K^X(\sigma)} + \sum_{s\in\omega} 2^{-\langle\sigma,s\rangle}$ to the sum. Therefore,

$$\sum_{n\in\omega} 2^{-g(n)} \leq 2 \sum_{\sigma\in 2^{<\omega}} 2^{-K^X(\sigma)} + \sum_{\langle\sigma,s\rangle\in\omega} 2^{-\langle\sigma,s\rangle} \leq 2 + 2 < \infty,$$

by Kraft's inequality. Applying the bounding lemma, $K(X \upharpoonright n) \leq^+ n + g(n)$. For $\sigma \in 2^{<\omega}$, choose the least s such that $K_s^X(\sigma) = K^X(\sigma)$. Then $K^X(\sigma) = g(\langle\sigma, s\rangle) \geq^+ K(X \upharpoonright \langle\sigma, s\rangle) - \langle\sigma, s\rangle$. Hence $K^X(\sigma) \geq^+ \min_{s\in\omega} K(X \upharpoonright \langle\sigma, s\rangle) - \langle\sigma, s\rangle$. □

Theorem 6.2 ([19]+[12]) *If X is 1-random and $X \leq_K Y$, then $Y \leq_{LK} X$.*

Proof Immediate from the lemma. □

The theorem allows us to apply results about the *LK*-degrees to the K-degrees. For example, Nies [21] proved that if $X \equiv_{LK} Y$, then $X' \equiv_T Y'$ (in fact, the jumps are truth-table equivalent). This means that if $X \equiv_K Y$ and X is 1-random (hence Y is too), then $X' \equiv_T Y'$. So 1-random K-degrees are countable. It is not hard to produce continuum many reals $X \in 2^\omega$ such that $K(X \upharpoonright n) =^+ n/2$; thus not all K-degrees are countable.

Section 5 also tells us something about the K-degrees.

Theorem 6.3 *The cone above a 2-random in the K-degrees is countable.*

Proof Let $X \in 2^\omega$ be 2-random. By Proposition 2.1, X is low for Ω. Assume $X \leq_K Y$, then $Y \leq_{LK} X$, so $Y \leq_T X'$ by Theorem 5.1. □

It is known that some 1-random reals have uncountable upper cones in the K-degrees [18]. On the other hand, it is open whether there are maximal K-degrees.

Open Question (See [17]) Is there a maximal K-degree? Is every 2-random K-degree maximal?

Notes

1. Strictly speaking, Kraft considered finite prefix-free codes.

2. Starting with Yu, Ding, and Downey [28], these notions have been called *strong Chaitin random* and *Kolmogorov random*, respectively. Since they were neither introduced by Kolmogorov nor Chaitin, and since Chaitin has used "strong Chaitin randomness" to denote one of his characterizations of 1-randomness, it seems reasonable to look for alternative names. One may even question the need for names, since both notions are equivalent to 2-randomness. For these reasons, we have adopted the descriptive—if artless—terms used in this paper.

3. This is the only place we use the fact that $\langle m, n\rangle \geq \max\{m, n\}$. We could do away with this restriction on our pairing function by simply stipulating that the use of $K_s^X(\sigma)$ is at most $\langle\sigma, s\rangle$.

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