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ON A CLASS OF REGULAR SETS

VLADETA VUČKOVIĆ

1. *Introduction*. In this paper we study a subclass of the class of all regular sets, the class of strictly (or strongly) regular sets. We introduced this class in [1], where it was shown that every regular set is some projection of a strictly regular set. Here we give a sample of theorems about strictly regular sets, as announced in [1], in order to show more closely their intrinsic properties. Methodologically, we insist on the use of the language of the recursive arithmetic or words, regarding this language as a natural device in the theory of finite automata.

We point that, independently of us, V. G. Bodnarčuk introduced in [9] also strictly regular sets, under the name of **R**-sets. This paper was written when the paper of Bodnarčuk appeared. As there are not many connections between our and Bodnarčuk's exposition, we did not rewrite our paper, but we made only following changes: as Bodnarčuk has the proof of our theorem 3.2 we eliminated our proof which followed the same lines; with theorems of Bodnarčuk's paper our paper has in common only this theorem and the theorem 4.2 (Here we bring the proof, as it differs from that one of Bodnarčuk).

In our paper [10] it was shown how strictly regular sets can be employed in the proofs of theorems about regular sets.

2. *Notations*. We shall use in some extent the notions of the recursive arithmetic or words, which were introduced in our paper [2]. As we need here a very minor part of the content of this paper, we give the necessary notions in a form which is suitable for our purposes.

Let

$$U = \{u_0, u_1, \ldots\ldots, u_{n-1}\} \tag{2.1}$$

be a finite alphabet (in the sequel, when we speak about an alphabet we suppose it always finite). The *words* in U are finite strings of letters of U, which we shall write by simple juxtaposition: $u_{i_1}u_{i_2}\ldots\ldots u_{i_k}$. By $\Omega(U)$ we denote the set of all words in U, inclusive the empty word O.

By $X + Y$, the *sum* of words X and Y, we denote the concatenation YX. By definition

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$$X + O = X,\ X + u_i Y = u_i(X + Y),_{i=0,1,..,n-1};$$

one can prove easily

$$O + X = X$$

and the associative law

$$X + (Y + Z) = (X + Y) + Z.$$

For every letter $u_i \epsilon U$ we call $u_i X = X + u_i$ the i-th *successor* of the word X.

We can imagine all words of $\Omega(U)$ as ordered partially in a tree, with O at bottom, its successors on the first level, their successors on the second level and so on. So, we can speak about all words Y which are *less* than or *equal* to the word, X, in sign: of all words Y, such that $O \leq Y \leq X$; that are exactly the words (including O and X) which are obtained from X by deleting from its beginning one letter after other, until the empty word O is reached.

$a(X)$ denotes the *beginning letter* of X; $a(O) = O$, $a(u_i X) = u_i$.

$\ell(X)$ denotes the *last letter* of X: $\ell(O) = O$, $\ell(Xu_i) = u_i$.

$v(X)$, the *predecessor* of X, is the rest of X after deleting its first letter: $v(O) = O$, $v(u_i X) = X$.

$V(X,Z)$, the *Z-th predecessor* of X, is the word obtained from X by deleting from its beginning as many letters as they are in the word Z: $V(X,O) = X$, $V(X, u_i Z) = v(V(X,Z))$.

$\bigwedge_{Y=O}^{X} p(Y)$, where $p(Y)$ is some word-predicate, means the predicate: For all words Y, such that $O \leq Y \leq X$, $p(Y)$.

$\bigvee_{Y=O}^{X} p(Y)$ means the predicate: There is a word Y, $O \leq Y \leq X$, such that $p(Y)$.

We employ other logical and mathematical symbols in their usual meaning.

By an initial *U-automaton* we understand a quadruple

(2.2) $$A = \langle Q, Q', \lambda, q_0 \rangle$$

where

(2.3) $$Q = \{q_0, q_1, \ldots\ldots, q_{t-1}\}$$

is some finite set (the set of *internal states*), $Q' \subset Q$ is the set of *final states* and λ (the *transition function*) is a total function mapping the set $Q \times U$ into the set Q. q_0 is called the *initial state*.

A word $u_{i_1} u_{i_2} \ldots u_{i_k} \epsilon \Omega(U) - \{O\}$ is *accepted* by A if and only if there is a word in Q, $q_{j_1} q_{j_2} \ldots\ldots q_{j_k}$, such that $\lambda(q_0, u_{i_1}) = q_{j_1}$, $\lambda(q_{j_\nu}, u_{i_{\nu+1}}) = q_{j_{\nu+1}}$ for all $\nu = 1,2,\ldots, k-1$ and $q_{j_k} \epsilon Q'$.

By convention, the empty word O is accepted by A if and only if $q_0 \epsilon Q'$.

The set of all accepted words of A is denoted by $T(A)$. Every set $\alpha \subset \Omega(U)$ for which there exists an automaton A, such that $\alpha = T(A)$, is called *U-regular*, (or simply, *regular*).

3. *Strong automata.*

Definition 3.1. Let U be the alphabet (2.1). The quadruple $A = <Q, Q', \lambda, q_0>$ is a *strong U-automaton* if its set of internal states

(3.1) $$Q = \{q_0, q_1, \ldots, q_{n-1}, q_n, q_{n+1}\}$$

contains exactly $n+2$ states, if its transition-function λ is subject of the conditions: for every $u_j \epsilon U$

(3.2) $\lambda(q_i, u_j) =$ either q_{j+1} or q_{n+1}, for all $i = 0, 1, \ldots, n-1, n$,

(3.2) $$\lambda(q_{n+1}, u_j) = q_{n+1},$$

and if $q_{n+1} \notin Q'$.

Characteristic for strong automata is the manner in which the accepted U-words are correlated with corresponding Q-words: If $u_{i_1} u_{i_2}, \ldots u_{i_k} \epsilon T(A)$ then the corresponding Q word is exactly the word $q_{i_1+1} q_{i_2+1} \ldots\ldots q_{i_k+1}$. Further, if a Q-word ends with q_{n+1}, then the corresponding U-word is surely not accepted (This is only a sufficient condition for non-acceptance).

Definition 3.2. If $\alpha \subset \Omega(U)$ and if there exists a strong U-automaton A, such that $\alpha = T(A)$, α is called a *strongly U-regular set.*

Obviously, the class of all strongly regular sets is a subclass of the class of all regular sets. We shall now proceed as to get an intrinsic criterion of strong regularity.

Definition 3.3. Let U be some alphabet and r a binary relation on it (i.e. a subset of the set $U \times U$). If we allow r to be fulfilled also for every single word of the length 1 (i.e. for every single letter of U), we call r a *birelation* on U.

Definition 3.4. Let r be a birelation on U. We extend it on the set $\Omega(U)$ by

(3.3) $$r(X,Y) \leftrightarrow r(a(X), a(Y)).$$

I.e. two words P and Q are r-related if and only if their beginning letters are r-related.

In the sequel we suppose always that every birelation on an alphabet is extended by (3.3) into the set of all words in this alphabet.

Definition 3.5. A set $\alpha \subset \Omega(U)$ is called *strictly U-regular* if there exist two subalphabets $U' \subset U$, $U'' \subset U$ and a birelation r on U such that

(3.4) $$X \epsilon \alpha - \{O\} \leftrightarrow a(X) \epsilon U' \wedge \bigwedge_{Z=O}^{v(x)} r(V(X,Z), V(X,Z + u_0)) \wedge \ell(X) \epsilon U''$$

The second condition on the right side of (3.4) means the following: if

$u_{i1}u_{i2}\dots u_{ik}\in\alpha-\{O\}$ then every two consecutive letters are r-related, i.e. $r(u_{i_\nu},u_{i_{\nu+1}})$ for $\nu = 1,2,\dots, k-1$.

In the case that X consists of only one letter, we suppose $\bigwedge_{Z=O}^{v(x)} r(V(X,Z).$ $V(X,Z + U_0))$ always fulfilled.

Theorem 3.1. *Every strongly U-regular set is strictly U-regular.*

Proof. Let $\alpha = T(A)$, where A is a strong U-automaton, say the automaton A from the definition 3.1. Define

$$U' = \{u_j \mid \lambda(q_0, u_j) = q_{j+1}, \tag{3.5}$$

$$U''= \{u_0 \mid q_{j+1}\in Q'\}, \tag{3.6}$$

and the birelation r in the following way:

$$r(u_i,u_j) \leftrightarrow \lambda(q_{i+1},u_j) = q_{j+1}. \tag{3.7}$$

Define the set β by

$$X\in\beta - \{O\} \leftrightarrow \alpha(\mathrm{x})\in U' \wedge \bigwedge_{Z=O}^{v(x)} r(V(X,Z),\ V(X,Z+u_0)) \wedge \ell(X)\in U'', \tag{3.8}$$

and add O to the set β if and only if $O\in\alpha$ (i.e. if $q_0\in Q'$). We shall prove $\alpha=\beta$.

Let first

$$u_{i_1}u_{i_2}\dots u_{i_k}\in\alpha - \{O\} \tag{3.9}$$

and let the length of it be greater than 1.

The corresponding Q-word is

$$q_{i_1+1}\, q_{i_2+1} \dots q_{i_k+1}, \quad q_{i_k+1}\in Q', \tag{3.10}$$

and for $\nu = 1,2,\dots,k-1$

$$\lambda(q_0,u_{i_1}) = q_{i_1+1},\ \lambda(q_{i_\nu},u_{i_\nu}) = q_{i_\nu+1}. \tag{3.11}$$

By the first condition in (3.11) and by the definition (3.5) we have

$$u_{i_1}\in U'; \tag{3.12}$$

by the condition before (3.11) and by the definition (3.6) we have

$$u_{i_k}\in U'', \tag{3.13}$$

and by the second condition in (3.11) and by the definition (3.7) we have

$$r(u_{i_\nu},\ u_{i_{\nu+1}}) \text{ for all } \nu = 1,2,\dots, k-1. \tag{3.14}$$

From (3.12), (3.13) and (3.14) follows: $u_{i_1}u_{i_2},\dots u_{i_k}\in \beta-\{O\}$.

If $u_i\in\alpha-\{O\}$ is of length 1, then the corresponding Q-word is q_{i+1}; so $u_i\in U'$ and, as then $q_{i+1}\in Q$, also $u_i\in U''$. By the definition of a birelation follows $u_i\in\beta-\{O\}$.

Let now

$$u_{i_1} u_{i_2} \cdots\cdots u_{i_k} \in \beta - \{O\} \tag{3.15}$$

and let the length of it be greater than *1*.

Form the Q-word

$$q_{i_1+1}\, q_{i_2+1} \cdots\cdot q_{i_k+1} \ . \tag{3.16}$$

As $u_{i_1} \in U'$ we have by (3.5)

$$\lambda(q_0, u_{i_1}) = q_{i_1+1}. \tag{3.17}$$

As $r(u_{i_\nu}, u_{i_{\nu+1}})$ for $\nu = 1,2,\ldots, k-1$ we have by (3.7)

$$\lambda(q_{i_\nu+1}, u_{i_{\nu+1}}) = q_{i_{\nu+1}+1} \text{ for } \nu = 1,2,\ldots, k-1, \tag{3.18}$$

and as $u_{i_k} \in U''$, we have by (3.6)

$$q_{i_k+1} \in Q'. \tag{3.19}$$

Therefore $u_{i_1} u_{i_2} \ldots u_{i_k} \in \alpha - \{O\}$.

If $u_i \in \beta - \{O\}$ is of the length *1*, then $u_i \in U'$, $u_i \in U''$, $\lambda(q_0, u_i) = q_{i+1}$ and $q_{i+1} \in Q'$, from where follows $u_i \in \alpha - \{O\}$.

Theorem 3.2. *Every strictly U-regular set α is strongly U-regular.*

As mentioned in the section 1, this theorem is proved by Bodnarčuk in [9], as theorem 5. As our original proof was almost the same we omit it.

Theorem 3.3. *The class of all strongly regular sets is effectively equal to the class of all strictly regular sets. Given a strong automaton one can effectively obtain the sets U' and U'' and the birelation r of the corresponding strictly regular set. Given such a set, one can effectively construct the strong automaton for the corresponding strongly regular set.*

On the ground of this theorem the predicates "strictly regular" and "strongly regular" are equivalent. We shall use them therefore interchangeable.

Nevertheless, to point the formal difference, we introduce

Definition 3.6. The quadruple

$$A = \langle U, U', U'', r \rangle, \tag{3.20}$$

where U is some alphabet, $U' \subset U, U'' \subset U$, and r a birelation on U, is called a *strict U-automaton.*

A word $X \in \Omega(U)$ is *accepted* by A if and only if

$$a(X) \in U' \wedge \bigwedge_{Z=O}^{v(x)} r(V(X,Z), V(X, Z + u_0)) \wedge \ell(X) \in U''. \tag{3.21}$$

The set of all accepted words is denoted by $T(A)$. A set $\alpha \subset \Omega(U)$ is *strictly regular* if there is some strict U-automaton such that $\alpha - \{O\} = T(A)$.

As every birelation r on an alphabet U can be put in the form of a finite disjunction

(3.22) $r(x,y) \leftrightarrow (x = u_{i_1} \wedge y = u_{j_1}) \vee (x = u_{i_2} \wedge y = u_{j_2}) \vee \ldots\ldots\ldots \vee (x = u_{i_1} \wedge y = u_{j_1})$

the predicate $r(X,Y)$ is primitive recursive; it is not difficult, by means of [2], to prove that (3.21) is primitive recursive also. So, we have

Theorem 3.3. *Every strictly (or strongly) regular set is primitive recursive.*

4. *Strictly regular sets.* In this section we shall give more insight into the structure of strictly regular sets. Much of the material of this section will consist of counterexamples, which show that many theorems for regular sets became invalid if the adjective "regular" is changed to "strictly regular".

Let V be the alphabet

(4.1) $$V = \{v_0, v_1, v_2\}.$$

Example 4.1. The set α, consisting of the only word $v_1 v_2 v_1 v_0$, is not strictly regular.

Proof. If α were strictly regular, we would have: $V' = \{v_1\}$, $V'' = \{v_0\}$ and $r = \{(v_1,v_2), (v_2,v_1), (v_1,v_0)\}$. The condition

$$X \epsilon \alpha - \{O\} \leftrightarrow a(X) \epsilon V' \wedge \bigwedge_{Z=O}^{v(x)} r(V(X,Z), V(X,Z+v_0)) \wedge \ell(X) \epsilon V''$$

is satisfied not only by $v_1 v_2 v_1 v_0$ but also by words $v_1 v_0$, $v_1 v_2 v_1 v_2 v_1 v_0$ and so on. So, with $v_1 v_2 v_1 v_0$ every strictly regular set must contain these words too. (Naturally, α is a regular set).

As known, the *product* $\alpha \cdot \beta$ of two sets α and β is defined by

$$\alpha \cdot \beta = \{XY | X \epsilon \alpha \wedge Y \epsilon \beta\}.$$

Example 4.2. There exist strictly regular sets whose product is not strictly regular.

Proof. Let $\alpha = \{v_1 v_2\}$ and $\beta = \{v_1 v_0\}$. Both sets are strictly regular: *for* $r_1 = \{(v_1,v_2)\}$, $r_2 = \{(v_1,v_0)\}$,

$$X \epsilon \alpha - \{O\} \leftrightarrow a(X) \epsilon \{v_1\} \wedge \bigwedge_{Z=O}^{v(x)} r_1(V(X,Z), V(X,Z+v_0)) \wedge \ell(X) \epsilon \{v_2\},$$

$$X \epsilon \beta - \{O\} \leftrightarrow a(X) \epsilon \{v_1\} \wedge \bigwedge_{Z=O}^{v(x)} r_2(V(X,Z), V(X,Z+v_0)) \wedge \ell(X) \epsilon \{v_0\}.$$

The direct product $\alpha \cdot \beta$ is $\{v_1 v_2 v_1 v_0\}$; by the foregoing example this set is not strictly regular.

Example 4.3. There exist strictly regular sets whose union is not strictly regular.

Proof. $\alpha = \{v_0 v_1\}$ and $\beta = \{v_1 v_2\}$ are strictly regular. If their union

$\alpha \cup \beta = \{v_0 v_1, v_1 v_2\}$ were strictly regular, for the corresponding strict automaton $A = \langle V, V', V'', r \rangle$ we had: $V' = \{v_0, v_1\}$, $V'' = \{v_1, v_2\}$ and $r = \{(v_0, v_1), (v_1, v_2)\}$. Then we would have $v_0 v_1 v_2 \in T(A)$, although this word is not in $\alpha \cup \beta$.

Example 4.4. There exists a strictly regular set α whose complement $\sim \alpha$ is not strictly regular.

Proof. Let $\alpha = \{v_0 v_1 v_2\}$, $\sim \alpha$ contains the word $v_0 v_1 v_2 v_0 v_1 v_2$, so its corresponding birelation r must contain the pairs (v_0, v_1), (v_1, v_2) also, and the corresponding set V' must contain the letter v_0 and the set V'' the letter v_2. Then obviously we would have $v_0 v_1 v_2 \in \sim \alpha$, what is impossible.

By foregoing examples we have

Theorem 4.1. *The complement of a strictly regular set is not necessary strictly regular; the union and the product of two strictly regular sets are not necessary strictly regular.*

We give now some positive results.

As known, the *iteration* α^* of a set α is the infinite union

$$\alpha^* = \{O\} \cup \alpha \cup \alpha \cdot \alpha \cup \alpha \cdot \alpha \cdot \alpha \cup \ldots\ldots$$

Theorem 4.2. *The iteration of a strictly regular set is strictly regular.*

Proof. Let

$$X \in \alpha - \{O\} \leftrightarrow a(X) \in U' \wedge \bigwedge_{Z=O}^{v(x)} r(V(X,Z), V(X, Z+u_0)) \wedge \ell(X) \in U''.$$

Define

$$r_1(x,y) \leftrightarrow r(x,y) \vee \{x \in U'' \wedge y \in U'\}$$

Let $O \in \beta$ and

$$X \in \beta - \{O\} \leftrightarrow a(X) \in U' \wedge \bigwedge_{Z=O}^{v(x)} r_1(V(X,Z), V(X, Z+u_0)) \wedge \ell(X) \in U''.$$

We shall prove: $\beta = \alpha^*$. If $P \in \alpha^*$, then obviously $P \in \beta$. Let now $P \in \beta - \{O\}$. If the length of P is 1 this is possible only if $U' \cap U'' \neq \emptyset$, so $P \in \alpha \subset \alpha^*$, i.e. $P \in \alpha^*$. Let therefore the length of P be greater than 1.

As $a(P) \in U'$ there exists a beginning part P_1 of P of maximal length, which is in α (In the case $a(P) \in U' \cap U'' \neq \emptyset$, we have eventually $P_1 = a(P)$). Therefore $P = P_1 Q_1$, where $P_1 \in \alpha$, $a(P_1) \in U'$, $\ell(P_1) \in U''$ and $r(\ell(P_1), a(Q_1)) \rightarrow a(Q_1) \notin U''$. Similar situation is with all consecutive letters of Q_1; but $\ell(Q_1) \in U''$. Therefore $a(Q_1) \in U'$. Repeat now the process with Q_1. We get

$$P = P_1 P_2 Q_2, \ P_1, P_2 \in \alpha, \ a(Q_2) \in U'.$$

Repeating the process, as the lengths of P_1, P_2 and the following P_v-s are ≥ 1, and as P is of finite length, we get at last

$$P = P_1 P_2 \ldots\ldots P_k, \ P_v \in \alpha, \ v = 1,2,\ldots,k \ \text{ i.e. } P \in \alpha^*.$$

Theorem 4.3. *The intersection of two strictly regular sets is strictly regular also.*

Proof. Let α and β be strictly U-regular and

$$X\epsilon\alpha - \{O\} \leftrightarrow a(X)\epsilon U'_1 \wedge \bigwedge_{Z=O}^{v(x)} r_1(V(X,Z),V(X,Z+u_0)) \wedge \ell(X)\epsilon U''_1 ; \tag{4.2}$$

$$X\epsilon\beta - \{O\} \leftrightarrow a(X)\epsilon U'_2 \wedge \bigwedge_{Z=O}^{v(x)} r_2(V(X,Z),V(X,Z+u_0)) \wedge \ell(X)\epsilon U''_2 . \tag{4.3}$$

Define the birelation ρ by

$$\rho(x,y) \leftrightarrow r_1(x,y) \wedge r_2(x,y)$$

and the set γ by:

$$O\epsilon\gamma \leftrightarrow O\epsilon\alpha \cap \beta ,$$

$$X\epsilon\gamma - \{O\} \leftrightarrow a(X)\epsilon U'_1 \cap U'_2 \wedge \bigwedge_{Z=O}^{v(x)} \rho(V(X,Z),V(X,Z+u_0)) \wedge \ell(X)\epsilon U''_1 \cap U''_2 .$$

Obviously $\gamma = \alpha \cap \beta$.

Definition 4.1. The *reversion-function* $R(X)$ is defined by: $R(O) = O$, $R(u_iX) = u_i + R(X)$.

The *reversion of the set* α is the set $R(\alpha) = \{R(X) \mid X\epsilon\alpha\}$. Obviously $R(u_{i_1}u_{i_2}\ldots, u_{i_k}) = u_{i_k}\ldots u_{i_2}u_{i_1}$.

Theorem 4.4. *The reversion of a strictly regular set is strictly regular.*

Proof. Let α be defined by (4.2.) Define the birelation ρ by $\rho(x,y) \leftrightarrow r_1(y,x)$. Then

$$X\epsilon R(\alpha) - \{O\} \leftrightarrow a(X)\epsilon U''_1 \wedge \bigwedge_{Z=O}^{v(x)} \rho(V(X,Z),V(X,Z+u_0)) \wedge \ell(X)\epsilon U'_1 .$$

(To this theorem compare [3], Th. 4, for regular sets).

Definition 4.2. If $X = Yu_iu_i \ldots\ldots u_i$, where $\ell(Y) \neq u_i$, then the *right u_i-truncation* of X is the word Y, in sign: $X^{u_i} = Y$. If α is a set of words, its right u_i-truncation is the set $\alpha^{u_i} = \{X^{u_i} \mid X\epsilon\alpha\}$. Similar for the left u_i-truncation. (Compare [4], Def. 4).

Theorem 4.5. *The truncation of every strictly regular set is strictly regular.*

Proof. (Only for the right u_i-truncation; for the left-similarly, or by Theorem 4.4.). Let α be defined by (4.2). Define

$$\tau_i = \{u_j \mid u_j\epsilon U \wedge r_1(u_j, u_i)\}$$

(If $r(u_i, u_i)$, the set τ can contain u_i). Then

$$X\epsilon\alpha^{u_i} - \{O\} \leftrightarrow a(X)\epsilon U_1' \wedge \bigwedge_{Z=O}^{v(x)} r_1(V(X,Z),V(X,Z+u_0)) \wedge$$
$$\ell(X)\epsilon(U_1'' - \{u_i\}) \cup \{\tau_i - \{u_i\}\} .$$

(For regular sets this theorem was proved by C. C. Elgot: [4], Lemmas 4.1 and 4.2).

5. *A proof of the primitive recursiveness of regular sets.* Although our end here is not the application of foregoing results onto the study of regular sets, we give here such an application by proving that every regular set is primitive recursive. Naturally, the theorem is not new, but the proof proceeds along completely new lines, employing only the devices of the recursive arithmetic of words. Exceptionally we suppose here that the reader is acquainted with the notions of the primitive recursive arithmetic of words in a greater extent than it was exposed in the section 2,; we suppose f.i. that he knows the first half of our paper [2], or of Asser's paper [6], or of Miss Peter's paper [7]. We employ the notations of [2], but chiefly that ones of the section 2.

Definition 5.1. Let U be the alphabet (2.1) and $S = \{S_0, S_1, \ldots, S_{p-1}\}$, $p \leq n$, another alphabet. Every mapping ϕ of U onto S, which is (uniquely) extended to a mapping of $\Omega(U)$ onto $\Omega(S)$ by

$$\phi(O) = O$$
$$\phi(u_i X) = \phi(u_i)\phi(X), \quad i = 0,1, \ldots, n-1$$

is called a *projection.*

We note that O represents the empty word in every alphabet. We shall employ

Theorem 5.1. *Every regular set α is some projection of a strictly regular set; if α is given, one can effectively find the corresponding strictly regular set and the projection.*

A proof of this theorem was given by the author in [1]. We point that this theorem is implicitely contained almost in every essential study about finite automata.

To simplify the treatment we suppose: if U is the alphabet

$$U = \{u_0, u_1, \ldots, u_{n-1}\} \tag{5.1}$$

then S is an initial subalphabet of U:

$$S = \{u_0, u_1, \ldots, u_{k-1}\}, \quad k \leq n . \tag{5.2}$$

This convention makes possible to regard every projection ϕ: $U \to S$ as a primitive recursive word-function in $\Omega(U)$: $\phi(u_i) = u_{\nu(i)}$, $i = 0,1, \ldots, n-1$, $\nu(i) \leq k-1$, is a finite application. Now the definition

$$\begin{aligned} \phi(O) &= O, \\ \phi(u_i X) &= \phi(X) + \phi(u_i), \quad i = 0,1, \ldots, n-1 \end{aligned} \tag{5.3}$$

makes ϕ a primitive recursive word-function in $\Omega(U)$, with values in the subset $\Omega(S) \subset \Omega(U)$.

Let α_1 be a strictly regular set in $\Omega(U)$, and let

$$(5.4)\quad X\epsilon\alpha_1 - \{O\} \leftrightarrow a(X)\epsilon U' \wedge \bigwedge_{Z=O}^{v(x)} r(V(X,Z), V(X,Z+u_0)) \wedge \ell(X)\epsilon U''$$

By theorem 3.3 the set α_1 is primitive recursive. But with this it is not proved that the set $\alpha = \phi(\alpha_1)$ is primitive recursive also.

To prove this, remark first that

$$(5.5)\qquad X\epsilon\alpha \leftrightarrow \bigvee_{Y} (Y\epsilon\alpha_1 \wedge \phi(Y) = X)$$

where $\bigvee_Y$ is the unbounded existential quantifiers: "there is an Y such that .....". To establish the primitive recursive character of α one has to bound this quantifier.

To this end we remark that Y in (5.5) is not to be searched in the whole set $\Omega(U)$, but only in the finite subset $\gamma(X)$ of all words which are of the same length as the word X. As the alphabet U consists of n letters, there are $n^{s_0(x)}$ such words, where $s_0(X)$ denotes the length of X: $s_0(O) = O$, $s_0(u_i X) = u_0 s_0(X)$.

A simple Gödelisation is best suited for our purpose. (Numerals are part of $\Omega(U)$: that are words written with the only letter u_0; see [2], the end of the section 10; we write 1 for u_0, 2 for u_0u_0, 3 for $u_0u_0u_0, \ldots$).

Correspond the numeral $\nu+1$ to the letter u_ν, $\nu = 0,1,\ldots,n-1$, (zero is corresponded to the empty word). To the word u_0u_0 correspond the numeral $n+1$, to the word u_1u_0 the numeral $n+2, \ldots\ldots$, to the word $u_{n-1}\,u_{n-1}$ the numeral n^2+n.

In general, to all words of the length k we correspond numerals, beginning with the word $\underbrace{u_0u_0\ldots u_0}_{k}$ (to which we correspond the numeral $\sum_{i=1}^{k-1} n_i + 1$) and ending with the word $\underbrace{u_{n-1}\,u_{n-1}\ldots u_{n-1}}_{k}$ (to which we correspond the numeral $\sum_{i=1}^{k} n^i$).

The correspondence of $\Omega(U)$ onto $\Omega(\{u_0\})$ is 1-1. Let $\mathbf{gn}(X)$ denote the numeral corresponding to the word X, and $\mathbf{exp}(n)$ the word to which corresponds the numeral n. It is not difficult to prove that $\mathbf{exp}(n)$ is a primitive recursive word function. (Our correspondence is not other than the known enumeration of all variations with repetition).

If k is the length of the word X, then all words of the same length have their Gödel numerals between $\psi_1(k,n) = \sum_{i=1}^{k-1} n^i + 1$ and $\psi_2(k,n) = \sum_{i=1}^{k} n^i$ (inclusive both numerals). Therefore the set

$$(5.6)\qquad \gamma(X) = \{\mathbf{exp}(Z) \mid \psi_1(s_0(X),n) \leq Z \leq \psi_2(s_0(X),n)\}$$

is primitive-recursive and can be enumerated effectively, Then

$$(5.7)\quad \begin{cases} X\epsilon\alpha \leftrightarrow \{\exp(\psi_1(s_0(X),n))\epsilon\alpha_1 \wedge X = \phi(\exp(\psi_1(s_0(X),n))\} \\ \vee\{\exp(\psi_1(s_0(X),n)+1)\epsilon\alpha_1 \wedge X = \phi(\exp(\psi_1(s_0(X),n)+1))\} \vee \\ \vee - \vee \\ \vee - \vee \\ \{\exp(\psi_2(s_0(X),n))\epsilon\alpha_1 \wedge X = \phi(\exp(\psi_2(s_0(X),n)))\}. \end{cases}$$

This predicate is obviously primitive recursive. So we have

Theorem 5.2. *Every regular set is primitive recursive.*

Accidentaly, our proof gives

Theorem 5.3. *Every projection of a primitive recursive set is primitive recursive.*

Obviously there are primitive recursive sets which are not regular. The simplest is the set of all words in the alphabet $\{S_0,S_1\}$ which have the same number of letters S_0 and S_1. (See [8] for the proof that it is not regular). Its characteristic function is (in notations of [2]).

$$\alpha(|s_0(X \dot{-} s_0(X)), X \dot{-} s_1(X)|)$$

what proves its primitive recursive character.

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University of Notre Dame
Notre Dame, Indiana