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Finiteness Axioms on Fragments of Intuitionistic Set Theory

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Abstract It is proved that in a suitable intuitionistic, locally classical, version of the theory **ZFC** deprived of the axiom of infinity, the requirement that every set be finite is equivalent to the assertion that every ordinal is a natural number. Moreover, the theory obtained with the addition of these finiteness assumptions is equivalent to a theory of hereditarily finite sets, developed by Previale in "Induction and foundation in the theory of hereditarily finite sets." This solves some problems stated there. The analysis is undertaken using for each of these results a limited fragment of the relevant theory.

1 Introduction

In [5], the author develops an intuitionistic theory of hereditarily finite sets **HS**, whose axiomatization will be recalled in Section 2. In this theory sets are built starting with the empty set with two operations for adding and removing one element at a time; it has as primitive notions the membership relation and its transitive closure. It is based on an induction axiom schema, essentially due to [3], of arithmetical flavor. The author then compares this with other theories, in particular, an intuitionistic, locally classical version $\mathbf{ZFC}^{\mathbf{int}} - \mathbf{Inf}$ of **ZFC** deprived of the axiom of infinity. He inquires about the equivalence in $\mathbf{ZFC}^{\mathbf{int}} - \mathbf{Inf}$ of two principles: every set is finite ($\mathbf{V} \subseteq \mathbf{Fin}$) and every ordinal number is a natural number ($\mathbf{Ord} \subseteq \mathbf{Nat}$), where ordinals are defined as those transitive sets all of whose members are transitive. Finally, after remarking that **HS** extends $\mathbf{ZFC}^{\mathbf{int}} - \mathbf{Inf} + (\mathbf{V} \subseteq \mathbf{Fin}) + (\mathbf{Ord} \subseteq \mathbf{Nat})$, he asks the question of their equivalence. In fact, these questions are raised after pointing out that the usual arguments for these equivalences in the corresponding classical theories involve an explicit use of the excluded middle.

This paper is dedicated to answering in the affirmative all these questions by developing alternative—though rather straightforward—arguments which are fully

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justified in the theories under consideration. The presentation is self-contained. It is based on the investigations on this subject carried out in [2] and [4].

In order to gauge the strength of the theories used to establish the equivalences, we will start in Section 3 with a rather detailed discussion of a basic theory **AS**, developing in it a good chunk of set theory.

Section 4 discusses the principle of $\in$-induction $\mathbf{F^{int}}$. With this schema a good amount of the usual arithmetics of ordinal and natural numbers can be carried out. However, here no efforts will be made to obtain general properties, rather the focus will be to get those facts to be used in the proof of the main results.

Section 5 deals with finiteness principles and answers two of the questions of [5] by showing that $\mathbf{AS} + \mathbf{F^{int}} + (\mathbf{V} \subseteq \mathbf{Fin})$ proves both $\mathbf{Ord} \subseteq \mathbf{Nat}$ and the induction axiom schema of **HS**. Whereas the former statement admits a direct proof, the latter will be established by describing the complexity of the construction of each set with the assignment of a natural number and then using an induction on this number.

Finally, Section 6 introduces the power set axiom **P**. This allows to define the cumulative hierarchy of sets and to settle the last question by showing that $\mathbf{AS} + \mathbf{F^{int}} + \mathbf{P} + (\mathbf{Ord} \subseteq \mathbf{Nat})$ proves $\mathbf{V} \subseteq \mathbf{Fin}$.

The notation used will be standard, with the convention that in a substitution $A(t)$ the bound variables of A do not occur in t. Moreover, whenever there will be the need of introducing a variable to be used also as a bound variable, it will be assumed that this does not occur in the expressions already considered. Also, when writing a formula A as $A(x, y, \ldots)$ no implicit assumptions will be made on the free variables of A.

2 The Theory HS

Theory **HS** has been introduced and studied in [5]. It is an intuitionistic theory of hereditarily finite sets. The language of **HS** consists of

1. a constant symbol $\varnothing$ (*empty set*);
2. binary function symbols W, L (*with* and *less*);
3. binary relation symbols $\in$, $<$.

The value of W, L on the pair (x, y) is denoted with $W_y x$, $L_y x$, respectively.

The nonlogical axioms of **HS** are

S.1 $u \notin \varnothing$;
S.2 $u \in W_y x \leftrightarrow u \in x \vee u = y$;
S.3 $u \in L_y x \leftrightarrow u \in x \wedge u \neq y$;
S.4 $\forall u\, (u \in x \leftrightarrow u \in y) \to x = y$;
S.5 $u \not< \varnothing$;
S.6 $u < W_y x \leftrightarrow u < x \vee u \leq y$;
S.7 (*principle of induction*) for every formula A:

$$A(\varnothing) \wedge \forall x, y\, (A(x) \wedge A(y) \wedge y \notin x \to A(W_y x)) \to \forall x\, A(x).$$

Note that the subpremise $y \notin x$ in S.7 is not irrelevant, as the instance $y \in x \vee y \notin x$ of excluded middle is not available as an axiom, though it turns out to be provable as is $A \vee \neg A$ for any Δ_0-formula A.

3 The Theory AS

Theory **AS** (*absolute set theory*) is an intuitionistic first-order theory with equality. The language of **AS** consists just of the binary relation symbol $\in$. The nonlogical axioms are

A.1 $x \in y \vee x \notin y$;
A.2 for every formula B, $\forall y\ (B(y) \vee \neg B(y)) \rightarrow \exists y \in x\ B(y) \vee \neg \exists y \in x\ B(y)$;
A.3 $\forall u\ (u \in x \leftrightarrow u \in y) \rightarrow x = y$;
A.4 $\exists z\ \forall u\ u \notin z$;
A.5 $\exists z\ \forall u\ (u \in z \leftrightarrow u = x \vee u = y)$;
A.6 $\exists z\ \forall u\ (u \in z \leftrightarrow \exists y \in x\ u \in y)$;
A.7 for every formula A,

$$\forall u \in x\ \exists! v\ A(u, v) \rightarrow \exists z\ (\forall u \in x\ \exists v \in z\ A(u, v) \wedge \forall v \in z\ \exists u \in x\ A(u, v)).$$

A few remarks may be useful here. Axioms A.1, A.2 are rather strong assumptions which, with the aid of A.3, imply the decidability of every bounded formula, making the theory *locally classical* (see Corollary 3.2 below). The fact that in **AS** it is possible to decide whether an element belongs to a set gives this theory quite a different flavor from other well-known intuitionistic set theories. Note, however, that **HS** proves A.1 and A.2 (as well as the other axioms of **AS**—see [5]), so any theory comparable with **HS** needs to be quite strong. Axioms A.3–A.7 are extensionality, empty set, (unordered) pair, union, and replacement, respectively. Note that, by extensionality, the sets z whose existence is postulated in A.4–A.7 are unique.

If A is a formula, we say that A is *determined* relative to a theory T if $T \vdash A \vee \neg A$. Reference to T will be omitted when the context is unambiguous.

Definitions and propositions of this section refer to **AS** and to extensions by definitions of **AS** that will be introduced throughout.

Proposition 3.1 *Let A be a formula; suppose that either A is atomic or it is of one of the forms $\neg B$, $B \gamma C$, $Q y \in x\ B(y)$ where γ is a propositional connective, Q is a quantifier, and B, C are determined formulas. Then A is determined.*

Proof If A is $x \in y$, the assertion is A.1. If A is $\neg B, B \gamma C, \exists y \in x\ B(y)$, $\forall y \in x\ B(y)$, then use logical rules, the hypotheses on B, C, and A.2. If A is $x = y$, then use extensionality and apply the preceding part of this proof. □

Corollary 3.2 (Excluded middle for bounded formulas) *For each bounded formula A, the formula $A \vee \neg A$ is a theorem.*

Proof By metamathematical induction on A, using Proposition 3.1. □

Proposition 3.3 (Separation scheme) *Let $A(y)$ be a formula for which $\forall y\ (A(y) \vee \neg A(y))$. Then $\exists z\ \forall u\ (u \in z \leftrightarrow u \in x \wedge A(u))$.*

Proof By A.2 one has $\exists y \in x\ A(y) \vee \neg \exists y \in x\ A(y)$. If there is $y \in x$ such that $A(y)$, let $C(u, v)$ be the formula $(A(u) \wedge v = u) \vee (\neg A(u) \wedge v = y)$. Then $\forall u\ \exists! v\ C(u, v)$; by replacement $\exists z\ (\forall u \in x\ \exists v \in z\ C(u, v) \wedge \forall v \in z\ \exists u \in x\ C(u, v))$. Hence, $u \in z \leftrightarrow u \in x \wedge A(u)$. If $\neg \exists y \in x\ A(y)$, apply the axiom of empty set. □

3.1 Extensions by definitions In the theory **AS** several definitions of new relation and function symbols will be introduced. A definition is called *determined* if its defining formula satisfies excluded middle. Note the following important fact.

Proposition 3.4 *Let T be a theory with respect to which equality is determined. Then the definitions of function symbols are determined.*

Proof Let $z = F(\vec{x}) \leftrightarrow A(\vec{x}, z)$ be the definition of the n-ary function symbol F. This means that

$$\forall x_1, \ldots, x_n \ \exists! z \ A(\vec{x}, z). \tag{1}$$

Thus, given $x_1, \ldots, x_n, y$, let z satisfy (1). Being equality determined, we can distinguish cases $y = z$, $y \neq z$, obtaining $A(\vec{x}, y)$, $\neg A(\vec{x}, y)$, respectively. □

In order to be able to use Proposition 3.1 for the extended language, we allow a new definition only if determined.

Proposition 3.5 *Let* $\mathbf{AS}^+$ *be an extension of* $\mathbf{AS}$ *by determinate definitions. Then, for any bounded formula A of the relative language, $A \vee \neg A$ is a theorem of* $\mathbf{AS}^+$.

Proof By metamathematical induction on A nested in a metamathematical induction on the number of definitions. If A is atomic the assertion follows from Proposition 3.1 or the meaning of determinate definition. Otherwise, argue as in Proposition 3.1. □

Thus it will be possible to still denote by $\mathbf{AS}$ any extension of $\mathbf{AS}$ by determined definitions.

3.2 Basic development of the theory We begin here to develop the part of the theory $\mathbf{AS}$ that will be needed for the main results. Most proofs will be standard verifications and will therefore be omitted.

Definition 3.6

1. For each formula $A(u, \vec{y})$ such that $\forall \vec{y} \ \forall u \ (A(u, \vec{y}) \vee \neg A(u, \vec{y}))$ is a theorem, $z = \{u \in x \mid A(u, \vec{y})\} \leftrightarrow \forall u \ (u \in z \leftrightarrow u \in x \wedge A(u, \vec{y}))$. Existence is by Proposition 3.3; uniqueness by extensionality. This is the definition of a function symbol of the form $F(x, \vec{y})$.
2. $\varnothing = \{u \in x \mid u \neq u\}$.
3. $L_y x = \{u \in x \mid u \neq y\}$.
4. $z = \{x, y\} \leftrightarrow \forall u \ (u \in z \leftrightarrow u = x \vee u = y)$. Existence by axiom of pairing; uniqueness by extensionality.
5. $\{x\} = \{x, x\}$.
6. $z = \bigcup x \leftrightarrow \forall u \ (u \in z \leftrightarrow \exists y \in x \ u \in y)$. Existence by the axiom of union; uniqueness by extensionality.
7. $x \cup y = \bigcup\{x, y\}$.
8. $W_y x = x \cup \{y\}$.
9. $Sx = W_x x$.
10. $x \cap y = \{u \in x \mid u \in y\}$.
11. $x - y = \{u \in x \mid u \notin y\}$.
12. $(x, y) = \{\{x\}, \{x, y\}\}$.
13. $\mathrm{OP}(z) \leftrightarrow \exists x \in \bigcup z \ \exists y \in \bigcup z \ z = (x, y)$.
14. $x \subseteq y \leftrightarrow \forall z \in x \ z \in y$.
15. $x \subset y \leftrightarrow x \subseteq y \wedge x \neq y$.

Proposition 3.7

(a) $u \notin \varnothing$.
(b) $u \in W_y x \leftrightarrow u \in x \vee u = y$.
(c) $u \in L_y x \leftrightarrow u \in x \wedge u \neq y$.
(d) $\{x\} = W_x \varnothing$.
(e) $\{x, y\} = W_y \{x\}$.
(f) $x \subset y \rightarrow \exists z \in y \; z \notin x$.
(g) $x \subseteq y \wedge y - x \neq \varnothing \rightarrow x \subset y$.
(h) $y \in x \leftrightarrow W_y x = x$.
(i) $y \notin x \leftrightarrow L_y x = x$.
(j) $y \in x \leftrightarrow W_y L_y x = x$.
(k) $y \notin x \leftrightarrow L_y W_y x = x$.
(l) $(x, y) = (u, v) \rightarrow x = u \wedge y = v$.

Proof

(f) Apply A.1, A.2, and extensionality.

(j) From the instance $u = y \vee u \neq y$ of excluded middle, $u \in x \rightarrow (u \in x \wedge u \neq y) \vee u = y$ follows. Then

$$\begin{aligned} y \in x &\rightarrow \forall u \; ((u \in x \wedge u \neq y) \vee u = y \leftrightarrow u \in x) \\ &\rightarrow \forall u \; (u \in W_y L_y x \leftrightarrow u \in x) \\ &\rightarrow W_y L_y x = x. \end{aligned}$$

The converse holds as $y \in W_y L_y x$.

(k) We have

$$\begin{aligned} y \notin x &\rightarrow \forall u \; ((u \in x \vee u = y) \wedge u \neq y \leftrightarrow u \in x) \\ &\rightarrow \forall u \; (u \in L_y W_y x \leftrightarrow u \in x) \\ &\rightarrow L_y W_y x = x. \end{aligned}$$

The converse obtains as $y \notin L_y W_y x$.

(l) A very short proof uses the fact that, by Proposition 3.1, it is possible to distinguish cases $x = y \vee u = v$, $x \neq y \wedge u \neq v$. However, there is also a proof that does not appeal to any instance of excluded middle (see [1]). □

Definition 3.8

$$\begin{aligned} x = \pi_1(z) &\leftrightarrow \exists y \in \bigcup z \; z = (x, y) \vee (\neg \mathrm{OP}(z) \wedge x = \varnothing); \\ y = \pi_2(z) &\leftrightarrow \exists x \in \bigcup z \; z = (x, y) \vee (\neg \mathrm{OP}(z) \wedge y = \varnothing). \end{aligned}$$

Existence is by the (determinate) definition of OP; uniqueness by Proposition 3.7(l).

For $a(y)$ a term, $z = \{a(y) \mid y \in x\} \leftrightarrow \forall u \; (u \in z \leftrightarrow \exists y \in x \; u = a(y))$; existence is by replacement, uniqueness from extensionality.

Proposition 3.9

(a) $\pi_j((x_1, x_2)) = x_j$.
(b) $\mathrm{OP}(z) \rightarrow (\pi_1(z), \pi_2(z)) = z$.
(c) $\exists z \; \forall u \; (u \in z \leftrightarrow \exists x \in v \; \exists y \in w \; u = (x, y))$.

Definition 3.10

1. $z = v \times w \leftrightarrow \forall u\ (u \in z \leftrightarrow \exists x \in v\ \exists y \in w\ u = (x, y))$. Existence by Proposition 3.9(c); uniqueness by extensionality.
2. $\mathrm{Rel}(r) \leftrightarrow \forall z \in r\ \mathrm{OP}(z)$.
3. $\mathrm{Dom}(r) = \{x \in \bigcup\bigcup r \mid \exists y \in \bigcup\bigcup r\ (x, y) \in r\}$.
4. $\mathrm{Rng}(r) = \{y \in \bigcup\bigcup r \mid \exists x \in \bigcup\bigcup r\ (x, y) \in r\}$.
5. $r^{-1} = \{(\pi_2(z), \pi_1(z)) \mid z \in r\}$.
6. $\mathrm{Fun}(g) \leftrightarrow \mathrm{Rel}(g) \wedge \forall x, y \in g\ (\pi_1(x) = \pi_1(y) \to \pi_2(x) = \pi_2(y))$.
7. $z = g(x) \leftrightarrow ((\neg\mathrm{Fun}(g) \vee x \notin \mathrm{Dom}(g)) \wedge z = \varnothing) \vee (\mathrm{Fun}(g) \wedge x \in \mathrm{Dom}(g) \wedge (x, z) \in g)$. Existence by properties of symbols involved; uniqueness by definition of Fun.
8. $g \restriction x = \{z \in g \mid \pi_1(z) \in x\}$.
9. $\mathrm{Inj}(g) \leftrightarrow \mathrm{Fun}(g) \wedge \forall x, y \in \mathrm{Dom}(g)\ (g(x) = g(y) \to x = y)$.
10. $\mathrm{Bij}(g, x, y) \leftrightarrow \mathrm{Inj}(g) \wedge \mathrm{Dom}(g) = x \wedge \mathrm{Rng}(g) = y$.
11. $\mathrm{Trans}(x) \leftrightarrow \forall y \in x\ y \subseteq x$.

Proposition 3.11

(a) $\mathrm{Rel}(r) \leftrightarrow r \subseteq \mathrm{Dom}(r) \times \mathrm{Rng}(r)$.
(b) *For* $A(u, v)$ *a formula,* $\forall u \in x\ \exists! v\ A(u, v) \to \exists g\ (\mathrm{Fun}(g) \wedge \mathrm{Dom}(g) = x \wedge \forall u \in x\ A(u, g(u)))$.

Proof (b) Assuming the premise $\forall u \in x\ \exists! v\ A(u, v)$ and letting $B(u, r)$ be the formula $\mathrm{OP}(r) \wedge \pi_1(r) = u \wedge A(u, \pi_2(r))$, one first gets $\forall u \in x\ \exists! r\ B(u, r)$. By replacement, $\exists g\ (\forall u \in x\ \exists r \in g\ B(u, r) \wedge \forall r \in g\ \exists u \in x\ B(u, r))$. This g works. □

Proposition 3.12

(a) $\mathrm{Trans}(x) \wedge \mathrm{Trans}(y) \to \mathrm{Trans}(x \cap y) \wedge \mathrm{Trans}(x \cup y)$.
(b) $\forall x \in z\ \mathrm{Trans}(x) \to \mathrm{Trans}(\bigcup z)$.
(c) $\mathrm{Trans}(\varnothing)$.
(d) $\mathrm{Trans}(x) \to \mathrm{Trans}(Sx)$.

4 The Principle of ∈-induction

Principle of ∈-induction constitutes a natural constructive version of foundation scheme.

$(\mathbf{F}^{\mathbf{int}})$ For A any formula, $\forall x\ (\forall y \in x\ A(y) \to A(x)) \to \forall x\ A(x)$.

In this section we work in the theory $\mathbf{AS} + \mathbf{F}^{\mathbf{int}}$ (or extensions by definitions of it).

4.1 Ordinal and natural numbers

Definition 4.1

1. $\mathrm{Ord}(x) \leftrightarrow \mathrm{Trans}(x) \wedge \forall y \in x\ \mathrm{Trans}(y)$.
2. $\mathrm{Suc}(x) \leftrightarrow \mathrm{Ord}(x) \wedge \exists y \in x\ x = Sy$.
3. $\mathrm{Lim}(x) \leftrightarrow \mathrm{Ord}(x) \wedge x \neq \varnothing \wedge \neg\mathrm{Suc}(x)$.
4. $\mathrm{Nat}(x) \leftrightarrow \mathrm{Ord}(x) \wedge \neg\mathrm{Lim}(x) \wedge \forall y \in x\ \neg\mathrm{Lim}(y)$.

Proposition 4.2

(a) $\mathrm{Ord}(x) \to \forall y \in x\, \mathrm{Ord}(y)$.
(b) $\mathrm{Ord}(x) \wedge \mathrm{Ord}(y) \to \mathrm{Ord}(x \cap y) \wedge \mathrm{Ord}(x \cup y)$.
(c) $\mathrm{Ord}(\varnothing)$.
(d) $\mathrm{Ord}(x) \to \mathrm{Ord}(Sx)$.
(e) $\forall x \in z\, \mathrm{Ord}(x) \to \mathrm{Ord}(\bigcup z)$.
(f) $\mathrm{Ord}(x) \to \forall u, v, w \in x\, (u \in v \in w \to u \in w)$.
(g) $\mathrm{Nat}(\varnothing)$.
(h) $\mathrm{Nat}(x) \to \forall y \in x\, \mathrm{Nat}(y)$.
(i) $\mathrm{Nat}(x) \to \mathrm{Nat}(Sx)$.

Note that the proof of Proposition 4.2 does not require the use of $\mathbf{F}^{\mathbf{int}}$.

Proposition 4.3

(a) $x \notin x$.
(b) $\neg x \in y \in x$.
(c) $Sx = Sy \to x = y$.
(d) $\mathrm{Ord}(x) \to \forall u, v \in x\, (u \in v \vee u = v \vee v \in u)$.
(e) $\mathrm{Ord}(x) \to \forall z\, (z \subseteq x \wedge z \neq \varnothing \to \exists u \in z\, u \cap z = \varnothing)$.
(f) $\mathrm{Ord}(x) \wedge \mathrm{Ord}(y) \to x \in y \vee x = y \vee y \in x$.
(g) $\mathrm{Ord}(x) \wedge \mathrm{Ord}(y) \to (x \subset y \leftrightarrow x \in y)$.
(h) $\mathrm{Ord}(x) \to \varnothing \in Sx$.
(i) $\mathrm{Ord}(v) \wedge \mathrm{Ord}(w) \wedge v \in w \to Sv \in Sw$.
(j) $\mathrm{Nat}(x) \wedge \mathrm{Nat}(y) \to \mathrm{Nat}(x \cap y) \wedge \mathrm{Nat}(x \cup y)$.

Proof (a) Apply $\in$-induction with A the formula $x \notin x$.

(b) By $\in$-induction it is enough to observe that $\forall z \in x\, \forall y\, \neg z \in y \in z \to \neg \exists v\, x \in v \in x$.

(c) Assuming $Sx = Sy$, by (a) and (b) $x \notin y \notin x$; the result then follows by extensionality.

(d) Let $A(u, v)$ be the formula $u \in v \vee u = v \vee v \in u$ and assume $\mathrm{Ord}(x)$ in order to prove $\forall u, v\, (u \in x \to (v \in x \to A(u, v)))$. By $\in$-induction it is enough to show

$$\forall y \in u\, \forall v\, (y \in x \to (v \in x \to A(y, v))) \to \forall v\, (u \in x \to (v \in x \to A(u, v))).$$

Assume thus $\forall y \in u\, \forall v\, (y \in x \to (v \in x \to A(y, v))), u \in x$ with the aim of proving $\forall v \in x\, A(u, v)$. By $\in$-induction again, it is enough to obtain $\forall z \in v\, (z \in x \to A(u, z)) \to (v \in x \to A(u, v))$, which means to assume $\forall z \in v\, (z \in x \to A(u, z)), v \in x$ and to prove $A(u, v)$. By assumptions, $\forall y \in u\, A(y, v) \wedge \forall z \in v\, A(u, z)$; using this and Proposition 3.1, the proof can be completed by distinguishing cases

$$\exists y \in u\, (y = v \vee v \in y), \quad \exists z \in v\, (u \in z \vee u = z), \quad \forall y \in u\, y \in v \wedge \forall z \in v\, z \in u,$$

corresponding by Proposition 4.2(f) and extensionality to

$$v \in u, \quad u \in v, \quad u = v.$$

(e) By $\in$-induction we can assume

$$\forall y \in x\ (\mathrm{Ord}(y) \to \forall z\ (z \subseteq y \wedge z \neq \varnothing \to \exists u \in z\ u \cap z = \varnothing)), \qquad \mathrm{Ord}(x), z \subseteq x \wedge z \neq \varnothing$$

in order to prove $\exists u \in z\ u \cap z = \varnothing$. By Proposition 3.1 we can distinguish cases $\exists y \in x\ z \cap y \neq \varnothing, \forall y \in x\ z \cap y = \varnothing$. If there is $y \in x$ such that $z \cap y \neq \varnothing$, by inductive hypothesis there exists $u \in z \cap y$ with $u \cap z \cap y = \varnothing$ and it follows $u \cap z = \varnothing$: indeed, if $r \in u \cap z$ then $r \in y$ too. If $\forall y \in x\ z \cap y = \varnothing$ then $\exists! r\ r \in z$; indeed, if $r \in z \wedge s \in z$, using $r \in s \vee s \in r \vee r = s$ the options $r \in s, s \in r$ are ruled out by the corresponding contradictions $z \cap s \neq \varnothing, z \cap r \neq \varnothing$. By (a) one deduces $r \cap z = \varnothing$.

(f) Apply (d) to the ordinal $Sx \cup Sy$.

(j) By (f) and (g), $x \cap y = x \vee x \cap y = y$; similarly, $x \cup y = x \vee x \cup y = y$. □

Propositions 4.2(f) and 4.3(a), (d), (e) state that every ordinal is well ordered by $\in$. Note that using transitive closures the conclusion of Proposition 4.3(e) can be shown to hold for any nonempty set z.

Definition 4.4 $y = Px \leftrightarrow (\mathrm{Suc}(x) \wedge x = Sy) \vee (\neg \mathrm{Suc}(x) \wedge y = \varnothing)$. Existence by (determinate) definition of Suc; uniqueness follows from Proposition 4.3(c).

Proposition 4.5 $\mathrm{Ord}(x) \to \mathrm{Ord}(Px)$.

Proposition 4.6 (Principles of induction)

(a) *(Principle of foundation on ordinals) For A a formula,*

$$\forall x\ (\mathrm{Ord}(x) \to A(x) \vee \neg A(x)) \to (\exists x\ (\mathrm{Ord}(x) \wedge A(x)) \to \exists x\ (\mathrm{Ord}(x) \wedge A(x) \wedge \forall y \in x\ \neg A(y))).$$

(b) *(Principle of transfinite induction) For A a formula,*

$$\forall x\ (\mathrm{Ord}(x) \wedge \forall y \in x\ A(y) \to A(x)) \to \forall x\ (\mathrm{Ord}(x) \to A(x)).$$

(c) *(Principle of transfinite induction by cases) For A a formula,*

$$A(\varnothing) \wedge \forall x\ (\mathrm{Ord}(x) \wedge A(x) \to A(Sx)) \wedge \forall x\ (\mathrm{Lim}(x) \wedge \forall y \in x\ A(y) \to A(x)) \to \forall x\ (\mathrm{Ord}(x) \to A(x)).$$

(d) *(Principle of* Nat*-induction) For A a formula,*

$$A(\varnothing) \wedge \forall x\ (\mathrm{Nat}(x) \wedge A(x) \to A(Sx)) \to \forall x\ (\mathrm{Nat}(x) \to A(x)).$$

(e) *(Principle of course of values* Nat*-induction) For A a formula,*

$$\forall x\ (\mathrm{Nat}(x) \wedge \forall y \in x\ A(y) \to A(x)) \to \forall x\ (\mathrm{Nat}(x) \to A(x)).$$

Proof

(a) Assume $\forall x\ (\mathrm{Ord}(x) \to A(x) \vee \neg A(x))$ and let x be such that $\mathrm{Ord}(x) \wedge A(x)$. Distinguish cases $\forall y \in x\ \neg A(y), \neg\forall y \in x\ \neg A(y)$. If $\forall y \in x\ \neg A(y)$ then $\mathrm{Ord}(x) \wedge A(x) \wedge \forall y \in x\ \neg A(y)$. So suppose $\neg\forall y \in x\ \neg A(y)$. Then, by the hypotheses and using A.2, one gets $\exists y \in x\ A(y)$. Let $z = \{y \in x \mid A(y)\} \neq \varnothing$.

By Proposition 4.3(e), there is $u \in z$ such that $u \cap z = \varnothing$; then $\mathrm{Ord}(u) \wedge A(u)$. Finally, if $y \in u$ then $\neg A(y)$, completing the proof.

(b) Assume the premise $\forall x\ (\mathrm{Ord}(x) \wedge \forall y \in x\ A(y) \to A(x))$ and prove the conclusion $\forall x\ (\mathrm{Ord}(x) \to A(x))$ by $\in$-induction.

(c) The asserted schema is equivalent to

$$A(\varnothing) \wedge \forall x\ (\mathrm{Ord}(x) \wedge \mathrm{Suc}(x) \wedge A(Px) \to A(x)) \wedge \\ \wedge \forall x\ (\mathrm{Lim}(x) \wedge \forall y \in x\ A(y) \to A(x)) \to \forall x\ (\mathrm{Ord}(x) \to A(x))$$

which holds by the principle of transfinite induction.

(d) Use (b).

(e) The equivalent form $\forall x\ (\forall y \in x\ (\mathrm{Nat}(y) \to A(y)) \to (\mathrm{Nat}(x) \to A(x))) \to \forall x\ (\mathrm{Nat}(x) \to A(x))$ is a particular case of $\in$-induction. □

In the sequel some arithmetical facts about natural numbers will be helpful. However, it will not be necessary to develop arithmetics entirely: the following mild properties will be enough.

Proposition 4.7

(a) $\mathrm{Nat}(x) \to \neg\exists y, g\ (y \subset x \wedge \mathrm{Bij}(g, x, y))$.
(b) $\mathrm{Nat}(w) \to (\mathrm{Nat}(z) \wedge \mathrm{Bij}(p, z, x) \wedge \mathrm{Bij}(g, w, y) \wedge x \subset y \to z \in w)$.
(c) $\mathrm{Nat}(x) \to (y \subseteq z \wedge \mathrm{Bij}(g, x, z) \to \exists v, p\ (\mathrm{Nat}(v) \wedge \mathrm{Bij}(p, v, y)))$.
(d) $\mathrm{Nat}(x) \to (\mathrm{Nat}(y) \wedge \mathrm{Bij}(p, x, u) \wedge \mathrm{Bij}(g, y, v) \to \exists q, w\ (\mathrm{Nat}(w) \wedge \mathrm{Bij}(q, w, u \cup v)))$.
(e) $\mathrm{Nat}(x) \to (\mathrm{Nat}(y) \wedge \mathrm{Bij}(p, x, u) \wedge \mathrm{Bij}(g, y, v) \to \exists q, w\ (\mathrm{Nat}(w) \wedge \mathrm{Bij}(q, w, u \times v)))$.
(f) $\mathrm{Nat}(y) \to (x \neq \varnothing \wedge \mathrm{Bij}(g, y, x) \wedge \forall u \in x\ \mathrm{Ord}(u) \to \exists u \in x\ \forall v \in x\ (v \in u \vee v = u))$.

Proof By Nat-induction. For (d) note that it is not restrictive to assume u, v disjoint. □

4.2 Transitive closure, primitive recursion

Proposition 4.8 (Transitive closure)

$$\exists! z\ (x \subseteq z \wedge \mathrm{Trans}(z) \wedge \forall y\ (x \subseteq y \wedge \mathrm{Trans}(y) \to z \subseteq y)).$$

Proof Let $A(x, z)$ be the formula $x \subseteq z \wedge \mathrm{Trans}(z) \wedge \forall y\ (x \subseteq y \wedge \mathrm{Trans}(y) \to z \subseteq y)$. Since $A(x, z) \wedge A(x, w) \to z = w$, uniqueness is immediate. Let $B(x, z)$ abbreviate

$$x \subseteq z \wedge \mathrm{Trans}(z) \wedge \forall r \in z\ \exists v\ \exists p\ (\mathrm{Nat}(v) \wedge \mathrm{Fun}(p) \\ \wedge \mathrm{Dom}(p) = Sv \wedge r = p(\varnothing) \wedge \forall q \in v\ p(q) \in p(Sq) \wedge p(v) \in x).$$

It is shown $B(x, z) \to A(x, z)$. Assume to this aim $B(x, z)$, so that in particular $x \subseteq z \wedge \mathrm{Trans}(z)$. First, by Nat-induction,

$$\mathrm{Nat}(v) \to (\mathrm{Fun}(p) \wedge \mathrm{Dom}(p) = Sv \wedge s = p(\varnothing) \\ \wedge \forall q \in v\ p(q) \in p(Sq) \wedge p(v) \in x \to s \in z). \quad (2)$$

It remains to prove $\forall y\ (x \subseteq y \wedge \mathrm{Trans}(y) \rightarrow z \subseteq y)$. Assume $x \subseteq y \wedge \mathrm{Trans}(y)$, to show $\forall r \in z\ r \in y$ and let $r \in z$, to obtain $r \in y$. It is then enough to prove

$$\mathrm{Nat}(v) \rightarrow (s \in z \wedge \mathrm{Fun}(p) \wedge \mathrm{Dom}(p) = Sv \wedge s = p(\varnothing) \wedge \forall q \in v\ p(q) \in p(Sq) \wedge p(v) \in x \rightarrow s \in y)$$

which is obtained by Nat-induction using (2) in the induction step. In particular, we have now $B(x,z) \wedge B(x,w) \rightarrow z = w$.

To conclude the proof it is then enough to show $\exists z\ B(x,z)$; this is achieved by $\in$-induction, that is, assuming $\forall u \in x\ \exists q\ B(u,q)$ to obtain $\exists z\ B(x,z)$. By the hypothesis of $\in$-induction we have $\forall u \in x\ \exists! q\ B(u,q)$; thus, by replacement, there is w such that

$$\forall u \in x\ \exists v \in w\ B(u,v) \wedge \forall v \in w\ \exists u \in x\ B(u,v).$$

If $z = x \cup \bigcup w$ then $B(x,z)$, completing the proof. □

Definition 4.9

$$z = \mathrm{TC}(x) \leftrightarrow x \subseteq z \wedge \mathrm{Trans}(z) \wedge \forall y\ (x \subseteq y \wedge \mathrm{Trans}(y) \rightarrow z \subseteq y).$$

Existence and uniqueness are stated in Proposition 4.8.

Proposition 4.10 $\mathrm{Trans}(x) \leftrightarrow \mathrm{TC}(x) = x$.

Proposition 4.11 $\mathrm{TC}(y) = y \cup \bigcup\{\mathrm{TC}(x) \mid x \in y\}$.

Proof First note that transitivity of the right-hand side of the equation is the unique interesting claim of the proposition, since the other two properties are immediate.

Let $w = \{\mathrm{TC}(x) \mid x \in y\}$. If $r \in y$, then $r \subseteq \mathrm{TC}(r) \in w$; thus $r \subseteq \mathrm{TC}(r) \subseteq \bigcup w$. If $r \in \bigcup w$, then there is $x \in y$ such that $r \in \mathrm{TC}(x)$; from $\mathrm{Trans}(\mathrm{TC}(x))$ it follows $r \subseteq \mathrm{TC}(x)$, hence $r \subseteq \bigcup w$ again. □

Proposition 4.12 (Induction on transitive closure) *Let $A(x)$ be a formula. Then*

$$\forall x\ (\forall y \in \mathrm{TC}(x)\ A(y) \rightarrow A(x)) \rightarrow \forall x\ A(x).$$

Proof Assume the premise $\forall x\ (\forall y \in \mathrm{TC}(x)\ A(y) \rightarrow A(x))$, in order to prove $\forall x\ \forall y \in \mathrm{TC}(x)\ A(y)$. This will do as $x \in \mathrm{TC}(\{x\})$.

By $\in$-induction it is enough to obtain $\forall w \in x\ \forall y \in \mathrm{TC}(w)\ A(y) \rightarrow \forall y \in \mathrm{TC}(x)\ A(y)$. Assuming also $\forall w \in x\ \forall y \in \mathrm{TC}(w)\ A(y)$, one gets $\forall w \in x\ A(w)$. Thus $\forall y \in x \cup \bigcup\{\mathrm{TC}(w) \mid w \in x\}\ A(y)$. By Proposition 4.11 this gives the claim. □

Proposition 4.13 (Definition by primitive recursion) *Let G be an $n+2$-ary function symbol. It is possible to define, in a unique way, an $n+1$-ary function symbol F in such a way that the formula*

$$F(\vec{x}, y) = G(\vec{x}, y, \{(r, F(\vec{x}, r)) \mid r \in \mathrm{TC}(y)\}) \tag{3}$$

is a theorem of the theory obtained from $\mathbf{AS}+\mathbf{F}^{\mathbf{int}}$ *(or some extension of it) by adding the defining axiom of F.*

Proof Let $A(\vec{x}, y, z, g)$ be the formula

$$\mathrm{Fun}(g) \wedge \mathrm{Dom}(g) = \mathrm{TC}(y) \wedge z = G(\vec{x}, y, g) \wedge \forall w \in \mathrm{TC}(y)\ g(w) = G(\vec{x}, w, g \restriction \mathrm{TC}(w)).$$

Proving $\forall x_1, \ldots, x_n, y\, \exists! z\, \exists g\, A(\vec{x}, y, z, g)$ it will yield the existence and uniqueness conditions to define

$$z = F(\vec{x}, y) \leftrightarrow \exists g\, A(\vec{x}, y, z, g); \tag{4}$$

more precisely, we are going to show

$$\begin{cases} A(\vec{x}, y, z, g) \wedge A(\vec{x}, y, r, p) \rightarrow z = r \wedge g = p \\ \forall y\, \exists z, g\, A(\vec{x}, y, z, g). \end{cases}$$

First we remark that

$$A(\vec{x}, y, z, g) \wedge w \in \mathrm{TC}(y) \rightarrow A(\vec{x}, w, g(w), g \restriction \mathrm{TC}(w)). \tag{5}$$

Now we are going to prove $A(\vec{x}, y, z, g) \wedge A(\vec{x}, y, r, p) \rightarrow z = r \wedge g = p$ by induction on $\mathrm{TC}(y)$. Thus suppose $\forall w \in \mathrm{TC}(y)\, (A(\vec{x}, w, u, q) \wedge A(\vec{x}, w, v, s) \rightarrow u = v \wedge q = s)$ and assume $A(\vec{x}, y, z, g) \wedge A(\vec{x}, y, r, p)$ in order to show $z = r \wedge g = p$. Since we have $z = G(\vec{x}, y, g) \wedge r = G(\vec{x}, y, p)$ it is enough to obtain $g = p$; as $\mathrm{Dom}(g) = \mathrm{Dom}(p) = \mathrm{TC}(y)$, this will consist in showing $\forall w \in \mathrm{TC}(y)\, g(w) = p(w)$. By the assumptions and what was observed earlier, for $w \in \mathrm{TC}(y)$, we have $A(\vec{x}, w, g(w), g \restriction \mathrm{TC}(w)) \wedge A(\vec{x}, w, p(w), p \restriction \mathrm{TC}(w))$. By induction hypothesis, $g(w) = p(w)$.

Then we show $\exists z, g\, A(\vec{x}, y, z, g)$ by induction on $\mathrm{TC}(y)$, so assume

$$\forall w \in \mathrm{TC}(y)\, \exists u, p\, A(\vec{x}, w, u, p).$$

By what was already observed, the pair (u, p) is unique; so, in particular,

$$\forall w \in \mathrm{TC}(y)\, \exists! u\, \exists! p\, A(\vec{x}, w, u, p).$$

From Proposition 3.11(b) there is g such that

$$\mathrm{Fun}(g) \wedge \mathrm{Dom}(g) = \mathrm{TC}(y) \wedge \forall w \in \mathrm{TC}(y)\, \exists! p\, A(\vec{x}, w, g(w), p).$$

So the existence condition will follow from $A(\vec{x}, y, G(\vec{x}, y, g), g)$, which is proved by obtaining $\forall v \in \mathrm{TC}(y)\, g(v) = G(\vec{x}, v, g \restriction \mathrm{TC}(v))$. Given $v \in \mathrm{TC}(y)$, let p_v be the unique object such that $A(\vec{x}, v, g(v), p_v)$, so that $g(v) = G(\vec{x}, v, p_v)$; so it remains to prove $g \restriction \mathrm{TC}(v) = p_v$. Let $r \in \mathrm{Dom}(p_v) = \mathrm{TC}(v)$. By (5) it follows $A(\vec{x}, r, p_v(r), p_v \restriction \mathrm{TC}(r))$. So, by the uniqueness part of this proof, $p_v(r) = g(r)$.

We have thus justified the definition introduced in (4). This definition satisfies (3). Uniqueness uses again induction on the transitive closure. □

Corollary 4.14 *Let G be an $n+2$-ary function symbol. It is possible to define in a unique way an $n+1$-ary function symbol F in such a way that the formula*

$$F(\vec{x}, y) = G(\vec{x}, y, \{(r, F(\vec{x}, r)) \mid r \in y\})$$

is a theorem of the theory obtained from $\mathbf{AS}+\mathbf{F}^{\mathbf{int}}$ *(or some extension of it) by adding the defining axiom of F.*

Proof Let H be the $n+2$-ary function symbol defined by $H(\vec{x}, y, g) = G(\vec{x}, y, g \restriction y)$ and apply Proposition 4.13. □

4.3 Ranks

Definition 4.15 $\operatorname{rank}(x) = \bigcup\{S\operatorname{rank}(y) \mid y \in x\}$. This is justified by Corollary 4.14.

Proposition 4.16

(a) $\operatorname{Ord}(\operatorname{rank}(x))$.
(b) $\operatorname{Ord}(x) \leftrightarrow \operatorname{rank}(x) = x$.

Proof (a) By $\in$-induction.

(b) The forward implication by transfinite induction; the converse by (a). □

5 The Axiom of Finiteness

The axiom of finiteness $\mathbf{V} \subseteq \mathbf{Fin}$ asserts that every set is in bijection with some natural number:

($\mathbf{V} \subseteq \mathbf{Fin}$) $\qquad \exists y, g\ (\operatorname{Nat}(y) \wedge \operatorname{Bij}(g, y, x))$.

In this section we work in $\mathbf{AS} + \mathbf{F}^{\mathbf{int}} + (\mathbf{V} \subseteq \mathbf{Fin})$ (or extensions by definitions).

Proposition 5.1 $\exists! y\ \exists g\ (\operatorname{Nat}(y) \wedge \operatorname{Bij}(g, y, x))$.

Proof By $\mathbf{V} \subseteq \mathbf{Fin}$ and Proposition 4.7(a). □

Theory $\mathbf{AS} + \mathbf{F}^{\mathbf{int}}$ is already enough to settle the first problem from [5]. Let

$\mathbf{Ord} \subseteq \mathbf{Nat}$) $\operatorname{Ord}(x) \to \operatorname{Nat}(x)$

be the assertion stating that every ordinal is a natural number.

Theorem 5.2 $\mathbf{AS} + \mathbf{F}^{\mathbf{int}} + (\mathbf{V} \subseteq \mathbf{Fin}) \vdash \mathbf{Ord} \subseteq \mathbf{Nat}$.

Proof Assume $\operatorname{Ord}(x)$. By finiteness, there are a natural number y and a bijection g from y onto x. As $\operatorname{Nat}(y) \to \operatorname{Ord}(y)$, by Proposition 4.3(f) we have $x \in y \vee x = y \vee y \in x$; it is then enough to check $x \notin y, y \notin x$. Formula $x \notin y$ holds by Proposition 4.7(a). On the other hand, admitting $y \in x$, one gets $Sy \subseteq x$ together with $\operatorname{Bij}(\{u \in g \mid \pi_2(u) \in Sy\}^{-1}, Sy, \{r \in y \mid g(r) \in Sy\})$ where $\{r \in y \mid g(r) \in Sy\} \subseteq y \subset Sy$. Since $\operatorname{Nat}(Sy)$ this contradicts Proposition 4.7(a). □

5.1 Deriving induction from finiteness The next step will be the derivation of the induction principle S.7 of **HS** in $\mathbf{AS} + \mathbf{F}^{\mathbf{int}} + (\mathbf{V} \subseteq \mathbf{Fin})$. The main tool will be the function K defined below. The idea of function K is to assign to each set x a number describing the complexity of building x from scratch by adding 1 to the sum of the complexities of all elements of x. Under the hypotheses of S.7 the conclusion will be drawn by course of values Nat-induction on this complexity number.

Definition 5.3

1. $z = E(r, w) \leftrightarrow (\operatorname{Fun}(w) \wedge r \in \operatorname{Dom}(w) \wedge z = \{r\} \times w(r)) \vee ((\neg \operatorname{Fun}(w) \vee r \notin \operatorname{Dom}(w)) \wedge z = \varnothing)$. Existence and uniqueness hold by the properties of functions and relations involved.
2. $I(x, w) = \bigcup\{E(r, w) \mid r \in x\}$.
3. $z = J(x, w) \leftrightarrow \exists v, g\ (\operatorname{Nat}(v) \wedge \operatorname{Bij}(g, v, I(x, w)) \wedge z = Sv)$. Existence by finiteness; uniqueness by Proposition 4.7(a).
4. $K(x) = J(x, \{(y, K(y)) \mid y \in x\})$. This is justified by Corollary 4.14.

Proposition 5.4 $\varnothing \in K(x)$.

Proposition 5.5 $x \subset y \to I(x, \{(z, K(z)) \mid z \in x\}) \subset I(y, \{(z, K(z)) \mid z \in y\})$.

Proof Assume $x \subset y$ and let

$$p \in I(x, \{(z, K(z)) \mid z \in x\}) = \bigcup\{E(r, \{(z, K(z)) \mid z \in x\}) \mid r \in x\};$$

hence there is $r \in x$ such that

$$\begin{aligned} p &\in E(r, \{(z, K(z)) \mid z \in x\}) = \{r\} \times K(r) = E(r, \{(z, K(z)) \mid z \in y\}) \\ &\subseteq \bigcup\{E(r, \{(z, K(z)) \mid z \in y\}) \mid r \in y\} = I(y, \{(z, K(z)) \mid z \in y\}); \end{aligned}$$

so we can conclude $I(x, \{(z, K(z)) \mid z \in x\}) \subseteq I(y, \{(z, K(z)) \mid z \in y\})$.

By Proposition 3.7(f), there is $q \in y$ such that $q \notin x$; moreover, by Proposition 5.4, $\varnothing \in K(q)$; hence

$$\begin{aligned} (q, \varnothing) &\in \{q\} \times K(q) = E(q, \{(z, K(z)) \mid z \in y\}) \\ &\subseteq \bigcup\{E(r, \{(z, K(z)) \mid z \in y\}) \mid r \in y\} = I(y, \{(z, K(z)) \mid z \in y\}). \end{aligned}$$

On the other hand $q \notin x$; so $\forall r \in x\ (q, \varnothing) \notin E(r, \{(z, K(z)) \mid z \in x\})$ and hence

$$(q, \varnothing) \notin \bigcup\{E(r, \{(z, K(z)) \mid z \in x\}) \mid r \in x\} = I(x, \{(z, K(z)) \mid z \in x\}).$$

Now apply Proposition 3.7(g). □

Proposition 5.6 $x \subset y \to K(x) \in K(y)$.

Proof Use Propositions 5.5, 4.7(b), and 4.3(i). □

Proposition 5.7 $K(\{x\}) = SK(x)$.

Proof By definition, there are v, g such that

$$\mathrm{Nat}(v) \wedge \mathrm{Bij}(g, v, I(\{x\}, \{(x, K(x))\})) \wedge K(\{x\}) = Sv$$

where $I(\{x\}, \{(x, K(x))\}) = E(x, \{(x, K(x))\}) = \{x\} \times K(x)$ and hence g is a bijection between v and $\{x\} \times K(x)$. Moreover, $\mathrm{Bij}(\{r \in K(x) \times (\{x\} \times K(x)) \mid \pi_1(r) = \pi_2(\pi_2(r))\}, K(x), \{x\} \times K(x))$. Since $\mathrm{Nat}(K(x))$, by Proposition 5.1 one gets $K(x) = v$, hence the claim. □

Proposition 5.8 $x \in y \to K(x) \in K(y)$.

Proof From the premise $x \in y$ one has $\{x\} \subseteq y$. By the instance $\{x\} = y \vee \{x\} \neq y$ of excluded middle, we can distinguish cases $\{x\} = y$, $\{x\} \neq y$. If $\{x\} = y$, from Proposition 5.7 it follows $K(x) \in SK(x) = K(\{x\}) = K(y)$. If $\{x\} \neq y$, then $\{x\} \subset y$. By Proposition 5.6, $K(x) \in SK(x) = K(\{x\}) \in K(y)$. The assertion follows as $\mathrm{Nat}(K(y))$ and thus $K(x) \in K(y)$. □

Now all ingredients are ready to prove the induction principle, providing the answer of another problem raised in [5].

Theorem 5.9 (Principle of induction) *Let A be a formula.*

$$\mathbf{AS} + \mathbf{F}^{\mathbf{int}} + (\mathbf{V} \subseteq \mathbf{Fin}) \\ \vdash A(\varnothing) \wedge \forall x, y\ (A(x) \wedge A(y) \wedge y \notin x \to A(W_y x)) \to \forall x\ A(x).$$

Proof Assume the premise $A(\varnothing)\wedge\forall x, y\ (A(x)\wedge A(y)\wedge y\notin x\rightarrow A(W_y x))$ in order to get the conclusion $\forall x\ A(x)$. Let $B(z)$ be the formula $\forall w\ (K(w)=z\rightarrow A(w))$. Since $\forall x\ \exists z\ (\mathrm{Nat}(z)\wedge K(x)=z)$, to prove $\forall x\ A(x)$ it is enough to show $\forall z\ (\mathrm{Nat}(z)\rightarrow B(z))$, which will be shown by course of values Nat-induction on z. So assume $\mathrm{Nat}(s)\wedge\forall y\in s\ B(y)$ in order to get $B(s)$ that is $\forall w\ (K(w)=s\rightarrow A(w))$; so assume $K(w)=s$ too, to obtain $A(w)$. By the instance $w=\varnothing\vee w\neq\varnothing$ of excluded middle, we can distinguish cases $w=\varnothing$, $w\neq\varnothing$.

If $w=\varnothing$, then $A(w)$ by the premise of the principle to be proved. If $w\neq\varnothing$, let $r\in w$. By Proposition 3.7(j), $W_r L_r w=w$; moreover, Propositions 5.8 and 5.6 yield $K(r)\in s\wedge K(L_r w)\in s$. By inductive assumption $B(K(r))\wedge B(K(L_r w))$ and so $A(r)\wedge A(L_r w)$. As $r\notin L_r w$, by the premise of the principle we got $A(W_r L_r w)$, which means $A(w)$. □

Remark 5.10 Since the instance $y\in x\vee y\notin x$ of excluded middle is an axiom of the theory $\mathbf{AS}+\mathbf{F^{int}}+(\mathbf{V}\subseteq\mathbf{Fin})$, clause $y\notin x$ in the premise of the principle of induction can be dropped. That is, principle of induction can be equivalently formulated as

$$A(\varnothing)\wedge\forall x, y\ (A(x)\wedge A(y)\rightarrow A(W_y x))\rightarrow\forall x\ A(x).$$

Recall from [5] that **HS** is an extension of $\mathbf{AS}+\mathbf{F^{int}}+(\mathbf{V}\subseteq\mathbf{Fin})$. Moreover, **HS** is equivalent to the theory obtained removing $<$ from the primitive vocabulary, omitting axioms S.5 and S.6, and adding $y<x\leftrightarrow y\in\mathrm{TC}(x)$ as defining axiom for $<$. Theorem 5.9 entails that the theories **HS** and $\mathbf{AS}+\mathbf{F^{int}}+(\mathbf{V}\subseteq\mathbf{Fin})$ are actually equivalent. Thus definitions and propositions concerning any of the two theories refer in fact to both.

6 The Power Set Axiom

Let $\mathbf{AS}+\mathbf{F^{int}}+\mathbf{P}$ be the theory obtained from $\mathbf{AS}+\mathbf{F^{int}}$ by adding the power set axiom

(P) $$\exists z\ \forall y\ (y\in z\leftrightarrow y\subseteq x).$$

In this section we work in $\mathbf{AS}+\mathbf{F^{int}}+\mathbf{P}$. By [5], **HS** extends $\mathbf{AS}+\mathbf{F^{int}}+\mathbf{P}$.

Definition 6.1

1. $z=\mathcal{P}(x)\leftrightarrow\forall y\ (y\in z\leftrightarrow y\subseteq x)$. Existence by the power set axiom; uniqueness by extensionality.
2. $x^y=\{g\in\mathcal{P}(y\times x)\mid\mathrm{Fun}(g)\}$.
3. $R(x)=\bigcup\{\mathcal{P}(R(y))\mid y\in x\}$. This is justified by Corollary 4.14.

Proposition 6.2 $\mathrm{Trans}(x)\leftrightarrow\mathrm{Trans}(\mathcal{P}(x))$.

Proposition 6.3

(a) $\mathrm{Nat}(x)\rightarrow(\mathrm{Bij}(g,x,y)\rightarrow\exists z, h\ (\mathrm{Nat}(z)\wedge\mathrm{Bij}(h,z,\mathcal{P}(y))))$.

(b) $\mathrm{Nat}(x)\wedge\mathrm{Nat}(y)\wedge\mathrm{Bij}(g,x,u)\wedge\mathrm{Bij}(p,y,v)$
$\rightarrow\exists w, h\ (\mathrm{Nat}(w)\wedge\mathrm{Bij}(h,w,u^v))$.

Proof (a) By Nat-induction on x. Since $\mathrm{Bij}(g,\varnothing,y)$ implies $y=\varnothing$, $\mathcal{P}(y)=\{\varnothing\}$, the basis is readily established. Assume the inductive hypothesis for natural x and suppose $\mathrm{Bij}(g,Sx,y)$. Since $\mathrm{Bij}(g\restriction x,x,L_{g(x)}y)$, there are a natural number z and a bijection between z and $\mathcal{P}(L_{g(x)}y)$. Moreover, $\mathrm{Bij}(\{(p,W_{g(x)}p)\mid p\in\mathcal{P}(L_{g(x)}y)\},$

$\mathcal{P}(L_{g(x)}y)$, $\{v \in \mathcal{P}(y) \mid g(x) \in v\}$), and thus there is also a bijection between z and $\{v \in \mathcal{P}(y) \mid g(x) \in v\}$. Finally, $\mathcal{P}(L_{g(x)}y) \cup \{v \in \mathcal{P}(y) \mid g(x) \in v\} = \mathcal{P}(y)$. Now apply Proposition 4.7(d).

(b) By (a) and Proposition 4.7(e), (c). □

Proposition 6.4 $\mathrm{Ord}(x) \to (y \in R(x) \leftrightarrow \mathrm{rank}(y) \in x)$.

Proof By transfinite induction on x. Assume the inductive hypothesis $\mathrm{Ord}(x) \wedge \forall z \in x\ (y \in R(z) \leftrightarrow \mathrm{rank}(y) \in z)$. Then we have

$$y \in R(x) \leftrightarrow \exists r \in x\ y \in \mathcal{P}(R(r)) \leftrightarrow \exists r \in x\ \forall z \in y\ \mathrm{rank}(z) \in r \leftrightarrow$$
$$\leftrightarrow \exists r \in x\ \mathrm{rank}(y) \in Sr \leftrightarrow \mathrm{rank}(y) \in x.$$

□

Proposition 6.5 $\mathrm{Ord}(y) \wedge x \in y \to R(x) \subset R(y)$.

Proof Assuming the premise, let $z \in R(x)$. By Proposition 6.4, $\mathrm{rank}(z) \in x$ and consequently $\mathrm{rank}(z) \in y$. By Proposition 6.4 again, $z \in R(y)$. Moreover, $x \in R(y) - R(x)$. □

Proposition 6.6 $\mathrm{Ord}(x) \to R(Sx) = \mathcal{P}(R(x))$.

Proof By definition $R(Sx) = \bigcup\{\mathcal{P}(R(v)) \mid v \in Sx\}$. By Proposition 6.5, the right-hand side of this equation is just $\mathcal{P}(R(x))$. □

Proposition 6.7 $\mathrm{Nat}(x) \to \exists y, g\ (\mathrm{Nat}(y) \wedge \mathrm{Bij}(g, y, R(x)))$.

Proof By Nat-induction on x, the basis holding as $R(\varnothing) = \varnothing$. Admitting the property for natural x and having $R(Sx) = \mathcal{P}(R(x))$, the assertion follows from Proposition 6.3(a). □

We are now ready to establish the following answer to the remaining question raised in [5].

Theorem 6.8 $\mathbf{AS} + \mathbf{F}^{\mathbf{int}} + \mathbf{P} + (\mathbf{Ord} \subseteq \mathbf{Nat}) \vdash \mathbf{V} \subseteq \mathbf{Fin}$.

Proof We want to prove $\exists y, g\ (\mathrm{Nat}(y) \wedge \mathrm{Bij}(g, y, x))$. By Propositions 6.4 and 6.6 we derive $x \in R(S\,\mathrm{rank}(x)) = \mathcal{P}(R(\mathrm{rank}(x)))$. Thus $x \subseteq R(\mathrm{rank}(x))$. By Proposition 6.7, $R(\mathrm{rank}(x))$ is finite and such is x by Proposition 4.7(c). □

To summarize, denote by $\mathbf{ZFC}^{\mathbf{int}}$ the theory $\mathbf{AS} + \mathbf{F}^{\mathbf{int}} + \mathbf{P} + \mathbf{Inf} + \mathbf{AC}$, where

(Inf) $\exists x\ (\varnothing \in x \wedge \forall y \in x\ Sy \in x)$,
(AC) $\exists g\ (\mathrm{Fun}(g) \wedge \mathrm{Dom}(g) = x \wedge \forall y \in x\ (y \neq \varnothing \to g(y) \in y))$

are the axioms of infinity and choice.

We have then proved that in $\mathbf{ZFC}^{\mathbf{int}} - \mathbf{Inf}$ the assertions $\mathbf{V} \subseteq \mathbf{Fin}$ and $\mathbf{Ord} \subseteq \mathbf{Nat}$ imply each other. Moreover, the theories **HS**, $\mathbf{ZFC}^{\mathbf{int}} - \mathbf{Inf} + (\mathbf{V} \subseteq \mathbf{Fin})$ and $\mathbf{ZFC}^{\mathbf{int}} - \mathbf{Inf} + (\mathbf{Ord} \subseteq \mathbf{Nat})$ are all equivalent. Note, however, that our analysis stressed the use of **P** for proving $\mathbf{V} \subseteq \mathbf{Fin}$ from $\mathbf{Ord} \subseteq \mathbf{Nat}$ (Theorem 6.8), whereas the other implications (Theorems 5.2 and 5.9) did not need this principle.

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