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On Lovely Pairs and the $(\exists y \in P)$ Quantifier

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Abstract Given a lovely pair $P \prec M$ of models of a simple theory T, we study the structure whose universe is P and whose relations are the traces on P of definable (in $\mathcal{L}$ with parameters from M) sets in M. We give a necessary and sufficient condition on T (which we call weak lowness) for this structure to have quantifier-elimination. We give an example of a non-weakly-low simple theory.

1 Introduction

In [1] Baisalov and Poizat introduce and discuss the notion of the elimination of the $\exists y \in P$ quantifier in elementary pairs $P \prec M$ of models of a complete first-order theory T. If T is stable, then (by definability of types) all pairs eliminate $\exists y \in P$. They prove that the same is true if T is o-minimal and M is saturated over P. They also state that "belle paires" of the theory of the random graph *do not* eliminate $\exists y \in P$. However, for T a simple theory, it is natural to consider lovely pairs rather than belles paires. In this paper we investigate whether lovely pairs of a simple theory eliminate $\exists y \in P$. In the process we come up with another combinatorial property of forking, which we call "weak-lowness".

Let us now give some more precise definitions. T will denote a complete first-order theory in a language $\mathcal{L}$. For a subset B of a model of T, $\mathcal{L}(B)$ denotes $\mathcal{L}$ together with constants for elements of B.

By a pair of models of T we will mean an elementary pair $P \prec M$ of models. We can view it as a structure (M, P) in the language $\mathcal{L}_P$ obtained by adding a new unary predicate symbol P to the language $\mathcal{L}$ of T. To specify the smaller model in the pair (M, P), we will often use notation $P(M)$ (rather than P). If $A \subset M$, $P(A)$ denotes $A \cap P(M)$, the P-part of A. When we write $(M, P) \subset (N, P)$, we mean $M \subset N$ and $M \cap P(N) = P(M)$.

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Definition 1.1

(i) The pair $P \prec M$ of models of T is said to eliminate the $\exists y \in P$ quantifier, if for every $\mathcal{L}$-formula $\varphi(x, y, z)$ and a in M there is an $\mathcal{L}$-formula $\psi(x, w)$ and b in M such that for all $c \in P$, $M \models \psi(c, b)$ if and only if there is $d \in P$ such that $M \models \varphi(c, d, a)$.

(ii) We say that the $\exists y \in P$ quantifier is uniformly eliminable for $P \prec M$ if in (i) above, $\psi(x, w)$ can be chosen to depend only on $\varphi(x, y, z)$ (not a).

Remark 1.2 Let $P \prec M$ be a pair of models of T. For each $\mathcal{L}$-formula $\varphi(x, y)$ and $a \in M$, let $R_{\varphi,a}(x)$ be a new predicate symbol. Let $\mathcal{L}^*$ be the resulting language (which depends of course on M). Let P^* denote the structure in the language $\mathcal{L}^*$, with universe P, where each predicate $R_{\varphi,a}(x)$ is interpreted by $\varphi(P, a)$. In other words, P^* is the structure induced on P by $\mathcal{L}$-formulas with parameters in M. Let $T^* = Th(P^*)$. Then $P \prec M$ eliminates the $\exists y \in P$ quantifier if and only if T^* has quantifier-elimination in the language $\mathcal{L}^*$.

We will assume knowledge of the basics of stability/simplicity theory, for which the reader is referred to Wagner [4]. We first recall the notions of lowness and lovely pairs which will play a role in this paper. The "low" property was introduced by Buechler and Shami. Lovely pairs appear in Ben-Yaacov et al. [2].

Suppose now T to be a simple theory and work in a saturated model $\bar{M}$. We say that T is *low* if for any $\mathcal{L}$-formula $\varphi(x, y)$ and possibly infinite tuple z of variables, the condition (on (y, z)) that $\varphi(x, y)$ forks over z is type-definable. Equivalently for any formula $\varphi(x, y)$ there is $k < \omega$ such that for any indiscernible sequence $(b_i : i < \omega)$, $\{\varphi(x, b_i) : i < \omega\}$ is inconsistent if and only if it is k-inconsistent.

A (κ-)lovely pair of models of T is a pair $P \prec M$ of models of T such that for any $A \subset M$ of cardinality $\leq |\mathcal{L}|$ ($< \kappa$) and complete $\mathcal{L}$-type $p(x)$ over A,

(i) some nonforking extension of $p(x)$ over $A \cup P$ is realized in M, and
(ii) if, moreover, $p(x)$ does not fork over $P(A)$ then $p(x)$ is realized in P.

If T is stable, lovely pairs coincide with Poizat's belles paires.

Definition 1.3 T is said to be weakly low if for any complete finitary type $tp(a/B)$, and any $\mathcal{L}(B)$-formula $\varphi(x, y, z)$ there is some $\mathcal{L}(aB)$-formula $\psi(y)$ such that for b independent from a over B, $\models \psi(b)$ holds if and only if $\varphi(a, b, z)$ does not fork over Bb.

Note that any low theory is weakly low.

Remark 1.4 The following are equivalent:

(i) T is weakly low;
(ii) for all a, B and $\mathcal{L}$-formula $\varphi(x, y, z)$ there is $k < \omega$ such that, whenever a is independent from b over B, and $(a_i : i < \omega)$ is a Morley sequence over Bb in $tp(a/Bb)$ such that $\{\varphi(a_i, b, z) : i < \omega\}$ is k-consistent, $\{\varphi(a_i, b, z) : i < \omega\}$ is consistent;
(iii) for any a, B, $\mathcal{L}$-formula $\varphi(x, y, z)$ and Morley sequence $(a_i : i < \omega)$ in $tp(a/B)$, there is $k < \omega$ such that whenever a is independent from b over B, and $(a_i : i < \omega)$ is also a Morley sequence in $tp(a/Bb)$ and $\{\varphi(a_i, b, z) : i < \omega\}$ is k-consistent, then it is consistent.

Proof

(i) implies (ii): Given a, B and $\mathcal{L}$-formula $\varphi(x, y, z)$ let $\psi(y)$ be the $\mathcal{L}(aB)$ formula given by weak lowness. Now the condition "y is independent from a over B and $(x_i : i < \omega)$ is a Morley sequence in $tp(a/By)$" is given by a partial type $\Sigma(y, x_i)_i$ over aB. By Kim's lemma (that a formula $\chi(z, c)$ does not fork over B if and only if for some (any) Morley sequence $(c_i : i < \omega)$ in $tp(c/B)$, $\{\chi(x, c_i) : i < \omega\}$ is consistent), we have the implication

$$\Sigma(y, x_i)_i \models \text{“}\{\varphi(x_i, y, z) : i < \omega\} \text{ is consistent”} \Leftrightarrow \psi(y).$$

By compactness we find the required k.

(ii) implies (iii) is immediate.

(iii) implies (i): Assume (iii). To prove weak lowness of T, we may assume in Definition 1.3 that $\varphi(x, y, z)$ is an $\mathcal{L}$-formula (as we can incorporate any parameters from B in b). Fix a Morley sequence $(a_i : i < \omega)$ in $tp(a/B)$. Let $r(x_i)_{i<\omega} = tp((a_i)_{i<\omega}/B)$. Let k be as given by (iii) (for the given choice of φ). Note that for any b which is independent from a over B there is a realization $(a'_i : i < \omega)$ of r such that $(a'_i : i < \omega)$ is a Morley sequence in $tp(a/Bb)$. By virtue of Kim's lemma again, we have that for any b which is independent from a over B, the following are equivalent:

(a) $\varphi(a, b, z)$ does not fork over Bb;
(b) there is a realization $(a'_i : i < \omega)$ of r which is a Morley sequence in $tp(a/Bb)$ and such that $\{\varphi(a'_i, b, z) : i < \omega\}$ is consistent;
(c) it is *not* the case that there is a realization $(a'_i : i < \omega)$ of r which is a Morley sequence in $tp(a/Bb)$ such that $\{\varphi(a'_i, b, z) : i < k\}$ is consistent.

As the relevant expressions are type-definable over aB, it follows by compactness that there is an $\mathcal{L}(aB)$ formula $\psi(y)$ such that for b independent of a over B, $\varphi(a, b, z)$ does not fork over bB if and only if $\models \psi(b)$. $\square$

In Section 2 we will show that (for T simple) T is weakly low if and only if some (any) lovely pair $P \prec M$ eliminates the $\exists y \in P$ quantifier. (This will be more or less tautological.) We will show that in this case any theory T^* (as described in Remark 1.2) is simple. Namely, the P-part of a lovely pair $P \prec M$, when equipped with traces of definable sets in M, has a simple theory. We also show that when T is low then lovely pairs uniformly eliminate the $\exists y \in P$ quantifier.

In Section 3 we recall an example due to Casanovas of a simple nonlow theory and point out that this example is weakly low.

In Section 4, we give an example of a non-weakly-low simple theory. The example will be a parametrized version of Casanovas's example.

2 Quantifier Elimination, Weak Lowness, and Simplicity

We continue with the conventions and notation of Section 1. So T is a complete simple theory in language $\mathcal{L}$. Let $P \prec M$ be a lovely pair of models of T, and let $\mathcal{L}^*$, P^*, and T^* be as in Remark 1.2.

Proposition 2.1 *T is weakly low if and only if T^* has quantifier-elimination.*

Proof First assume T to be weakly low. Let $\varphi(x, y, z)$ be an $\mathcal{L}$-formula, and $a \in M$. We want to find an $\mathcal{L}(M)$-formula $\psi(y)$ such that for $b \in P$, $\models \psi(b)$ if and only if there is $c \in P$ such that $\models \varphi(a, b, c)$.

Let $B \subset P$ be of cardinality $\leq |\mathcal{L}|$ such that $tp(a/P)$ does not fork over B. Let $\psi(y)$ be the $\mathcal{L}(aB)$-formula given (for $tp(a/B)$ and $\varphi(x, y, z)$) by weak lowness of T. Suppose $b \in P$. Then a is independent from b over B. Moreover, by the second clause in the definition of lovely pairs (the "coheir property") $\varphi(a, b, z)$ does not fork over Bb if and only if $\varphi(a, b, z)$ is realized in P. Hence $\models \psi(b)$ if and only if $\varphi(a, b, z)$ is realized in P.

Conversely, suppose that T^* has quantifier-elimination. Let $\varphi(x, y, z) \in \mathcal{L}$ and let a, B be from some big model $\bar{M}$ of T. Clearly we may assume B to be of cardinality $\leq |\mathcal{L}|$. As $P \prec M$ is a lovely pair we may also assume that $B \subset P$ and a is independent from P over B. Let $\psi(y)$ be the $\mathcal{L}(M)$-formula which is equivalent, on P, to $\exists z \in P(\varphi(a, y, z))$. Write $\psi(y)$ as $\psi'(d, y)$ for $\psi'(w, y)$ an $\mathcal{L}$-formula and $d \in M$.

Claim 2.2 *For b independent from ad over B, $\varphi(a, b, z)$ does not fork over Bb if and only if $\models \psi'(d, b)$.*

Proof of Claim If b (in $\bar{M}$) is independent from ad over B, then as $P \prec M$ is a lovely pair we may realize $tp(b/adB)$ in P and so may assume $b \in P$. But then we have, by loveliness of $P \prec M$, that $\varphi(a, b, z)$ does not fork over Bb if and only if it is realized in P. By choice of $\psi'(d, y)$, we get the claim. □

The only problem with deducing weak lowness of T is the additional parameter d in Claim 2.2. But this is easily dealt with. As the condition "$\varphi(a, y, z)$ does not fork over By" is type-definable over aB (for y independent of a over B) we only have to show that for y independent of a over B, the condition "$\varphi(a, y, z)$ forks over yB" is type-definable over aB. If not, there is some index set I and for each $i \in I$ some b_i which is independent from a over B such that $\varphi(a, b_i, z)$ forks over Bb_i, but there is b' realizing some ultraproduct of $(tp(b_i/aB))_i$ such that $\varphi(a, b', z)$ does not fork over aB. Without loss of generality b_i is independent from ad over B and b' realizes the corresponding ultraproduct of $(tp(b_i/adB))_i$. So b' is also independent from ad over B. But, by Claim 2.2, $\neg\psi'(d, b_i)$ for all i; hence $\neg\psi'(d, b')$; hence again by the claim, $\varphi(a, b', z)$ forks over $b'B$, a contradiction. □

Proposition 2.3 *If T is low, then T^* has uniform* QE.

Proof Let T be low, $P \prec M$ be a lovely pair of models of T, and let T^* be the corresponding $\mathcal{L}^*$-theory. Since T is low, then, as shown in [2], a saturated ($\mathcal{L}_P$-)elementary extension (N, P) of (M, P) has the "coheir property" (the second clause in the definition of a lovely pair). Note that from the proof of Proposition 2.1 it follows that for a weakly low T and any pair of models of T satisfying the coheir property (not necessarily lovely), the corresponding theory T^* has QE. Thus $P(N)$ with the structure induced by N has QE, that is, for any $\varphi(x, y, z)$ in $\mathcal{L}$ and $a \in N$ there is an $\mathcal{L}$-formula $\psi_a(x, t, y)$ and a parameter $d \in N$ such that

$$\exists z \in P\ \varphi(a, P(N), z) = \psi_a(a, d, P(N)).$$

By ($\mathcal{L}_P$-)saturation of (N, P) and compactness, there are $\mathcal{L}$-formulas

$$\psi_1(x, t, y), \ldots, \psi_n(x, t, y)$$

such that for any $a \in N$ there is $1 \leq k \leq n$ such that

$$\exists z \in P\ \varphi(a, P(N), z) = \psi_k(a, d, P(N))$$

for some parameter d. Let $z_1, \ldots, z_{k+1}$ be new variables of the same sort, and let

$$\psi(x, t, z_1, \ldots, z_n, z_{n+1}, y) = \bigwedge_{k=1}^{n} (z_k = z_{n+1} \rightarrow \psi_k(x, t, y)).$$

Then for any $a \in N$ there is $d \in N$ and $e_1, \ldots, e_{n+1} \in N$ such that

$$\exists z \in P\ \varphi(a, P(N), z) = \psi(a, d, e_1, \ldots, e_{n+1}, P(N))$$

(simply choose distinct $e_1, \ldots, e_n$, and let $e_{n+1} = e_k$ for a suitable k).

But then it holds, in particular, for any $a \in M$. Since $(M, P) \prec (N, P)$, for any $a \in M$ there are $d, e_1, \ldots, e_{n+1} \in M$ such that

$$\exists z \in P\ \varphi(a, P(M), z) = \psi(a, d, e_1, \ldots, e_{n+1}, P(M)).$$

Thus $P(M)$ with the structure induced by M (and the corresponding theory T^*) has uniform QE. □

Finally in this section, we look at the question of preservation of simplicity when passing to the "externally induced" structure.

Theorem 2.4 *Let T be a simple weakly low theory. Then the theory T^* is simple.*

Proof Let $P \prec M$ be a lovely pair of models of T, and T^* the corresponding $\mathcal{L}^*$-theory. As shown in [2], we can embed (M, P) in a κ-lovely pair (M_1, P) of models of T, for a sufficiently large κ, so that $M \mathop{\smile\!\!\!\!\!\!\mid}_{P(M)} P(M_1)$, and such an embedding is always $\mathcal{L}_P$-elementary. Take a sufficiently saturated $\mathcal{L}_P$-elementary extension (N, P) of (M_1, P). So also $(M, P) \prec (N, P)$, and thus also $M \mathop{\smile\!\!\!\!\!\!\mid}_{P(M)} P(N)$. Considering the structure induced on $P(N)$ by parameters from M, we can view $P(N)$ as an $\mathcal{L}^*$-structure. Then clearly $P(M)$ is an $\mathcal{L}^*$-elementary substructure of $P(N)$, and $P(N)$ is a saturated model of T^*.

Since T is weakly low, T^* has quantifier elimination. Therefore the $\mathcal{L}^*$-type of any $a \in P(N)$ is determined by the $\mathcal{L}$-type of a over M (and vice versa). We will show that for $A \subset B \subset P(N)$ and $a \in P(N)$, $p^*(x, B) = tp_{\mathcal{L}^*}(a/B)$ does not divide over A (in the sense of T^*) if and only if $p(x, BM) = tp_{\mathcal{L}}(a/BM)$ does not divide over AM (in the sense of T). Note that only the "if" direction is needed to show simplicity of T^*. First note that for $B_i \subset P(N)$, $B_0 = B$, $(B_i : i \in \omega)$ is $\mathcal{L}^*$-indiscernible over A if and only if $(B_i : i \in \omega)$ is $\mathcal{L}$-indiscernible over AM, and for $a' \in P(N)$, $a' \models \bigcup p^*(x, B_i)$ if and only if $a' \models \bigcup p(x, B_i M)$.

Now, assume $p(x, BM) = tp_{\mathcal{L}}(a/BM)$ does not divide over AM. Let $(B_i : i \in \omega)$ be an $\mathcal{L}^*$-indiscernible over A sequence in $P(N)$, with $B_0 = B$. Since $M \mathop{\smile\!\!\!\!\!\!\mid}_{P(M)} P(N)$, $tp_{\mathcal{L}}((B_i : i \in \omega)/M)$ does not fork over $P(M)$, so by the coheir property and the characterization of $\mathcal{L}^*$-types in $P(N)$, we may assume that B_i are all in $P(M_1)$. Since $p(x, BM) = tp_{\mathcal{L}}(a/BM)$ does not divide over AM, $(B_i : i \in \omega)$ is AM-indiscernible, and M_1 is sufficiently saturated as an $\mathcal{L}$-structure, we can find $a' \in M_1$ such that $a' \models \bigcup p(x, B_i M)$ and $a' \mathop{\smile\!\!\!\!\!\!\mid}_{AM} M \cup \bigcup B_i$. Since $a \in P(N)$, $B \subset P(N)$, and $P(N) \mathop{\smile\!\!\!\!\!\!\mid}_{P(M)} M$, $p(x, BM)$ does not divide over $BP(M)$. Thus $a' \mathop{\smile\!\!\!\!\!\!\mid}_{BP(M)} M \cup \bigcup B_i$. So, by the coheir property again, we may assume that $a' \in P(M_1)$. But then, by the characterization of $\mathcal{L}^*$-types in $P(N)$,

$a' \models \bigcup p^*(x, B_i)$. So $p^*(x, B)$ does not divide over A. Simplicity of T^* now follows from simplicity of T.

For the converse, assume that $p^*(x, B)$ does not divide over A. Since $B \subset P(N)$ and $P(N) \downarrow_{P(M)} M$, we may assume (by the coheir property) that $B \subset P(M_1)$. Let $(B_i : i \in \omega) \subset N$ be a Morley sequence in $tp_{\mathcal{L}}(B/AM)$, with $B_0 = B$. Then since $B \downarrow_{AP(M)} AM$, we have $\bigcup B_i \downarrow_{AP(M)} AM$. Again, by the coheir property, we may assume that all the B_i are in $P(M_1)$. In particular, $B_i \subset P(N)$ for all i. Note that $(B_i : i \in \omega)$ is $\mathcal{L}^*$-indiscernible over A. So there is $a' \in P(N)$ realizing $\bigcup p^*(x, B_i)$. But then, by characterization of $\mathcal{L}^*$-types in $P(N)$, a' realizes $\bigcup p(x, B_i M)$. By Kim's lemma, $p(x, BM)$ does not divide over AM. $\square$

3 Weak Lowness of the Casanovas Example

We will show that an example of a simple nonlow theory, due to Casanovas [3], satisfies the weak lowness property.

First, we recall the construction. The structure consists of two sorts: P (points) and I (indexes), unary predicates I_n, $0 < n < \omega$ and a binary predicate $R \subset I \times P$ such that I_n define disjoint infinite subsets of I such that

(1) $\forall p \in P \exists^{=n} i \in I_n R(i, p)$;
(2) if $A, B \subset I$ are disjoint and finite and $|A \cap I_n| \leq n$ for all n, then there is $p \in P$ such that $R(i, p)$ holds for all $i \in A$ and does not hold for all $i \in B$ (in particular, this implies that P is infinite);
(3) if $A, B \subset P$ are disjoint and finite, then there is $i \in I$ such that $R(i, p)$ holds for all $p \in A$ and does not hold for all $p \in B$.

The theory T axiomatized by these statements is simple and nonlow. The algebraic closure is given by $acl(A) = A \cup \bigcup_n \{i \in I_n : \exists a \in A R(i, a)\}$, so, in particular, is disintegrated. The independence relation is given by $A \downarrow_B C \iff acl(AB) \cap acl(CB) \subset acl(B)$. Any formula in one variable over an algebraically closed set A is equivalent to a quantifier-free formula over A.

For a variable p of sort P and $1 \leq n < \omega$, let $(p_1^n, \ldots, p_n^n)$ be a tuple of new variables. For a tuple of variables $\bar{x}$, let $\bar{x}^n$ be an extension of the tuple $\bar{x}$ obtained by adding new variables $(p_1^m, \ldots, p_m^m)$, $1 \leq m \leq n$, for each variable p in $\bar{x}$ of sort P. Now, for any formula $\psi(\bar{w}, z)$ (where z is a single variable) and a tuple $\bar{a}$ there is $1 \leq n < \omega$ and a quantifier-free formula $\theta(\bar{w}^n, z)$ such that $\psi(\bar{a}, z)$ is equivalent to $\theta(\bar{a}^n, z)$. By compactness, we can choose one θ and n which will work for all choices of $\bar{a}$.

We will show that T is weakly low. Note that for quantifier elimination in T^*, it is enough to show elimination of a single existential quantifier. Thus, since the proof of Proposition 2.1 is done on a formula-by-formula basis, in the weak lowness condition it is enough to consider formulas $\varphi(\bar{x}, \bar{y}, z)$ where z is a single variable. Let $(M, R, P, I, I_n)_{1 \leq n < \omega}$ be a saturated model of the theory T, $\varphi(\bar{x}, \bar{y}, z)$ an $\mathcal{L}(T)$-formula where z is a single variable, $B \subset M$ $(\bar{a}_k : k < \omega)$ a Morley sequence over B. We need to find $n \in \omega$ such that for any $\bar{b} \downarrow_B \bar{a}_0$ such that $(\bar{a}_k : k < \omega)$ is a Morley sequence over $B\bar{b}$, if the family $(\varphi(\bar{a}_i, \bar{b}, z) : i < \omega)$ is n-consistent, then it is consistent.

Augmenting $\bar{b}$s with a fixed tuple from B (if needed), we may assume that $\bar{a}_k$ are disjoint from each other and from B. Let $1 \leq m < \omega$ and $\theta(\bar{x}^m, \bar{y}^m, z)$ be such that θ is quantifier free, and for any $\bar{a}$ and $\bar{b}$, $\varphi(\bar{a}, \bar{b}, z)$ is equivalent to $\theta(\bar{a}^m, \bar{b}^m, z)$.

Changing $\bar{a}_k$ to $\bar{a}_k^m$ and B to $acl(B)$, and considering $\bar{b}^m$ instead of $\bar{b}$, we may assume that φ is quantifier free. Also note that if $\varphi_1(\bar{x}, \bar{y}, z)$ and $\varphi_2(\bar{x}, \bar{y}, z)$ both satisfy the weak lowness condition relative to n_1 and n_2, respectively, then so does $\varphi_1 \vee \varphi_2$, relative to $n_1 + n_2$. So we may assume that $\varphi(\bar{x}, \bar{y}, z)$ is a conjunction of atomic formulas or their negations. We may also assume that we consider only such $\bar{b}$ that $\models \exists z \varphi(\bar{a}_0, \bar{b}, z)$.

Case 1: z is of sort P. If $\varphi(\bar{x}, \bar{y}, z)$ contains positively a formula of the form $z = x_j$ or $z = y_j$, then it is easy to see that the family $(\varphi(\bar{a}_k, \bar{b}, z) : k \in \omega)$ is either 2-inconsistent or consistent. Assume if $\varphi(\bar{x}, \bar{y}, z)$ contains positively a formula of the form $R(x_j, z)$ where $\bar{a}_{0_j}$ (the jth component of the tuple $\bar{a}_0$) is in I_n for some n. Let N be the maximal of such ns. Then $(\varphi(\bar{a}_k, \bar{b}, z) : k \in \omega)$ is N-inconsistent, and N does not depend on the choice of $\bar{b}$. If $\varphi(\bar{x}, \bar{y}, z)$ does not contain (positively) formulas of the form $R(x_j, z)$ where $\bar{a}_{0_j} \in I_n$ for some n or of the form $z = x_j, z = y_j$, then it is of the form

$$\begin{gathered}
\bigwedge_{j \in J_0} R(x_j, z) \wedge \bigwedge_{j \in J_1} \neg R(x_j, z) \\
\wedge \bigwedge_{j \in K_0} R(y_j, z) \wedge \bigwedge_{j \in K_1} \neg R(y_j, z) \wedge \\
\bigwedge_{j \in L_0} z \neq x_j \wedge \bigwedge_{j \in L_1} z \neq y_j \wedge \xi(\bar{x}, \bar{y}),
\end{gathered}$$

where for any $j \in J_0$, $\bar{a}_{0_j}$ is not in I_n for any n, $J_0 \cap J_1 = \varnothing$, $K_0 \cap K_1 = \varnothing$, and $\models \theta(\bar{a}_k, \bar{b})$ (since $\models \exists z \varphi(\bar{a}_0, \bar{b}, z)$, and $(\bar{a}_k : k \in \omega)$ is $\bar{b}$-indiscernible). Then $(\varphi(\bar{a}_k, \bar{b}, z) : k \in \omega)$ is consistent.

Case 2: z is of the sort I. Since there are only finitely many I_ms occurring in φ, and I_ms are disjoint, we may assume that

$$\varphi(\bar{x}, \bar{y}, z) = z \in I_m \wedge \theta(\bar{x}, \bar{y}, z)$$

or

$$\varphi(\bar{x}, \bar{y}, z) = z \notin I_{n_1} \cup \cdots \cup I_{n_r} \wedge \theta(\bar{x}, \bar{y}, z),$$

where $\theta(\bar{x}, \bar{y}, z)$ is a conjunction of atomic formulas or their negations. Similarly to Case 1, if θ has positive occurrences of $z = x_j$ or $z = y_j$, then in both cases, the family $(\varphi(\bar{a}_k, \bar{b}, z) : k \in \omega)$ is either consistent or 2-inconsistent. If no such formulas occur, then the only occurrences of z in $\theta(\bar{x}, \bar{y}, z)$ are of the form $R(z, x_j)$, $R(z, y_j)$, $\neg R(z, x_j)$, $\neg R(z, y_j)$, $\neg z = x_j$, $\neg z = y_j$. Therefore, by disjointness of a_ks and the axioms of T, $(z \notin I_{n_1} \cup \cdots \cup I_{n_r} \wedge \theta(\bar{a}_k, \bar{b}, z) : k \in \omega)$ is consistent. Now, it is easy to see that any definable subset of I_m is either finite or cofinite. So in the case when $\varphi(\bar{x}, \bar{y}, z) = z \in I_m \wedge \theta(\bar{x}, \bar{y}, z)$, there is N such that for any parameters $\bar{a}, \bar{b}$, $\varphi(\bar{a}, \bar{b}, z)$ is either cofinite, or of size $\leq N$. The existence of n now easily follows.

4 A Non-weakly-low Simple Theory

In this section we will give an example of a non-weakly-low simple theory, which is, in some sense, a parametrized version of the Casanovas example. Our proofs are similar to the ones in [3], with some modifications.

Our language $\mathcal{L}$ has 3 sorts: P (points), K (classes), and I (indices), unary function $f : P \to K$, binary relations $I_n \subset I \times K$, $1 \leq n < \omega$, and a binary relation $R \subset I \times P$. The axioms are as follows.

First we define an $\mathcal{L}$-theory T_0, axiomatized as follows.

(1) I, K, and P are infinite.
(2) For any $b \in K$, $f^{-1}(b)$ is infinite and $f(P) = K$ (so P is divided into infinitely many infinite classes by the relation $f(x) = f(x')$, and K is the set of "names" of these classes).
(3) For any $b \in K$, $I_n(-, b)$ define infinite disjoint subsets of I.
(4) For any $1 \leq n < \omega$, $\forall y \in K \forall z \in f^{-1}(y) \exists^{=n} x (I_n(x, y) \wedge R(x, z))$.

Note that the theory T_0 is inductive ($\forall\exists$-axiomatizable).

Let $(M, I, K, P, R, I_n)_{1 \leq n < \omega}$ be an existentially closed model of T_0.

Lemma 4.1 *M satisfies the following axiom schemes.*

(5) *For any $b \in K$, and any finite disjoint $A_0, A_1 \subset I$ such that $|\{i \in A_0 : I_n(i, b)\}| \leq n$ for all n, there is $c \in f^{-1}(b)$ such that $R(a, c)$ holds for all $a \in A_0$, and $\neg R(a, c)$ holds for all $a \in A_1$. Moreover, there are infinitely many such c.*
(6) *For any $n_1, \ldots, n_m$ and distinct $a_1, \ldots, a_m \in I$ there exists $b \in K$ such that $I_{n_1}(a_1, b), \ldots, I_{n_m}(a_m, b)$. Moreover, there are infinitely many such b.*
(7) *For any $1 \leq n_1, \ldots, n_m, n < \omega$, distinct $b_1, \ldots, b_m, b_{m+1}, \ldots, b_l \in K$, and disjoint finite sets $C_0, C_1 \subset P$ such that $f^{-1}(\{b_1, \ldots, b_m\}) \cap C_0 = \varnothing$, there exists $a \in I$ such that*

$$I_{n_1}(a, b_1), \ldots, I_{n_m}(a, b_m), \bigwedge_{k=1}^{n} \neg I_k(a, b_{m+1}), \ldots, \bigwedge_{k=1}^{n} \neg I_k(a, b_l),$$

$R(a, c)$ for all $c \in C_0$ and $\neg R(a, c)$ for all $c \in C_1$. Moreover, there are infinitely many such a.

Proof Easy. □

Note that the axiom schemes (1 – 7) are first-order. Let T be axiomatized by (1 – 7). So, any existentially closed model of T_0 satisfies T. We will show that T is complete.

Let M be a model of T_0. For $A \subset M$, let $cl(A) = A \cup \{i \in I : I_n(i, b) \wedge R(i, c)$ for some $b \in A \cap K$, $c \in f^{-1}(b)$ and $1 \leq n < \omega\}$. Note that cl is a disintegrated closure operator and $cl(A) \subset acl(A)$.

For convenience, by $I_\omega(-, b)$ we denote the partial type $\{\neg I_n(-, b) : 1 \leq n < \omega\}$.

Lemma 4.2 *Let M and N be ω_1-saturated models of T, $A = cl(A) \subset M$, $A' = cl(A') \subset N$ countable, and let $g : A \to A'$ be a partial isomorphism. Let $a \in M$. Then there is $a' \in N$ and a partial isomorphism $g' : cl(aA) \to cl(a'A')$ extending $g \cup \{(a, a')\}$.*

Proof If $a \in A$, there is nothing to prove. So assume $a \notin A = cl(A)$.

Case 1: $a \in I$. Let $C_0 = \{c \in A \cap P : R(a, c)\}$, $C_1 = \{c \in A \cap P : \neg R(a, c)\}$. Since $a \notin cl(A) = A$, for any $c \in C_0$, we have $I_\omega(a, f(c))$. So we need to find $a' \in N \setminus A'$ realizing $R(x, g(c))$ for all $c \in C_0$, $\neg R(x, g(c))$ for all $c \in C_1$, and $I_n(x, g(b))$ or $\neg I_n(x, g(b))$ for $b \in A \cap K$ and $1 \leq n < \omega$ such that we have $I_n(a, b)$ or $\neg I_n(a, b)$, respectively. Since for any $c \in C_0$, $I_n(a, f(c))$ does not hold for any $n < \omega$, it follows from axiom scheme (7) that such type is consistent. By saturation, we can find such a'. Finally, note that $Aa = cl(Aa)$ and $A'a' = cl(A'a')$.

Case 2: $a \in K$. For any $i \in A \cap I$ let $1 \leq n_i < \omega + 1$ be such that we have $I_{n_i}(i, a)$. By axiom scheme (6), we can find $a' \in N \backslash A'$ such that we have $I_{n_i}(g(i), a')$ for all $i \in A \cap I$ (if $n_i = \omega$, we use the fact that $I_{n+1}(x, b)$ implies $\neg I_1(x, b) \wedge \cdots \wedge \neg I_n(x, b)$ plus compactness). Then $g \cup \{(a, a')\}$ is a partial isomorphism. Note again that $cl(Aa) = Aa$ and $cl(A'a') = A'a'$.

Case 3: $a \in P$. By Cases 1 and 2, we may assume that $cl(a) \backslash \{a\} \subset A$ (so, $cl(Aa) = Aa$. Let $A_0 = \{i \in A \cap I : R(i, x)\}$, $A_1 = \{i \in A \cap I : \neg R(i, x)\}$. Note that for any $1 \leq n < \omega$, $|\{i \in A_0 : I_n(i, f(a))\}| \leq n$ (and $f(a) \in A$). So, for any $1 \leq n < \omega$, $|\{i \in g(A_0) : I_n(i, g(f(a)))\}| \leq n$. Also, $g(A_0)$ and $g(A_1)$ are disjoint. So, by axiom scheme (5) and saturation, we can find $a' \in N \backslash A$ such that $f(a') = g(f(a))$, $R(a', i)$ holds for $i \in g(A_0)$, and $R(a', i)$ does not hold for $i \in g(A_1)$. Then $cl(A'a') = A'a'$, and $g \cup \{a, a'\}$ is a partial isomorphism, as needed. $\square$

Proposition 4.3

(i) *T is complete.*

(ii) *The complete type of a single element a over a cl-closed set A in a model of T is determined by the quantifier-free type of a over A.*

(iii) *In T, $acl = cl$.*

Proof (i) and (ii) follow easily from Lemma 4.2. For (iii), note that if $a \notin A = cl(A)$, then by the axiom schemes, there are infinitely many realizations of the quantifier free type of a over A. $\square$

Our next goal is to show simplicity of T. We are working inside a saturated model of T. As in [3], we define an independence relation

$$A \mathop{\smile\!\!\!\!\!\!\raise4pt\hbox{$|$}}_C B \iff acl(AC) \cap acl(BC) \subset acl(C),$$

and note that it is easy to check that $\mathop{\smile\!\!\!\!\!\!\raise4pt\hbox{$|$}}$ is invariant under automorphisms and satisfies local and finite character, monotonicity, transitivity, symmetry, and extension, and it suffices to show the independence theorem over an algebraically closed set. Exactly as in Lemma 5.4 of [3], we can see that in our case it suffices to check the independence theorem for types of single elements. So let $C = acl(C) \subset A, B$, $A \mathop{\smile\!\!\!\!\!\!\raise4pt\hbox{$|$}}_C B$, $a \mathop{\smile\!\!\!\!\!\!\raise4pt\hbox{$|$}}_C A$, $b \mathop{\smile\!\!\!\!\!\!\raise4pt\hbox{$|$}}_C B$, $a \equiv_C b$. We need to find c such that $c \equiv_A a$, $c \equiv_B b$ and $c \mathop{\smile\!\!\!\!\!\!\raise4pt\hbox{$|$}}_C AB$. Since acl is disintegrated, the latter will follow from $a \mathop{\smile\!\!\!\!\!\!\raise4pt\hbox{$|$}}_C A$, $b \mathop{\smile\!\!\!\!\!\!\raise4pt\hbox{$|$}}_C B$, so it suffice to find c such that $c \equiv_A a$, $c \equiv_B b$. We may assume that A, B are algebraically closed, and $a \notin A$, $b \notin B$. Note that $A \cap B = C$. We need to show that $qftp(a/A) \cup qftp(b/B)$ is consistent. Let A^K, A^P, A^I denote $A \cap K$, $A \cap P$, and $A \cap I$, respectively, and similarly for C and B.

Case 1: $a, b \in I$. Then $qftp(a/A) \cup qftp(b/B)$ is of the form

$$\{I_{n_k}(x, k) : k \in A_0^K\} \cup \bigcup_{k \in A_1^K} I_\omega(x, k) \cup$$

$$\{I_{m_k}(x, k) : k \in B_0^K\} \cup \bigcup_{k \in B_1^K} I_\omega(x, k) \cup$$

$$\{R(x, p) : p \in A_0^P\} \cup \{\neg R(x, p) : p \in A_1^P\}$$

$$\cup \{R(x, p) : p \in B_0^P\} \cup \{\neg R(x, p) : p \in B_1^P\} \cup x \notin A \cup B,$$

where A^K is a disjoint union of A_0^K and A_1^K, B^K is a disjoint union of B_0^K and B_1^K, and $A_0^K \cap C = B_0^K \cap C$ (so also $A_1^K \cap C = B_1^K \cap C$). Clearly $f(A_0^P \cup B_0^P) \cap (A_0^K \cup B_0^K) = \varnothing$ (since $a \notin acl(A) = A$ and $b \notin acl(B) = B$), by axiom scheme (7) (and its "moreover" part), this type is consistent.

Case 2: $a, b \in K$. Then $qftp(a/A) \cup qftp(b/B)$ is of the form

$$\{I_{n_i}(i,x) : i \in A_0^I\} \cup \bigcup_{i \in A_1^I} I_\omega(i,x) \cup$$

$$\{I_{n_i}(i,x) : i \in B_0^I\} \cup \bigcup_{i \in B_1^I} I_\omega(i,x) \cup x \notin A \cup B$$

where A^I is a disjoint union of A_0^I and A_1^I, B^I is a disjoint union of B_0^I and B_1^I, and $A_0^I \cap C = B_0^I \cap C$ (so also $A_1^I \cap C = B_1^I \cap C$). Similarly to Case 2 in the proof of Lemma 4.2, consistency follows from the axiom scheme (6).

Case 3: $a, b \in P$. Note that in this case, $f(a) = f(b) = k_0 \in C^K$. Also, if we have $I_n(i, k_0)$ and $R(i,a)$ or $R(i,b)$, then $i \in C^I$. Then $qftp(a/A) \cup qftp(b/B)$ is of the form

$$\{f(x) = k_0\} \cup \{R(i,x) : i \in A_0^I\} \cup \{\neg R(i,x) : i \in A_1^I\}$$

$$\{R(i,x) : i \in B_0^I\} \cup \{\neg R(i,x) : i \in B_1^I\} \cup x \notin A \cup B$$

where A^I is a disjoint union of A_0^I and A_1^I, B^I is a disjoint union of B_0^I and B_1^I, and $A_0^I \cap C = B_0^I \cap C$ (so also $A_1^I \cap C = B_1^I \cap C$), and for any $1 \leq n < \omega$, $\{i \in A_0^I : I_n(i,k_0)\} = \{i \in B_0^I : I_n(i,k_0)\}$ is a subset of C^I of size $\leq n$. Consistency now follows from the axiom scheme (5).

Thus we have proved the following proposition.

Proposition 4.4 *T is simple, with independence given by $A \downarrow_C B \iff acl(AC) \cap acl(BC) \subset acl(C)$.*

Note that T is not supersimple, since for any point $p \in P$, $acl(P) \backslash \{p\}$ is infinite but not finite generated in the sense of acl.

Proposition 4.5 *T is not weakly low.*

Proof Let $(a_l : l \in \omega)$ be a sequence of distinct elements of I, and let $1 \leq n < \omega$. Note that $(a_l : l \in \omega)$ is a Morley sequence in the unique 1-type of sort I over $\varnothing$. Let $\varphi(x,y,z) = R(x,z) \wedge y = f(z)$. We can find $b \in K$ such that $I_n(a_l, b)$ holds for any l. Then $(a_l : l \in \omega)$ is still a Morley sequence over b, and $b \downarrow_\varnothing a_0$. But $(\varphi(a_l, b, z) : i \in \omega)$ is clearly n-inconsistent. Since n was arbitrary, this shows that T is not weakly low. $\square$

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