Notes on Singular Cardinal Combinatorics

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Abstract   We present a survey of combinatorial set theory relevant to the study of singular cardinals and their successors. The topics covered include diamonds, squares, club guessing, forcing axioms, and PCF theory.

1   Introduction

In May 2004 I gave a series of three 45-minute survey talks at a workshop on Singular Cardinal Combinatorics held at the Banff International Research Station, under the blanket title "ZFC Combinatorics." These notes were worked up from my slides for those talks; they retain to some extent the telegraphic style of the originals.

Several related themes are prominent in these notes:

1. The universe of set theory V is surprisingly L-like in the sense that weak versions of Jensen's combinatorial principles, diamond and square, are provable outright as theorems of ZFC.

2. The extent to which L-like combinatorial principles hold in V can be measured by constructing certain "canonical invariants" which are typically ideals or stationary sets; examples which are important in these notes include the ideal $I[\lambda]$ and the stationary set of good points (qv). Understanding these invariants is the key to many combinatorial problems, especially those involving singular cardinals and their successors.

3. PCF theory had its origins in questions involving the Singular Cardinals Hypothesis. However, the theory has much broader applicability, in particular, PCF is a fertile source of the sort of canonical invariants discussed above.

4. There is a tension in set theory between "compactness" and "incompactness". If the universe is sufficiently L-like then there are many examples of incompactness, such as nonreflecting stationary sets or κ-Aronszajn trees. By contrast, in the presence of large cardinals or strong forcing axioms there are

typically fewer examples of incompactness; moreover, for a regular uncountable cardinal κ compactness statements such as "there are no κ-Aronszajn trees" can have a high consistency strength, for example, when κ is the successor of a singular cardinal or we demand the statement be true for several successive values of κ at once. The canonical invariants are especially useful in exploring the tension between compactness and incompactness.

Here is a section by section guide to the results that are proved in this paper. Many of the early sections can (hopefully) be read by anyone who has completed a beginning course in set theory; some later sections presume some knowledge of large cardinals, strong forcing axioms, and the basic facts about PCF.

Section 2	The principles $\diamondsuit_\kappa(S)$ and $\diamondsuit'_\kappa(S)$ are equivalent. Under GCH, $\diamondsuit_{\kappa^+}(S)$ holds for any stationary set $S \subseteq \{\alpha \in \kappa^+ : \mathrm{cf}(\alpha) \neq \mathrm{cf}(\kappa)\}$. The collection of sets which do not carry a diamond sequence forms a normal ideal.
Section 3	If κ and λ are regular with $\kappa^+ < \lambda$, any stationary subset of $\lambda \cap \mathrm{cof}(\kappa)$ carries a club guessing sequence.
Section 4	$\square_\mu$ implies that every stationary subset of μ^+ contains a nonreflecting stationary set. Under GCH, $\square_\mu$ implies the existence of a μ^+-Souslin tree.
Section 5	The weak square $\square_\mu^*$ is equivalent to the existence of a special μ^+-Aronszajn tree without any cardinal arithmetic assumptions.
Section 6	If κ is strongly compact then $\square_\mu$ fails for all cardinals $\mu \geq \kappa$, and $\square_\mu^*$ fails for all cardinals μ such that $\mathrm{cf}(\mu) < \kappa < \mu$.
Section 7	If $\lambda < \mu$ with λ and μ regular then $\mu^+ \cap \mathrm{cof}(\lambda)$ is the union of μ sets each carrying a partial square sequence.
Section 8	If λ is regular and uncountable then $I[\lambda]$ is a normal ideal. If $S \in I[\lambda]$ is a stationary subset of $\lambda \cap \mathrm{cof}(\kappa)$ then the stationarity of S is preserved by κ^+-closed forcing. Under GCH, there is for each regular $\kappa < \lambda$ a stationary subset of $\lambda \cap \mathrm{cof}(\kappa)$ which is the maximal subset of $\lambda \cap \mathrm{cof}(\kappa)$ lying in $I[\lambda]$ and is also the maximal stationary subset whose stationarity is preserved by any κ^+-closed forcing.
Section 9	If $S \in I[\lambda]$ then almost every point $\gamma \in S$ has form $\sup(M \cap \lambda)$ for an IA structure of length and cardinality $\mathrm{cf}(\gamma)$. The ideal $I[\lambda]$ is nontrivial on any cofinality κ such that $\kappa^+ < \lambda$.
Section 10	If a $<_I$-increasing sequence in a reduced product ${}^X On/I$ has stationarily many good points of cofinality $\kappa > |X|$, then there is an exact upper bound h for the sequence such that $\mathrm{cf}(h(x)) > \kappa$ for all x. The converse is also true.
Section 11	There is a scale of length $\aleph_{\omega+1}$ in some reduced product $\prod_{n \in A} \aleph_n/\mathit{finite}$. The set of approachable points is contained in the set of good points.
Section 12	If $\mathrm{NPT}(\mu, \lambda)$ and κ has a nonreflecting stationary set of cofinality μ ordinals, then $\mathrm{NPT}(\kappa, \lambda)$. For every singular λ, $\mathrm{NPT}(\lambda, \aleph_1)$.
Section 13	If $\mathrm{cf}(\kappa) = \omega$ and there is a good scale of length κ^+ then $\mathrm{NPT}(\kappa^+, \aleph_1)$.

Section 14 If $\square_{\aleph_\omega,\aleph_n}$ holds for some $n < \omega$ there is a very good scale of length $\aleph_{\omega+1}$. The existence of such a scale implies that among any infinite family of stationary subsets of $\aleph_{\omega+1}$, there are infinitely many which do not reflect simultaneously.

Section 15 If $\square^*_{\aleph_{\omega+1}}$ holds then there is a better scale of length $\aleph_{\omega+1}$. The existence of such a scale limits the extent of stationary reflection in $[\aleph_{\omega+1}]^{\aleph_0}$.

Section 16 If PFA holds then $\square_{\kappa,\aleph_1}$ fails for every uncountable κ.

Section 17 If MM holds then in a scale of length $\aleph_{\omega+1}$ there are stationarily many nongood points of cofinality $\aleph_1$.

Section 18 If κ is supercompact then there are no good scales of length μ^+ when $\mathrm{cf}(\mu) < \kappa < \mu$. By suitable Lévy collapses this can be used to produce a model of GCH in which there is a scale of length $\aleph_{\omega+1}$ with stationarily many nongood points of cofinality $\aleph_1$.

Section 19 The Trichotomy Theorem. In a scale of length $\aleph_{\omega+1}$ points of cofinality greater than the continuum are good. The theorem of Section 13 is true in ZFC for $\kappa = \aleph_\omega$. If SCH fails at $\aleph_\omega$ there is a better scale of length $\aleph_{\omega+1}$.

I have tried to give at least an impression of the various powerful methods which are useful in singular cardinal combinatorics. Among the notable techniques are the use of internally approaching chains of submodels, of determinacy results for infinite games, and of various "goodness" properties for PCF-theoretic scales.

2 Diamonds

2.1 Basic facts We start by recalling Jensen's classical diamond principle for a regular uncountable κ and $S \subseteq \kappa$:

$\diamondsuit_\kappa(S)$: There exists $\langle S_\alpha : \alpha \in S \rangle$ such that for all $X \subseteq \kappa$, $\{\alpha \in S : X \cap \alpha = S_\alpha\}$ is stationary.

Jensen [10] showed that the principle $\diamondsuit_\kappa(S)$ holds in L for all κ and all stationary S. It easily implies that $\kappa^{<\kappa} = \kappa$. Diamond is typically used in inductive constructions of objects of size κ, where an object is built up via a κ-sequence of approximations with size less than κ; the key idea is that all of the 2^κ many subsets of the final object are anticipated at many stages. For example, Jensen showed that $\diamondsuit_{\omega_1}$ implies there is an ω_1-Souslin tree, where the diamond sequence is used to anticipate maximal antichains of the final ω_1-tree.

Diamond can be used to anticipate subsets of any set X with $|X| = \kappa$. If we fix a decomposition $X = \bigcup_{\alpha < \kappa} X_\alpha$ where the sequence of X_α is increasing and continuous with $|X_\alpha| < \kappa$, and a bijection $f : \kappa \longleftrightarrow X$, then $f \upharpoonright \alpha : \alpha \longleftrightarrow X_\alpha$ for a club set C of $\alpha < \kappa$. If $\langle S_\alpha : \alpha \in S \rangle$ is a $\diamondsuit_\kappa(S)$-sequence and we let $T_\alpha = f_\alpha``S_\alpha$ for $\alpha \in C \cap S$, then we can obtain a sequence $\langle T_\alpha : \alpha \in S \rangle$ such that $T_\alpha \subseteq X_\alpha$ and for every $Y \subseteq X$ the set $\{\alpha \in S : X_\alpha \cap Y = T_\alpha\}$ is stationary.

2.2 From GCH to diamond We noted above that instances of diamond imply instances of the GCH. It is a surprising fact that there is a partial converse. More precisely, let $\kappa = \lambda^+$, $\mu = \mathrm{cf}(\lambda)$, $T = \{\alpha < \kappa : \mathrm{cf}(\alpha) \neq \mu\}$. Suppose that GCH

holds. Then $\diamondsuit_\kappa(S)$ holds for all stationary $S \subseteq T$; this result is due to Gregory for regular λ and was extended by Shelah to cover the case of λ singular.

We will sketch the proof, but before we do that we introduce some variants of diamond.

Definition 2.1   Let $\langle \mathcal{S}_\alpha : \alpha \in T \rangle$ be a sequence such that $\mathcal{S}_\alpha \subseteq P(\alpha)$, $|\mathcal{S}_\alpha| \leq \lambda$ for all $\alpha \in T$; we say that the sequence is a $\diamondsuit'_{\lambda^+}(T)$-*sequence* if $\{\alpha \in T : X \cap \alpha \in \mathcal{S}_\alpha\}$ is stationary for all $X \subseteq \lambda^+$. If the sequence has the stronger property that for every X almost every $\alpha \in T$ (modulo the club filter) has $X \cap \alpha \in \mathcal{S}_\alpha$, we say that the sequence is a $\diamondsuit^*_{\lambda^+}(T)$-*sequence*.

Clearly $\diamondsuit^*_{\lambda^+}(T)$ implies $\diamondsuit'_{\lambda^+}(S)$ for every stationary $S \subseteq T$. We claim that for every S, $\diamondsuit'_{\lambda^+}(S)$ is equivalent to $\diamondsuit_{\lambda^+}(S)$ where the nontrivial direction is to go from $\diamondsuit'_{\lambda^+}(S)$ to $\diamondsuit_{\lambda^+}(S)$.

As we pointed out above, we may alter a $\diamondsuit'_{\lambda^+}(S)$-sequence to guess subsets of $\lambda \times \lambda^+$; so we assume that we are given $\langle S^i_\alpha : \alpha \in S \rangle$ for $i < \lambda$ such that $S^i_\alpha \subseteq \lambda \times \alpha$ and for every $X \subseteq \lambda \times \lambda^+$ $\{\alpha \in S : \exists i \ X \cap (\lambda \times \alpha) = S^i_\alpha\}$ is stationary. We let $T^i_\alpha = \{\beta < \alpha : (i, \beta) \in S^i_\alpha\}$ and claim that for some i, $\langle T^i_\alpha : \alpha \in S \rangle$ is a $\diamondsuit_{\lambda^+}(S)$-sequence. If not then we fix for every i a set $X_i \subseteq \lambda^+$ and a club set $C_i \subseteq \lambda^+$ such that $T^i_\alpha \neq X_i \cap \alpha$ for $\alpha \in C_i \cap S$, and define $X = \{(i, \alpha) : \alpha \in X_i\}$ and $C = \bigcap_i C_i$. Since C is club, there exists $\alpha \in C \cap S$ and $i < \lambda$ such that $X \cap (\lambda \times \alpha) = S^i_\alpha$; in particular, for $\beta < \alpha$ we have $\beta \in T^i_\alpha \iff (i, \beta) \in S^i_\alpha \iff (i, \beta) \in X \iff \beta \in X_i$, so that $T^i_\alpha = X_i \cap \alpha$ in contradiction to the choice of X_i and C_i. So $\diamondsuit_{\lambda^+}(S)$ holds as claimed.

Theorem 2.2 (Gregory [7] for λ regular, Shelah [16] for λ singular)   *Let $\kappa = \lambda^+$, $\mu = \mathrm{cf}(\lambda)$, $T = \{\alpha < \kappa : \mathrm{cf}(\alpha) \neq \mu\}$. Suppose that* GCH *holds. Then $\diamondsuit^*_\kappa(T)$ holds, and so $\diamondsuit_\kappa(S)$ holds for all stationary $S \subseteq T$.*

Proof   We fix for each $\alpha < \lambda^+$ a representation $\alpha = \bigcup_{j < \mu} a^\alpha_j$ where the a^α_j increase with j and $|a^\alpha_j| < \lambda$. We also fix an enumeration of the bounded subsets of λ^+ as $\langle x_i : i < \lambda^+ \rangle$.

Suppose we are given $X \subseteq \lambda^+$. Clearly the set $C = \{\delta : \forall \gamma < \delta \ \exists i < \delta \ X \cap \gamma = x_i\}$ is club in λ^+. It follows that for any $\alpha \in C$ we may find a sequence $\langle \alpha_i : i < \mathrm{cf}(\alpha) \rangle$ in α such that $X \cap \alpha = \bigcup_i x_{\alpha_i}$.

Suppose now that $\alpha \in C \cap T$. We may choose $\langle \alpha_i : i < \mathrm{cf}(\alpha) \rangle$ in α such that $X \cap \alpha = \bigcup_i x_{\alpha_i}$, and then since $\mathrm{cf}(\alpha) \neq \mu$, we may thin out this sequence so that in addition there is some j such that $\{\alpha_i : i < \mathrm{cf}(\alpha)\} \subseteq a^\alpha_j$.

If we now set

$$\mathcal{S}_\alpha = \{x \in P(\alpha) : \exists j \ \exists y \subseteq a^\alpha_j \ x = \bigcup_{i \in y} x_i\},$$

then by GCH $|\mathcal{S}_\alpha| \leq \lambda$ and we have just argued that in fact $\langle \mathcal{S}_\alpha : \alpha \in T \rangle$ is a $\diamondsuit^*_{\lambda^+}(T)$-sequence. $\qquad \square$

The only instances of GCH we have used are that $2^{<\lambda} = \lambda$ and $2^\lambda = \lambda^+$. There are consistency results showing that in general GCH does not imply that some stationary subset of $\lambda^+ \cap \mathrm{cof}(\lambda)$ carries a diamond sequence. This is the first example of a phenomenon we shall encounter repeatedly, the special status of the "critical cofinality" $\mathrm{cf}(\lambda)$ when we study the combinatorics of the successor cardinal λ^+.

2.3 The diamond ideal   As was mentioned in the introduction, we can often define an ideal which measures the extent to which some combinatorial principle holds in V. As a first example we describe a natural ideal which measures the extent to which diamond holds.

For κ regular and uncountable, let I_{diamond} be the set of $S \subseteq \kappa$ such that $\diamondsuit_\kappa(S)$ fails. Clearly I_{diamond} contains all the nonstationary sets and is downward closed. We will show that I_{diamond} is a normal ideal by verifying that it is closed under diagonal unions; the argument is similar to the one for the equivalence of $\diamondsuit$ and $\diamondsuit'$.

Let $S_i \in I_{\text{diamond}}$ for $i < \kappa$ and let $S = \{\alpha : \exists i < \alpha \ \alpha \in S_i\}$. Suppose for a contradiction that $\diamondsuit_\kappa(S)$ holds. We may assume that we have a diamond sequence which guesses subsets of $\kappa \times \kappa$; to be more precise we have $\langle A_\alpha : \alpha \in S \rangle$ such that $A_\alpha \subseteq \alpha \times \alpha$ and for every $X \subseteq \kappa \times \kappa$ the set $\{\alpha \in S : X \cap (\alpha \times \alpha) = A_\alpha\}$ is stationary.

For each i let $T_\alpha^i = \{\beta < \alpha : (i, \beta) \in A_\alpha\}$ for $\alpha \in S_i$. Since $S_i \in I_{\text{diamond}}$ we may fix $X_i \subseteq \kappa$ and C_i a club subset of κ such that $X_i \cap \alpha \neq T_\alpha^i$ for all $\alpha \in S_i \cap C_i$. Let $X = \{(i, \beta) : \beta \in X_i\}$, let C be the diagonal intersection of the club sets C_i and fix $\alpha \in C \cap S$ such that $X \cap (\alpha \times \alpha) = A_\alpha$; find $i < \alpha$ with $\alpha \in S_i$, and note that for $\beta < \alpha$ we have $\beta \in T_\alpha^i \iff (i, \beta) \in A_\alpha \iff (i, \beta) \in X \iff \beta \in X_i$. So $T_\alpha^i = X_i \cap \alpha$, which is a contradiction to the choice of the C_i and X_i since $\alpha \in C_i$.

The extent of the diamond ideal can vary considerably. To fix ideas let $\kappa = \aleph_1$. Even with CH there is a wide range of possibilities; at the extremes $I_{\text{diamond}} = NS_{\aleph_1}$ in L and $I_{\text{diamond}} = P(\aleph_1)$ in Jensen's model of "CH + no $\aleph_1$-Souslin tree".

3 Club Guessing

Since instances of diamond imply instances of GCH which are consistently false, it is hopeless to ask that any instance of the full diamond principle be provable in ZFC. In this section we look at a weak version of diamond known as "club guessing", which is provable in ZFC. Club guessing differs from the full diamond in two respects: we only guess club subsets C of κ, and we lower the bar further by only asking at α for a club subset of $C \cap \alpha$. More precisely we have the following definition.

Definition 3.1   *Club guessing* holds for κ and $S \subseteq \kappa$ if there exists $\langle C_\alpha : \alpha \in S \rangle$ such that C_α is club in α, and for all club subsets $C \subseteq \kappa$ the set $\{\alpha \in S : C_\alpha \subseteq C \cap \alpha\}$ is stationary.

Shelah showed that many instances of club guessing are provable in ZFC.

Theorem 3.2 (Shelah [17])   *Let λ and κ be regular with $\lambda^+ < \kappa$, and let $S \subseteq \kappa \cap \mathrm{cof}(\lambda)$ with S stationary. Then club guessing holds for κ and S.*

Proof   The idea is to start with an "attempt" at a club guessing sequence and then improve it repeatedly until it becomes a successful attempt. Given E and F club subsets of α we define a club subset

$$\mathrm{pd}(E, F) = \{\sup(\gamma \cap F) : \gamma \in E \text{ and } \gamma \cap F \neq \varnothing\}.$$

Think of this as the club obtained by *pushing down E onto F*.

We choose an arbitrary sequence $\langle C_\alpha : \alpha \in S \rangle$ with C_α club in α of order type λ. We will define a decreasing sequence of club subsets $E_i \subseteq \kappa$, and for each i we will define $\langle C_\alpha^i : \alpha \in \lim(E_i) \cap S \rangle$ by setting $C_\alpha^i = \mathrm{pd}(C_\alpha, E_i \cap \alpha)$. To start the construction we let $E_i = \kappa$.

If $\langle C_\alpha^i : \alpha \in \lim(E_i) \cap S \rangle$ fails to have the club guessing property then we choose $E_{i+1} \subseteq \lim(E_i)$ a club set so that for all $\alpha \in E_{i+1} \cap S$, $C_\alpha^i \nsubseteq E_{i+1} \cap \alpha$. For i limit we let $E_i = \bigcap_{j<i} E_j$.

We claim this process terminates before stage λ^+, so producing a sequence $\langle C_\alpha^i : \alpha \in \lim(E_i) \cap S \rangle$ which has the club guessing property. Otherwise let $E = \bigcap_{i<\lambda^+} E_i$ and fix some $\alpha \in E \cap S$. Then $\alpha \in \lim(E_i) \cap S$ and $C_\alpha^i = \mathrm{pd}(C_\alpha, E_i \cap \alpha)$ for all $i < \lambda^+$. For each $\gamma \in C_\alpha$ the sequence of suprema $\langle \sup(\gamma \cap E_i) : i < \lambda^+ \rangle$ is (nonstrictly) decreasing, so must stabilize. Since C_α has size λ we may find $i < \lambda^+$ so large that this stabilization has happened for every $\gamma \in C_\alpha$, so that $C_\alpha^i = C_\alpha^{i+1} \subseteq E_{i+1}$. Since $\alpha \in E_{i+1} \cap S$, this contradicts our choice of E_{i+1}. $\square$

As in the discussion of diamond and GCH, we note that there is a "critical cofinality" issue here. In this case when $\kappa = \mu^+$ for μ regular the theorem does not guarantee the existence of club guessing on any stationary subset of $\kappa \cap \mathrm{cof}(\mu)$, and indeed it is consistent that there is no stationary subset of $\kappa \cap \mathrm{cof}(\mu)$ which carries a club guessing sequence.

There are many interesting variations: for example, we may add demands on the order types of the C_α or insist that their successor points be of large cofinality. An especially interesting variant is *strong club guessing* where we weaken the guessing condition at each α by asking only that $C_\alpha \setminus C$ is bounded, but ask that this should hold for almost every $\alpha \in S$. Ishiu's paper on the precipitousness of club-guessing ideals [9] contains a wealth of information on these matters.

4 Squares

4.1 Basic facts   We recall Jensen's classical square principle for an uncountable cardinal μ.

Principle 4.1 ($\square_\mu$)   There exists $\langle C_\alpha : \alpha < \mu^+ \rangle$ such that C_α is club in α, $\mathrm{ot}(C_\alpha) \leq \mu$, $C_\beta = C_\alpha \cap \beta$ for all $\alpha \in \mu^+$, and all $\beta \in \lim(C_\alpha)$.

The principle $\square_\mu$ is often used to get through limit stages of uncountable cofinality in inductive constructions of length μ^+. We see an example of this shortly. Jensen [10] showed that in L the principle $\square_\mu$ holds for all μ.

We note that the property of being a $\square_\mu$-sequence is upward absolute to any universe in which μ and μ^+ are preserved. It is sometimes useful to know that $\square_\mu$ implies a stronger version of itself in which we add the demand that $\mathrm{ot}(C_\alpha) < \mu$ for all α such that $\mathrm{cf}(\alpha) < \mu$. To see this fix $C \subseteq \mu$ a club set with $\mathrm{ot}(C) = \mathrm{cf}(\mu)$, and then replace C_α by $\{\beta \in C_\alpha : \mathrm{ot}(C_\alpha \cap \beta) \in C\}$ whenever $\mathrm{ot}(C_\alpha) \in \lim(C) \cup \{\mu\}$.

The principle $\square_\mu$ is generally used to construct "incompact" or "nonreflecting" objects. This is natural when we consider that there can be no club $C \subseteq \mu^+$ such that $C \cap \alpha = C_\alpha$ for $\alpha \in \lim(C)$, so that a square sequence is itself an incompact object. A basic example is the construction of a nonreflecting stationary set.

Let $S \subseteq \mu^+$ be any stationary set and use Fodor's Lemma to find $T \subseteq S$ a stationary set such that $\mathrm{ot}(C_\alpha) = \beta$ for all $\alpha \in T$. If $\gamma < \mu^+$ has uncountable cofinality $\lim(C_\gamma)$ meets T at most once, since $C_\alpha = C_\gamma \cap \alpha$ for $\alpha \in \lim(C_\gamma)$. It follows that $T \cap \gamma$ is nonstationary in γ. Note that if we set $D_\alpha = C_\alpha$ when $\mathrm{ot}(C_\alpha) \leq \beta$, and $D_\alpha = \{\gamma \in C_\alpha : \mathrm{ot}(C_\alpha \cap \gamma) > \beta\}$ when $\mathrm{ot}(C_\alpha) > \beta$ we have produced a

$\square_\mu$-sequence $\langle D_\alpha : \alpha < \mu^+ \rangle$ with the additional property that $\lim(D_\alpha) \cap T = \varnothing$ for all $\alpha \in T$.

4.2 Square in action: Souslin trees   To give more of a feeling for the principle $\square_\mu$, we sketch a proof of Jensen's theorem that GCH and the principle $\square_\mu$ imply the existence of a μ^+-Souslin tree for μ uncountable. We choose a regular $\kappa < \mu$ with $\kappa \neq \mathrm{cf}(\mu)$, and by the considerations above we find a stationary $U \subseteq \mu^+ \cap \mathrm{cof}(\kappa)$, a $\diamondsuit_{\mu^+}(U)$-sequence and a $\square_\mu$-sequence such that $U \cap \lim(C_\alpha) = \varnothing$ for all α".

We then build up a normal μ^+-tree T in the standard way, using ordinals in $[\mu.\alpha, \mu.(\alpha+1))$ for points on level α. If at $\beta \in U$ the diamond sequence guesses a maximal antichain in $T \upharpoonright \beta$ we arrange that every point on level β is above some point of this antichain. As usual this "seals off" the antichain, and diamond implies that the final tree is a μ^+-Souslin tree.

The key point is to maintain the hypotheses that every level has size at most μ and that every point x in $T \upharpoonright \beta$ is below some point $y \in T_\beta$. We do this as follows: we build a branch $c(x, \beta)$ starting at x and working up the levels in C_β above x, choosing at each stage the least point above points so far. The construction will be organized so this "canonical branch" is always cofinal in $T \upharpoonright \beta$.

We define level β by completing every canonical branch unless $\beta \in U$ and the diamond sequence guessed a maximal antichain, in which case we only complete canonical branches for those points which lie above some point of the maximal antichain. The remaining issue is to see that every canonical branch is cofinal: the point is that if γ is a limit stage in the construction of $c(x, \beta)$ then $C_\gamma = C_\beta \cap \gamma$ and $\gamma \notin U$, so that $c(x, \gamma)$ is an initial segment of $c(x, \beta)$ and it gets completed at stage γ.

5   Weak Squares

Jensen also introduced a principle "weak square" in which μ many club sets are allowed at each $\alpha < \mu^+$. We will consider a generalization due to Schimmerling.

Principle 5.1 ($\square_{\mu,\lambda}$)   There exists $\langle \mathcal{C}_\alpha : \alpha < \mu^+, \alpha \text{ limit} \rangle$ such that $\mathcal{C}_\alpha$ is a nonempty set of club subsets of α, $|\mathcal{C}_\alpha| \leq \lambda$, $\mathrm{ot}(C) \leq \mu$, and $C \cap \beta \in \mathcal{C}_\beta$ for all $\alpha < \mu^+$, all $C \in \mathcal{C}_\alpha$, and all $\beta \in \lim(C)$.

We note that the "silly square" principle $\square_{\mu,\mu^+}$ is always true, since we may just fix D_α club in α for every $\alpha < \mu^+$ and let

$$\mathcal{C}_\beta = \{D_\alpha \cap \beta : \beta \in \lim(D_\alpha) \cup \{\alpha\}\}.$$

Jensen showed that $\square_{\mu,\mu}$ (the principle usually denoted by $\square_\mu^*$) is equivalent to the existence of a special μ^+-Aronszajn tree. We give a version of this proof using ideas of Todorčević [20] to construct the Aronszajn tree from the square sequence.

5.1 From weak square to a special tree   Let η be an arbitrary ordinal and let $\langle C_\zeta : \zeta < \eta \rangle$ be such that C_ζ is club in ζ for all ζ. By convention we let $C_{\alpha+1} = \{\alpha\}$ for $\alpha + 1 < \eta$. We define for $\alpha < \beta < \eta$ a "minimal walk from β down to α". This is a decreasing sequence of ordinals given by the recursion $\beta_0 = \beta$ and $\beta_{n+1} = \min(C_{\beta_n} \setminus \alpha)$, halting when we reach (as we must do in a finite number of steps) an n such that $\beta_n = \alpha$.

We associate two sequences with the minimal walk from β down to α. If this walk is the sequence $\beta_0 = \beta > \cdots > \beta_n = \alpha$, then the *projection* of the walk $pr(\alpha, \beta)$ is

the sequence $C_{\beta_0} \cap \alpha, \ldots, C_{\beta_n} \cap \alpha$ and the *trace* of the walk $\mathrm{tr}(\alpha, \beta)$ is the sequence $\mathrm{ot}(C_{\beta_0} \cap \alpha), \ldots, \mathrm{ot}(C_{\beta_n} \cap \alpha)$.

The key point: we suppose that $\alpha < \alpha^* < \beta$ and compare the walks from β down to α and α^*. Suppose that these walks are given by sequences $\beta_0 = \beta > \cdots > \beta_m = \alpha$ and $\beta_0^* = \beta > \cdots > \beta_n^* = \alpha^*$.

1. If $\beta_i = \beta_i^* > \alpha^*$ and $C_{\beta_i} \cap [\alpha, \alpha^*) = \varnothing$ then clearly $\beta_{i+1} = \beta_{i+1}^*$. It follows that exactly one of the following possibilities must occur:
   (i) there is $i < \min\{m, n\}$ such that $C_{\beta_i} \cap [\alpha, \alpha^*) \neq \varnothing$. If i is minimal with this property then $\beta_j = \beta_j^*$ for $j \leq i$, and $\beta_{i+1} < \alpha^* \leq \beta_{i+1}^*$;
   (ii) there is no such i; in this case $n < m$, $\beta_j = \beta_j^*$ for $j \leq n$.
   We note that in either case $\mathrm{tr}(\alpha, \beta)$ is below $\mathrm{tr}(\alpha^*, \beta)$ in the Kleene-Brouwer linear ordering of all finite sequences of ordinals.

2. We claim that if we are given $\langle C_\zeta : \zeta < \eta \rangle$ and $\mathrm{pr}(\alpha^*, \beta)$ then we can compute $\mathrm{pr}(\alpha, \beta)$ (and hence $\mathrm{tr}(\alpha, \beta)$) for all $\alpha < \alpha^*$ *without knowing the value of* β. Let $\mathrm{pr}(\alpha^*, \beta) = D_0, \ldots, D_n$ and note that α^* can be computed from D_n. Let $j \leq n$ be least such that $D_j \cap [\alpha, \alpha^*) \neq 0$; we have $\beta_i = \beta_i^*$ for $i \leq j$ so that we can compute the first $j + 1$ terms of $\mathrm{pr}(\alpha, \beta)$ as $D_0 \cap \alpha, \ldots, D_j \cap \alpha$. Also $\beta_{j+1} = \min(D_j \setminus \alpha)$ and so we can compute the rest of the walk from β down to α, and hence the remaining terms of $\mathrm{pr}(\alpha, \beta)$.

We are now almost ready to construct a special μ^+-Aronszajn tree from a $\square_\mu^*$-sequence $\langle \mathcal{C}_\alpha : \alpha < \mu^+ \rangle$. We observe that for any $\gamma < \mu^+$ the set $\{C \cap \gamma : C \in \bigcup_{\alpha \geq \gamma} \mathcal{C}_\alpha\}$ has size at most μ, because if $\delta \leq \gamma$ is maximal with $\sup(C \cap \delta) = \delta$ then $C \cap \delta \in \mathcal{C}_\delta$ and $(C \cap \gamma) \setminus \delta$ is a finite subset of γ; since $\bigcup_{\delta \leq \gamma} \mathcal{C}_\delta$ and $[\gamma]^{<\omega}$ both have size at most μ, this gives at most μ possibilities for $C \cap \gamma$. Choose C_α as above with $C_\alpha \in \mathcal{C}_\alpha$ for α limit.

We now define the tree: the elements are sequences of the form $\langle tr(\alpha, \beta) : \alpha \leq \alpha^* \rangle$ where $\alpha^* < \beta < \mu^+$, so that level α^* consists of $\alpha^* + 1$-sequences. The tree element $\langle \mathrm{tr}(\alpha, \beta) : \alpha \leq \alpha^* \rangle$ is determined by $\mathrm{pr}(\alpha^*, \beta)$ which is a finite sequence from $\{C \cap \alpha^* : C \in \bigcup_{\alpha^* \leq \beta < \mu^+} \mathcal{C}_\alpha\}$, so every level of the tree has size at most μ. The map which takes $\langle \mathrm{tr}(\alpha, \beta) : \alpha \leq \alpha^* \rangle$ to $\mathrm{tr}(\alpha^*, \beta)$ is a specializing function, in fact the specializing function is an order-preserving map from the tree to $[\mu + 1]^{<\omega}$ ordered by the reverse lexicographic ordering.

5.2 From a special tree to weak square

For the converse, suppose that we are given a special μ^+-tree T together with a specializing function $f : T \longrightarrow \mu$. We will show how to associate with each point $x \in T$ of limit height an unbounded set $A_x \subseteq \mathrm{ht}(x)$ such that $\mathrm{ot}(A_x) \leq \mu$, and if $\sup(A_x \cap \delta) = \delta$ for some $\delta < \mathrm{ht}(x)$ then $A_x \cap \delta = A_y$ for y the unique point of height δ below x. It should then be clear that if we set $C_x = A_x \cup \{\gamma < \mathrm{ht}(x) : \sup(A_x \cap \gamma) = \gamma\}$ and $\mathcal{C}_\alpha = \{C_x : \mathrm{ht}(x) = \alpha\}$ we have defined a $\square_\mu^*$-sequence.

We fix $x \in T$ with $\mathrm{ht}(x) = \gamma$ for some limit γ and define by induction an increasing sequence of ordinals $\gamma_i^x < \gamma$. We define γ_j^x to be $\mathrm{ht}(y)$ for the unique $y <_T x$ such that $f(y)$ is the minimum element of the set

$$\{f(z) : z <_T x \text{ and } \gamma_i^x < \mathrm{ht}(z) \text{ for all } i < j\},$$

halting the induction when we reach j_x such that $\{\gamma_i^x : i < j_x\}$ is unbounded in γ, and set $A_x = \{\gamma_i^x : i < j_x\}$.

Let $y_i <_T x$ be the unique point with $\mathrm{ht}(y_i) = \gamma_i^x$ for each $i < j_x$. If $i_0 < i_1 < j_x$ then by definition $f(y_{i_0})$ is the minimal value of f attained at any point $z <_T x$ such that $\mathrm{ht}(z) > \gamma_i$ for all $i < i_0$; y_{i_1} is such a point so $f(y_{i_0}) < f(y_{i_1})$. It follows easily that $\mathrm{ot}(A_x) = j_x \le \mu$.

Finally suppose that $\delta < \mathrm{ht}(x) = \gamma$, $\delta = \sup(A_x \cap \delta)$. Let $A_x \cap \delta = \{\gamma_i^x : i < j\}$, and let $y <_T x$ be the unique point below x with $\mathrm{ht}(y) = \delta$. We will show by induction that $\gamma_i^x = \gamma_i^y$ for all $i < j$, so that $A_x \cap \delta = A_y$. Suppose that $i^* < j$ and $\gamma_i^x = \gamma_i^y$ for $i < i^*$; the minimum value of f on $\{z : z <_T x$ and $\gamma_i^x < \mathrm{ht}(z)$ for all $i < i^*\}$ is attained at the point of height γ_{i*}^x. Since $\gamma_i^x = \gamma_i^y$ for all $i < i^*$ and $\gamma_{i*}^x < \delta = \mathrm{ht}(y)$, this is the same point where f attains its minimum on $\{z : z <_T y$ and $\gamma_i^y < \mathrm{ht}(z)$ for all $i < i^*\}$, so that $\gamma_{i*}^x = \gamma_{i*}^y$ as required.

5.3 Cosmetic improvements.   It will be useful later to know that weak square sequences can be "improved." The value of this will be apparent when we introduce the ideal $I[\lambda]$ on a regular cardinal λ in Section 8; a suitably improved $\square_\mu^*$-sequence will there be used as a witness that $\mu^+ \in I[\mu^+]$, so that we can see the assertion that $I[\mu^+]$ is trivial as a "weak weak square".

We claim that if $\square_\mu^*$ holds then there is a $\square_\mu^*$-sequence with the additional property that for all $\alpha < \mu^+$ there is $C \in \mathcal{C}_\alpha$ with $\mathrm{ot}(C) = \mathrm{cf}(\alpha)$. To see this we fix $\{D_j : j < \mu\}$ with $\mathrm{ot}(D_j) = \mathrm{cf}(j)$ and D_j club in j. For all $C \in \mathcal{C}_\alpha$ and all j with $\mathrm{ot}(C) \in \lim(D_j) \cup \{j\}$ we will add the set $\{\alpha \in C : \mathrm{ot}(C \cap \alpha) \in D_j\}$ to $\mathcal{C}_\alpha$. It is routine to check that this works.

By the same trick that we used above for $\square_\mu^*$ (Principle 4.1), we may also prove that if μ is singular and $\square_{\mu,\lambda}$ (Principle 5.1) holds then this is witnessed by a sequence in which every club set which appears has order type less than μ. This is useful later in Sections 14 and 15 where we want to use $\square_{\lambda,\mu}$ to produce PCF-theoretic scales with various extra properties.

6 The Extent of Square

There are consistency results which place sharp limits on what we can hope for in terms of square principles provable from ZFC or even ZFC + GCH. The cases of singular and regular μ are quite different, and for μ singular there is a sharp dividing line between $\square_{\mu,\lambda}$ for $\lambda < \mathrm{cf}(\mu)$ and for $\lambda \ge \mathrm{cf}(\mu)$.

If μ is regular and $\mu^{<\mu} = \mu$ then it is easy to see that $\square_\mu^*$ holds. On the other hand if $\kappa > \mu$ is weakly compact and we force with the Lévy collapse $\mathrm{Coll}(\mu, < \kappa)$ then (by work of Baumgartner [2]) in the extension every stationary subset of $\mu^+ \cap \mathrm{cof}(< \mu)$ reflects, and so $\square_\mu$ fails. Harrington and Shelah [8] showed that with more work a Mahlo cardinal suffices.

Dropping GCH, models with μ regular where $\square_\mu^*$ fails can be obtained. For example, Mitchell [14] showed how to collapse a Mahlo cardinal to $\aleph_2$ in such a way that in the extension $2^{\aleph_0} = \aleph_2$ and $\square_{\aleph_1}^*$ fails.

It is known that a Mahlo cardinal is the optimal hypothesis for the failure of $\square_\mu$ (or even $\square_\mu^*$) when μ is regular. For μ singular the picture is much different because of core models and the covering lemma. As a simple example, if $0^\sharp$ does not exist and μ is a singular cardinal in V, then $\mu^+ = \mu_L^+$, so that a $\square_\mu$-sequence in L is still a $\square_\mu$-sequence in V.

Sufficiently large cardinals are incompatible with $\square_\mu$ and $\square_\mu^*$ for μ singular. To be more precise if κ is a strongly compact cardinal then

(i) $\square_\mu$ fails for every cardinal $\mu \geq \kappa$ (Solovay [18]);
(ii) $\square_\mu^*$ fails for every singular cardinal μ such that $\mathrm{cf}(\mu) < \kappa < \mu$ (Shelah [16]).

Before we give these arguments, we note that in Section 18 we will give an alternative PCF-theoretic argument for the failure of weak square (and indeed of much weaker principles) at singular cardinals above a supercompact cardinal. We will also give a consistency proof for the failure of $\square_{\aleph_\omega}^*$ in that section.

Let $\kappa < \lambda$ where λ is regular and κ is λ-strongly compact. We recall that this large cardinal assumption can be formulated as follows: there is an elementary embedding $j : V \longrightarrow M$ such that $\mathrm{crit}(j) = \kappa$ and there is $X \in M$ such that $j``\lambda \subseteq X$ and $M \models |X| < j(\kappa)$. We fix such an embedding $j$, define $\gamma = \sup j``\lambda$ and note that $\lambda \leq \mathrm{cf}^M(\gamma) < j(\kappa)$ and $\gamma < j(\lambda)$. It is easy to see that if $S \subseteq \lambda \cap \mathrm{cof}(< \kappa)$ is stationary in λ, then $j(S) \cap \gamma$ is stationary in γ, so that by elementarity the stationarity of S reflects at some point in $\lambda \cap \mathrm{cof}(< \kappa)$. Accordingly $\square_\mu$ fails at every μ above a strongly compact cardinal.

We now use an idea of Solovay. If $\vec{D} = \langle D_\alpha : \alpha < \lambda \rangle$ is any sequence such that D_α is club in α and $\mathrm{ot}(D_\alpha) = \mathrm{cf}(\alpha)$, then let $\vec{E} = j(\vec{D})$ and $E = E_\gamma$. Since j is continuous at points of cofinality less than κ, the set $D = \{\alpha < \lambda : j(\alpha) \in E\}$ is $< \kappa$-club in λ. So for every $x \subseteq D$ of order type less than κ, $j(x) = j``x \subseteq E_\gamma$ and by reflection $x \subseteq D_\alpha$ for some α with $\mathrm{cf}(\alpha) < \kappa$.

This has some interesting consequences. First there are $\lambda^{<\kappa}$ possibilities for x, and each D_α for $\mathrm{cf}(\alpha) < \kappa$ has size less than κ. So $\lambda^{<\kappa} = \lambda$. In particular, if μ is singular of cofinality less than κ and $\lambda = \mu^+$ then we can argue that $\mu^{\mathrm{cf}(\mu)} = \mu^+$. This is the heart of the argument that SCH holds above a strongly compact cardinal.

Now suppose for contradiction that $\mathrm{cf}(\mu) < \kappa$, $\lambda = \mu^+$, and $\square_\mu^*$ holds. We may assume $\square_\mu^*$ is witnessed by $\langle \mathcal{C}_\alpha : \alpha < \lambda \rangle$ such that each $\mathcal{C}_\alpha$ contains some D_α of order type $\mathrm{cf}(\alpha)$ and then construct $\vec{D}$, $\vec{E}$ and sets D, E as above. Let ζ be the supremum of the first μ elements of D, and note that $j(\zeta)$ is a limit point of E and hence $E \cap j(\zeta) \in j(\mathcal{C})_{j(\zeta)} = j(\mathcal{C}_\zeta)$. By a similar reflection argument to the one above, if $x \subseteq D \cap \zeta$ with $\mathrm{ot}(x) < \kappa$, then $x \subseteq F$ for some $F \in \mathcal{C}_\zeta$ with $\mathrm{ot}(F) < \kappa$. This is impossible as $\mu^{<\kappa} > \mu$ but there are only μ possibilities for F, each with fewer than κ subsets.

7 Partial Squares

We now consider another weakening of $\square_\mu$ (Principle 4.1), in which we only assign club sets to certain ordinals less than μ^+. In the spirit of our discussion of diamonds and club guessing, we focus on particular cofinalities.

Let $S \subseteq \{\alpha \in \mu^+ : \mathrm{cf}(\alpha) = \lambda\}$. We say S *carries a partial square* if there exists $\langle C_\alpha : \alpha \in S \rangle$ such that C_α is club in α, $\mathrm{ot}(C_\alpha) = \lambda$, and $C_\alpha \cap \beta = C_{\alpha^*} \cap \beta$ for any α and α^* in S and any common limit point β of the club sets C_α and C_{α^*}.

A straightforward induction argument shows that if $\square_\mu$ holds for all μ then $\{\alpha \in \mu^+ : \mathrm{cf}(\alpha) = \lambda\}$ carries a partial square for all μ and all regular $\lambda \leq \mu$.

Theorem 7.1 (Shelah [17])   *Let $\lambda < \mu$ with λ and μ both regular. Then $\{\alpha < \mu^+ : \mathrm{cf}(\alpha) = \lambda\}$ is the union of μ sets each carrying a partial square.*

Before proving the theorem, we note that in general $\mu^+ \cap \mathrm{cof}(\mu)$ is not the union of μ sets each carrying a partial square, and that in the conclusion of Theorem 7.1 at least one set carrying a partial square will be stationary.

Proof   The proposition is trivial for $\lambda = \omega$ so we assume $\lambda > \omega$. We fix some large regular θ and some well-ordering $<_\theta$ of H_θ and work inside the structure $(H_\theta, \in, <_\theta)$. For each $\alpha < \mu^+$ of cofinality λ and each $\zeta < \mu$ we let $M(\alpha, \zeta)$ be the Skolem hull of $\{\alpha\} \cup \zeta$, where we note that for fixed α the model $M(\alpha, \zeta)$ is increasing and continuous as a function of ζ.

For each α choose $\zeta(\alpha)$ as the least $\zeta \geq \lambda$ such that $M(\alpha, \zeta) \cap \mu \in \mu$ and $\mathrm{cf}(M(\alpha, \zeta) \cap \mu) > \omega$. Let $N_\alpha = M(\alpha, \zeta(\alpha))$ and note that N_α is unbounded in α. Let $E_\alpha = N_\alpha \cap \alpha$ and let D_α be the closure of E_α.

Key point: E_α is closed under suprema of length ω. To see this let $x \subseteq E_\alpha$ have order type ω, let $\beta = \sup(x)$ and $\gamma = \min(E_\alpha - \beta)$, and suppose for a contradiction that $\beta < \gamma$. Note that if $f \in N_\alpha$ is an increasing cofinal map from $\mathrm{cf}(\gamma)$ to γ, then $f \restriction N_\alpha \cap \mathrm{cf}(\gamma)$ is increasing and cofinal in $N_\alpha \cap \gamma$. If $\mathrm{cf}(\gamma) < \mu$ then since $N_\alpha \cap \mu \in \mu$ we have N_α unbounded in γ, so necessarily $\mathrm{cf}(\gamma) = \mu$. Then we must have $\mathrm{cf}(N_\alpha \cap \gamma) = \mathrm{cf}(N_\alpha \cap \mu)$, which is impossible since $\mathrm{cf}(N_\alpha \cap \mu) > \omega$ while $\mathrm{cf}(N_\alpha \cap \gamma) = \mathrm{cf}(N_\alpha \cap \beta) = \omega$.

Given ρ and σ in μ, let $S(\rho, \sigma) = \{\alpha : N_\alpha \cap \mu = \rho, \mathrm{ot}(D_\alpha) = \sigma\}$. We argue that each of these sets carries a partial square.

Suppose that α and α^* are in $S(\rho, \sigma)$ and γ is a common limit point of D_α and D_{α^*}; we claim that $D_\alpha \cap \gamma = D_{\alpha^*} \cap \gamma$. If γ has cofinality ω then $\gamma \in N_\alpha \cap N_{\alpha^*}$; since $|\gamma| \leq \mu$ and $N_\alpha \cap \mu = \rho = N_{\alpha^*} \cap \mu$ we see easily that $N_\alpha \cap \gamma = N_{\alpha^*} \cap \gamma$. If γ has uncountable cofinality then there are unboundedly many $\eta < \gamma$ such that η has cofinality ω and η is a common limit point of D_α and D_{α^*}; as we just saw $N_\alpha \cap \eta = N_{\alpha^*} \cap \eta$ for each such η, so $N_\alpha \cap \gamma = N_{\alpha^*} \cap \gamma$.

To finish we fix $C \subseteq \sigma$ a club set of order type λ and let $C_\alpha = \{\gamma \in D_\alpha : \mathrm{ot}(D_\alpha \cap \gamma) \in C\}$. It is routine to check that this thinning out of the D_α preserves coherence at limit points and so yields a partial square sequence on $S(\rho, \sigma)$. $\square$

The exact extent of partial squares at the successor of a singular cardinal is an interesting open problem. By contrast with Theorem 7.1, there are consistency results showing that for μ singular and regular $\lambda < \mu$ it may not be the case that $\mu^+ \cap \mathrm{cof}(\lambda)$ is the union of μ sets with squares; such results are given in Sections 17 and 18.

8   Approachability and $I[\lambda]$

8.1 Basic facts   We now discuss *approachability*, a squarelike principle weak enough to be provable in ZFC (even at successors of singulars) yet strong enough to do useful work.

Let κ be regular and $\langle a_\alpha : \alpha < \kappa \rangle$ be some sequence of bounded subsets of κ. We say that a limit ordinal $\gamma < \kappa$ is *approachable with respect to the sequence* if and only if there is $A \subseteq \gamma$ unbounded in γ with $\mathrm{ot}(A) = \mathrm{cf}(\gamma)$ and $\{A \cap \beta : \beta < \gamma\} \subseteq \{a_\beta : \beta < \gamma\}$. We note that modulo clubs the set of approachable points depends only on the set $\{a_\alpha : \alpha < \kappa\}$. Given κ regular and uncountable, Shelah defined an ideal $I[\kappa]$ on κ as follows.

Definition 8.1   For $S \subseteq \kappa$, $S \in I[\kappa]$ if and only if there exists a sequence $\langle a_\alpha : \alpha < \kappa \rangle$ of bounded subsets of κ and a club subset C of κ such that every $\gamma \in S \cap C$ is approachable with respect to the sequence.

A useful alternative view of $I[\kappa]$ can be given using elementary submodels. As usual θ denotes some very large regular cardinal. Given $\mathcal{A}$ some expansion of $(H_\theta, \in, <_\theta)$ by countably many constants, functions, and relations, γ is *approachable with respect to* $\mathcal{A}$ if there is $A \subseteq \gamma$ unbounded of order type $\mathrm{cf}(\gamma)$ such that every initial segment of A lies in $Sk^{\mathcal{A}}(\gamma)$.

If $\langle a_\alpha : \alpha < \kappa \rangle$ is an enumeration of the bounded subsets of κ lying in $Sk^{\mathcal{A}}(\kappa)$, then it is easy to see that almost every γ which is approachable with respect to $\mathcal{A}$ is approachable with respect to $\langle a_\alpha : \alpha < \kappa \rangle$. Conversely, if $\langle a_\alpha : \alpha < \kappa \rangle$ is any sequence of bounded subsets of κ and we let $\mathcal{A}$ be $(H_\theta, \in, <_\theta, \vec{a})$, then almost every γ which is approachable with respect to $\langle a_\alpha : \alpha < \kappa \rangle$ is approachable with respect to $\mathcal{A}$. So we could have defined $I[\kappa]$ as the collection of S such that for some $\mathcal{A}$, almost every $\gamma \in S$ is approachable with respect to $\mathcal{A}$.

We will argue that $I[\kappa]$ is a normal ideal. Downward closure is immediate, so we need only to show that $I[\kappa]$ is closed under diagonal unions. Let $S_i \in I[\kappa]$ for all $i < \kappa$ and fix for each i a club set C_i and a structure $\mathcal{A}_i$ such that every $\gamma \in C_i \cap S_i$ is approachable with respect to $\mathcal{A}_i$.

We now fix for each i a function $F_i : {}^{<\omega}\kappa \longrightarrow H_\theta$ such that for any $X \subseteq \kappa$, $Sk^{\mathcal{A}_i}(X) = F_i``({}^{<\omega}X)$. We combine them into a single function $F : {}^{<\omega}\kappa \longrightarrow H_\theta$ by defining $F(i, \vec{\alpha}) = F_i(\vec{\alpha})$ and then let $\mathcal{A}$ equal $(H_\theta, \in, <_\theta, F)$.

Let $S = \{\gamma < \kappa : \exists i < \gamma \ \gamma \in S_i\}$ and let $C = \{\gamma < \kappa : \forall i < \gamma \ \gamma \in C_i\}$. Let $\gamma \in C \cap S$ and find $i < \gamma$ such that $\gamma \in S_i$. Since $\gamma \in C$ we have $\gamma \in C_i \cap S_i$, so γ is approachable with respect to $\mathcal{A}_i$ and we may find $A \subseteq \gamma$ unbounded such that $\mathrm{ot}(A) = \mathrm{cf}(\gamma)$ and all proper initial segments of A lie in $Sk^{\mathcal{A}_i}(\gamma)$. It is immediate from the definitions of F_i and $\mathcal{A}$ that $Sk^{\mathcal{A}_i}(\gamma) = F_i``({}^{<\omega}\gamma) \subseteq F``({}^{<\omega}\gamma) \subseteq Sk^{\mathcal{A}}(\gamma)$, so that γ is approachable with respect to $\mathcal{A}$. Since C is club we have showed that $S \in I[\kappa]$.

We collect some easy facts about $I[\kappa]$ with sketchy proofs.

(i) If $\square_\mu^*$ holds then $\mu^+ \in I[\mu^+]$.

   Let $\langle \mathcal{C}_\alpha : \alpha < \mu^+ \rangle$ be a $\square_\mu^*$-sequence with the additional property that every $\mathcal{C}_\alpha$ contains a set of order type $\mathrm{cf}(\alpha)$. If $\mathcal{A}$ has a predicate for this weak square sequence then every limit ordinal α such that $\mu \le \alpha < \mu^+$ is approachable with respect to $\mathcal{A}$.

(ii) If $S \subseteq \{\alpha < \kappa : \mathrm{cf}(\alpha) = \lambda\}$ for any λ and κ, and S carries a partial square, then $S \in I[\kappa]$.

   Let $\langle C_\alpha : \alpha \in S \rangle$ be a partial square sequence. Let T be the set of $\gamma \in \kappa \cap \mathrm{cof}(< \lambda)$ such that γ is a limit point of some C_α, and let $\langle D_\gamma : \gamma \in T \rangle$ be such that $C_\alpha \cap \gamma = D_\gamma$ for all α and all $\gamma \in \lim(C_\alpha)$. If $\mathcal{A}$ has a predicate for $\langle D_\gamma : \gamma \in T \rangle$ then every ordinal $\alpha \in S$ is approachable with respect to $\mathcal{A}$.

(iii) If μ is regular then $\mu^+ \cap \mathrm{cof}(< \mu) \in I[\mu^+]$.

   For each regular $\lambda < \mu$, $\mu^+ \cap \mathrm{cof}(\lambda)$ is the union of μ many sets with partial squares. Each of these sets is in $I[\mu^+]$ and the ideal $I[\mu^+]$ is μ^+-complete.

8.2 Interlude on forcing It is well known that for any regular κ and any stationary set $S \subseteq \kappa \cap \mathrm{cof}(\omega)$, the stationarity of S is preserved by countably closed forcing. In general it is not the case that a stationary subset of $S \subseteq \kappa \cap \mathrm{cof}(\mu)$ is preserved by μ^+-closed forcing (see the discussion after Theorem 8.2).

Shelah studied this kind of question and in fact the ideal $I[\kappa]$ was originally introduced by him to clarify the question of which stationary sets are preserved.

Theorem 8.2 (Shelah [16]) *If $S \subseteq \kappa \cap \mathrm{cof}(\mu)$ is stationary and $S \in I[\kappa]$, then μ^+-closed forcing preserves the stationarity of S.*

Proof Let $S \subseteq \kappa \cap \mathrm{cof}(\mu)$ be stationary with $S \in I[\kappa]$, let $\vec{a}$ witness this, and let $\mathbb{P}$ be μ^+-closed. Let $p \in \mathbb{P}$ force $\dot{C}$ is club in κ.

Expand H_θ by $\vec{a}$, p, and $\dot{C}$ to get a structure $\mathcal{A}$. Find $\gamma \in S$ such that $\gamma = Sk^{\mathcal{A}}(\gamma) \cap \kappa$, and fix $A \subseteq \gamma$ of order type μ witnessing that γ is approachable. Build a decreasing chain $\langle p_j : j < \mu \rangle$ below p choosing p_j to be least below $\langle p_i : i < j \rangle$ forcing some ordinal β_i greater than the ith point of A into $\dot{C}$. Easily all p_i are in $Sk^{\mathcal{A}}(\gamma)$, all β_i are less than γ, and any lower bound for $\langle p_j : j < \mu \rangle$ forces $\gamma \in \dot{C}$. $\square$

There is also a kind of converse to the last result, showing that under the right hypotheses we may destroy stationarity with highly closed forcing. If $\mu = \mathrm{cf}(\mu) < \kappa$ and $\kappa^{<\mu} = \kappa$, then the restriction of $I[\kappa]$ to points of cofinality μ is generated by a single stationary set. To see this, enumerate $[\kappa]^{<\mu}$ as $\langle a_\alpha : \alpha < \kappa \rangle$ and let S be the set of points of cofinality μ approachable with respect to this enumeration. It should be clear that S is (modulo clubs) the maximal subset of $\kappa \cap \mathrm{cof}(\mu)$ lying in $I[\kappa]$, and we will prove in Section 9 that S is stationary.

Assume now that $\mu^{<\mu} < \kappa$. We may as well assume that each set in $[\kappa]^{<\mu}$ appears κ times in the enumeration $\vec{a}$. With this understanding consider the poset $\mathbb{Q}$ whose conditions are closed bounded $c \subseteq \kappa$ of order type less than μ^+, where $c \cap \mathrm{cof}(\mu) \subseteq S$ and every bounded subset of c of size less than μ is enumerated as a_α for $\alpha < \max(c)$. $\mathbb{Q}$ is μ^+-closed and in $V^{\mathbb{Q}}$ the set $(\kappa \cap \mathrm{cof}(\mu)) \setminus S$ is nonstationary.

Remark 8.3 Some kind of cardinal arithmetic assumption is needed here. Mitchell's model with no special $\aleph_2$-Aronszajn trees has $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$. Since $\aleph_2^{\aleph_1} = \aleph_2$ there is a maximal stationary set $S \subseteq \aleph_2 \cap \mathrm{cof}(\aleph_1)$ with $S \in I[\aleph_2]$, and it can be shown that $(\aleph_2 \cap \mathrm{cof}(\aleph_1)) \setminus S$ is also stationary. However, κ-closed forcing always preserves stationarity for subsets of κ, in particular, $\aleph_2$-closed forcing preserves the stationarity of $(\aleph_2 \cap \mathrm{cof}(\aleph_1)) \setminus S$. This does not contradict the conclusion of the preceding paragraph, because now we are in a situation where $\aleph_1^{<\aleph_1} = \aleph_2$.

9 IA Chains and More On $I[\lambda]$

9.1 IA chains Let θ be a large regular cardinal, and let $\mathcal{A}$ be some expansion of $(H_\theta, \in, <_\theta)$ by countably many functions, constants, and relations. Let γ be a limit ordinal. Then an *IA (internally approachable) chain of substructures of $\mathcal{A}$ of length γ* is a continuous and increasing sequence $\langle M_i : i < \gamma \rangle$ of elementary substructures of $\mathcal{A}$ such that $\langle M_i : i \le j \rangle \in M_{j+1}$ for all $j < \gamma$. It is worth noting that automatically $j \subseteq M_j$, and $M_i \in M_j$. The following fact is key.

Theorem 9.1 (Foreman and Magidor [6]) *Let $S \in I[\kappa]$ and let $\mathcal{A}$ be as above. Then there is a club subset $C \subseteq \kappa$ such that for every $\gamma \in C \cap S$, setting $\lambda = \mathrm{cf}(\gamma)$*

there exists an IA chain $\langle N_i : i < \lambda \rangle$ of substructures of $\mathcal{A}$ such that $|N_i| < \lambda$ for all $i < \lambda$, and if $N = \bigcup_i N_i$ then $\sup(N \cap \kappa) = \gamma$.

Proof Expand $\mathcal{A}$ to $\mathcal{B}$ by adding a predicate for a sequence of bounded subsets of κ witnessing that $S \in I[\kappa]$. Build $\langle M_\zeta : \zeta < \kappa \rangle$ an IA chain of substructures of $\mathcal{B}$ with $|M_\zeta| < \kappa$ and $M_\zeta \cap \kappa \in \kappa$. Let $\gamma \in S$ with $M_\gamma \cap \kappa = \gamma$, let $\lambda = \mathrm{cf}(\gamma)$. We fix $\langle \gamma_j : j < \lambda \rangle$ an increasing continuous and cofinal sequence in γ such that $\langle \gamma_i : i \leq j \rangle \in M_{\gamma_{j+1}}$ for $j < \lambda$. Let $P_j = \langle M_{\gamma_i} : i \leq j \rangle$ and note that $P_j \in M_{\gamma_{j+1}}$; define N_i to be the hull in M_γ (or equivalently in M_{γ_i}) of the set of parameters $\{P_j : j < i\}$. The main point is to verify that $\langle N_i : i \leq j \rangle \in N_{j+1}$. This is true because $P_j \in N_{j+1}$, so that $M_{\gamma_j} \in N_{j+1}$ and also $\langle P_i : i < j \rangle \in N_{j+1}$.

We note for use later that $N \subseteq M_\gamma$ and $M_i \in N$ for unboundedly many $i \in \gamma$. □

If $\kappa < \lambda$ with κ and λ both regular and $\lambda^{<\kappa} = \lambda$, then there is a stationary subset of $\lambda \cap \mathrm{cof}(\kappa)$ which is in $I[\lambda]$ and is (modulo clubs) the largest subset of $\lambda \cap \mathrm{cof}(\kappa)$ which lies in $I[\lambda]$. To see this fix an enumeration $\langle a_\alpha : \alpha < \lambda \rangle$ of $[\lambda]^{<\kappa}$ and let S be the set of $\gamma \in \lambda \cap \mathrm{cof}(\kappa)$ which are approachable with respect to this enumeration. It should be clear that S is the maximal element of $I[\lambda]$ restricted to cofinality κ; it remains to see that S is stationary. If we build an IA chain $\langle M_i : i < \kappa \rangle$ of substructures of $(H_\theta, \in, <_\theta, C, \vec{a})$ each with size less than κ, and let $\gamma_i = \sup(M_i \cap \lambda)$ and $\gamma = \sup_i \gamma_i$, then it is routine to check that $\gamma \in C$ and every proper initial segment of $\langle \gamma_i : i < \kappa \rangle$ is enumerated on $\vec{a}$ before stage γ. Abusing language slightly we refer to S as *the set of approachable points of cofinality κ in λ.* The set S is well defined modulo the club filter.

9.2 $I[\lambda]$ is nontrivial   We just saw that under some cardinal arithmetic assumption $I[\lambda]$ is nontrivial. Shelah proved that $I[\lambda]$ is nontrivial for any regular $\lambda > \aleph_1$ without any assumption on cardinal arithmetic. The following theorem states this result more precisely.

Theorem 9.2 (Shelah [17])   *Let κ, θ, and λ be regular with $\kappa < \kappa^+ < \theta < \lambda$. Then there is a set $A \in I[\lambda]$ which consists of cofinality κ points and is stationary; in fact, it satisfies the stronger property that $A \cap \delta$ is stationary for stationarily many $\delta < \lambda$ of cofinality θ.*

Proof Let $\langle C_\zeta : \zeta \in \theta \cap \mathrm{cof}(\kappa) \rangle$ be a club guessing sequence. Fix $\langle M_i : i < \lambda \rangle$ an IA chain such that $\{C_\zeta : \zeta \in \theta \cap \mathrm{cof}(\kappa)\} \subseteq M_0$ and $|M_i| < \lambda$ for all i. Let A be the set of $\gamma < \lambda$ such that $\mathrm{cf}(\gamma) = \kappa$ and there is $c \subseteq \gamma$ a club set of order type κ, all of whose proper initial segments lie in M_γ. If $\langle a_i : i < \lambda \rangle$ enumerates all the bounded subsets of λ lying in $\bigcup_i M_i$ then $\{a_i : i < \gamma\}$ enumerates the bounded subsets lying in M_γ for a club set of $\gamma < \lambda$, so $A \in I[\lambda]$.

Suppose for a contradiction that $A \cap \delta$ is nonstationary in δ for almost all $\delta \in \lambda \cap \mathrm{cof}(\theta)$. We may then build an IA chain $\langle N_j : j < \theta \rangle$ such that $\langle M_i : i < \lambda \rangle \in N_0$, $\{C_\zeta : \zeta \in \theta \cap \mathrm{cof}(\kappa)\} \subseteq N_0$, $|N_j| = \theta$ for all $j < \theta$, and setting $\delta = \sup(\bigcup_j N_j \cap \lambda)$ we have $A \cap \delta$ nonstationary in δ.

Let $\alpha_j = \sup(N_j \cap \lambda)$, so that $\langle \alpha_i : i \leq j \rangle \in N_{j+1}$ for $j < \theta$, and the α_i are increasing, continuous, and cofinal in δ. Choose $\langle \beta_j : j < \theta \rangle \in M_{\delta+1}$ increasing, continuous, and cofinal in δ. Let $e = \{j < \theta : \alpha_j = \beta_j\}$ and note that since e is club in θ and $\langle C_\zeta : \zeta \in \theta \cap \mathrm{cof}(\kappa) \rangle$ is a club guessing sequence, $C_\zeta \cup \{\zeta\} \subseteq e$ for stationarily many $\zeta \in \theta \cap \mathrm{cof}(\kappa)$.

For such a ζ let $c = \{\alpha_j : j \in C_\zeta\} = \{\beta_j : j \in C_\zeta\}$. Every proper initial segment of c can be computed from C_ζ and a proper initial segment of $\{\alpha_j : j < \zeta\}$ and so lies in N_ζ; similarly every proper initial segment of c lies in $M_{\delta+1}$.

If x is a proper initial segment of c, then N_ζ sees that $x \in \bigcup_i M_i$, so $x \in M_i$ for $i \in N_\zeta \cap \lambda$ and hence $x \in M_{\alpha_\zeta}$. So c witnesses that $\alpha_\zeta \in A$. Hence $A \cap \delta$ is stationary. Contradiction! $\qquad\square$

10 Scales, Good Points, and Eubs

Now we begin to consider the interaction of the principles we have discussed with PCF theory. PCF theory is a very large subject and we do not pretend to give anything like a comprehensive account here. We refer the reader to Abraham and Magidor's survey paper [1] and Shelah's book [17] for a detailed treatment.

10.1 Scales and eubs   Here are some of the basic definitions. Given an index set X and an ideal I on X, we can order $^X ON$ by a relation $<_I$ given by

$$ f <_I g \iff \{x : f(x) \geq g(x)\} \in I. $$

We will also define relations $=_I$ and $\leq_I$ in the obvious way. An important point is that $<_I$ is *not* the strict part of the preordering $\leq_I$ unless I happens to be a prime ideal.

A $<_I$-increasing sequence $\langle f_i : i < \alpha \rangle$ has an *exact upper bound (eub)* if there is f such that $f_i <_I f$ for all i, and every $g <_I f$ has $g <_I f_i$ for large enough i. Such a function f may not exist but is unique modulo I if it does exist.

Given a function $f \in {}^X ON$, a *scale of length α in* $\prod_x f(x)/I$ is a $<_I$-increasing sequence $\langle f_i : i < \alpha \rangle$ in $\prod_x f(x)$ which is cofinal in $\prod_x f(x)$ under the relation $<_I$. It is clear that in this case f is an eub for $\langle f_i : i < \alpha \rangle$, and that conversely if f is an eub then we can alter the f_i on I-small sets to produce a scale in $\prod_x f(x)$.

It is easy to see that if $\langle f_i : i < \alpha \rangle$ has f as an eub then f is also an lub for the sequence in the partial ordering $\leq_I$. The converse is false in general. The function f is an lub in the ordering $\leq_I$ if and only if $f_i <_I f$ for all i, and every function which is below f on some I-*positive* set is also below some f_i on an I-*positive* set.

Scales are central objects in PCF theory, and accordingly a central technical problem is to produce increasing sequences which have eubs. The key idea is that if we can build a sequence such that along the way there are many stages where a "simple" eub exists, then at the end we are guaranteed the existence of a (possibly "complex") eub.

We can sometimes simplify questions about eubs as follows: if $\langle f_i : i < \gamma \rangle$ and $\langle g_j : j < \delta \rangle$ are two $<_I$-increasing sequences of limit length, we say they are *cofinally interleaved* if every function in one sequence is dominated modulo I by some function from the other sequence, or equivalently,

$$ \{h : \exists i \ h <_I f_i\} = \{h : \exists j \ h <_I g_j\}. $$

It is easy to see that if two sequences are cofinally interleaved and one has an eub, then the other one does too and the eubs are equal modulo I.

10.2 Good points   Let I be an ideal on the index set X and let $\langle f_i : i < \gamma \rangle$ be a $<_I$-increasing sequence. We say that a limit ordinal $\alpha \leq \gamma$ is a *good (or flat)* point for the sequence if and only if $\mathrm{cf}(\alpha) > |X|$ and there exists an eub h for $\langle f_i : i < \alpha \rangle$ such that $\mathrm{cf}(h(x)) = \mathrm{cf}(\alpha)$ for all x. This is equivalent to asserting that there is

a *pointwise increasing* sequence $\langle h_j : j < \mathrm{cf}(\alpha) \rangle$ cofinally interleaved mod I with $\langle f_i : i < \alpha \rangle$.

Motivation: we show in the next theorem that if a sequence has enough good points, it has an eub. We will then show how to use the nontriviality of $I[\lambda]$ to manufacture such sequences.

Theorem 10.1 (Shelah [17])   *Let $|X| < \kappa < \lambda$ with κ and λ regular. Suppose that $\langle f_i : i < \lambda \rangle$ is a $<_I$-increasing sequence with stationarily many good points of cofinality κ. Then there exists an eub h such that $\mathrm{cf}(h(x)) > \kappa$ for all x.*

Proof   The proof breaks into two phases. We use the assumption that there are stationarily many good points to construct an lub for the sequence in the ordering $\leq_I$, then use it again to argue that the lub is in fact an eub.

Phase I :   We will construct by induction a sequence of functions g_j such that $f_i <_I g_j$ for all i and j, and for $j_1 < j_2$ we have $g_{j_2} \leq_I g_{j_1}$, $g_{j_2} \neq_I g_{j_1}$. We choose g_0 to be any bound, and for all j if g_j fails to be an lub we choose g_{j+1} to witness this failure. We will show that the construction halts for some $j < |X|^+$.

At limit stages μ we let $S_\mu(x) = \{g_j(x) : j < \mu\}$, and define $h^i_\mu(x) = \min(S_\mu(x) \setminus f_i(x))$. We claim that for $\mu < |X|^+$, h^i_μ is eventually constant modulo I. If not we find γ good of cofinality κ such that h^i_μ does not stabilize for large $i < \gamma$, and fix $\langle H_\zeta : \zeta < \kappa \rangle$ pointwise increasing and cofinally interleaved with $\langle f_i : i < \gamma \rangle$. The function $x \mapsto \min(S_\mu(x) \setminus H_\zeta(x))$ cannot stabilize for large $\zeta < \kappa$, but this is impossible because $|S_\mu(x)| \leq |X| < \kappa$. We now choose g_μ so that $g_\mu =_I h^i_\mu$ for all large i.

Now we suppose for a contradiction that the construction runs for $|X|^+$ steps. For each x and each i, the value of $h^i_\mu(x)$ will stabilize for large limit $\mu < |X|^+$ since the smallest value which will ever appear must turn up at some point. So for each $i < \lambda$, the function h^i_μ stabilizes for large limit μ. So there is an unbounded set $B \subseteq \lambda$ and a fixed ν such that for all $i \in B$, h^i_μ is constant for limit $\mu \geq \nu$. If $\nu \leq \mu_1 < \mu_2$ we may choose $i \in B$ so large that $g_{\mu_1} =_I h^i_{\mu_1}$ and $g_{\mu_2} =_I h^i_{\mu_2}$, so that $g_{\mu_1} =_I g_{\mu_2}$, contradicting the choice of the function g_j.

We conclude that the construction halts at some stage before $|X|^+$ with the construction of an lub g.

Phase II :   Suppose for a contradiction that our lub g from Phase I is not an eub. Then we may find $h <_I g$ such that the set $S_i = \{x : f_i(x) \leq h(x)\}$ is I-positive for all i. We claim that this sequence of sets is eventually constant modulo I. If not then we find γ a good point of cofinality κ such that S_i does not stabilize modulo I for large $i < \gamma$, and fix $\langle H_\zeta : \zeta < \kappa \rangle$ pointwise increasing and cofinally interleaved with $\langle f_i : i < \gamma \rangle$. If $D_\zeta = \{x : H_\zeta(x) \leq h(x)\}$ then (by the cofinal interleaving of a pointwise increasing sequence) D_ζ cannot stabilize for large ζ, but this is impossible because D_ζ decreases with ζ and $|X| < \kappa$.

Let S be such that $S_i =_I S$ for all large i, and define g^* so that g^* agrees with h on S and with g on the complement of S. Then by construction g^* is a bound the f_i and g^* is below g on a positive set, which is impossible since g is an lub.

To finish we should check that $\mathrm{cf}(g(x)) > \kappa$ for almost all x. This follows from an argument similar to that we gave in Phase I that h^i_μ stabilizes for large i. $\square$

It is interesting to note that the converse of the last result is also true. To see this let C be a club subset of λ and build an IA chain $\langle M_j : j < \kappa \rangle$ of structures of size less than κ such that if $M = \bigcup_j M_j$, then $\gamma = \sup(M \cap \lambda) \in C$. It is routine to check that the function $\bar{h}$ given by $\bar{h} : x \mapsto \sup(M \cap h(x))$ is an eub for $\langle f_i : i < \gamma \rangle$, with $\mathrm{cf}(\bar{h}(x)) = \kappa$ for all x.

11 Building Scales, Goodness, and Approachability

11.1 Building scales   Now we show how to use the nontriviality of $I[\lambda]$ to construct increasing sequences with eubs. For the sake of simplicity we work in the setting of $\prod_{n < \omega} \aleph_n / \mathit{finite}$, and work toward a proof of the basic fact (Shelah [17]) that there is an infinite $A \subseteq \omega$ with a scale of length $\aleph_{\omega+1}$ in $\prod_{n \in A} \aleph_n / \mathit{finite}$.

It is easy to see that $\prod_{n < \omega} \aleph_n / \mathit{finite}$ is $\aleph_{\omega+1}$-*directed*, that is to say, any $\aleph_\omega$ functions can be dominated mod finite by a single function. We build an IA chain in $(H_\theta, \in, <_\theta)\langle M_i : i < \aleph_{\omega+1} \rangle$, consisting of structures of size $\aleph_\omega$ with $M_i \cap \aleph_{\omega+1} \in \aleph_{\omega+1}$. We define g_α to be the $<_\theta$-least function such that g_α dominates all functions in M_α mod finite. Note that if $\beta < \alpha$ then $M_\beta \in M_\alpha$, so that $g_\beta \in M_\alpha$ and g_β is dominated by g_α.

For each $k < \omega$ it follows from Theorem 9.2 there is a stationary subset of $\aleph_{\omega+1} \cap \mathrm{cof}(\aleph_k)$ in $I[\aleph_{\omega+1}]$. So for stationarily many γ of cofinality $\aleph_k$ we may find $N \subseteq M_\gamma$ such that N is the union of an IA chain $\langle N_i : i < \aleph_k \rangle$ with $|N_i| < \aleph_k$. Also $M_\alpha \in N$ for unboundedly many $\alpha < \gamma$. If we define $h_i : m \mapsto \sup(N_i \cap \aleph_m)$ and $h : m \mapsto \sup(N \cap \aleph_m)$, then easily for $m > k$ the sequence $\langle h_i(m) : i < \aleph_k \rangle$ is increasing with limit $h(m)$. So h is an eub for $\langle h_i : i < \aleph_k \rangle$.

We claim that the sequences $\langle h_i : i < \aleph_k \rangle$ and $\langle g_\alpha : \alpha < \gamma \rangle$ are cofinally interleaved. Once we have seen this, it easily follows that h is an eub for $\langle g_\alpha : \alpha < \gamma \rangle$. On the one hand each h_i is defined from the corresponding N_i, so $h_i \in M_\gamma$, so $h_i \in M_\beta$ for some $\beta < \gamma$ and hence $h_i <^* g_\beta$ by construction. On the other hand, for cofinally many $\alpha < \gamma$ we have $M_\alpha \in N$, so $M_\alpha \in N_i$ for some $i < \aleph_k$, so $g_\alpha \in N_i$, and thus easily $g_\alpha < h_i$.

Appealing to Theorem 10.1, we have produced a sequence $\langle g_\alpha : \alpha < \aleph_{\omega+1} \rangle$ which is increasing mod finite and has an eub g. Moreover, we know that for each $k < \omega$ the set $\{n : \mathrm{cf}(g(n)) = \aleph_k\}$ is finite. Let $A = \{k : \exists n \; \mathrm{cf}(g(n)) = \aleph_k\}$. For each $k \in A$ and each n with $\mathrm{cf}(g(n)) = \aleph_k$ fix $\langle \beta_i^n : i < \aleph_k \rangle$ increasing and cofinal in $g(n)$. If we now define $f_\alpha(k)$ to be the least i such that $g_\alpha(n) < \beta_i^n$ for all n with $\mathrm{cf}(g(n)) = \aleph_k$, then it is routine to check that the sequence of f_αs can be thinned out to give a scale of length $\aleph_{\omega+1}$ in $\prod_{k \in A} \aleph_k / \mathit{finite}$.

Remark 11.1   It is routine to check that if X is the set of good points for the increasing sequence $\langle g_\alpha : \alpha < \aleph_{\omega+1} \rangle$, then the resulting scale is good at almost every point in X.

11.2 Goodness and approachability   From now on we will mostly focus on the special case of the combinatorics of $\aleph_\omega$ and $\aleph_{\omega+1}$. As we just saw, there is an infinite $A \subseteq \omega$ such that $\prod_{n \in A} \aleph_n / \mathit{finite}$ has a scale of length $\aleph_{\omega+1}$. In fact, Shelah showed there is a maximal choice for A (a "PCF generator") which is well defined modulo finite. Let B be this maximal set, let $\langle f_i : i < \aleph_{\omega+1} \rangle$ be a scale in $\prod_{n \in B} \aleph_n / \mathit{finite}$, and define G to be the set of good points for this scale.

The set G seems to depend on the choice of the scale $\langle f_i : i < \aleph_{\omega+1} \rangle$. However, it is easy to see that if $\langle f_i' : i < \aleph_{\omega+1} \rangle$ is another scale then for a club set of α the sequences $\langle f_i : i < \alpha \rangle$ and $\langle f_i' : i < \alpha \rangle$ are cofinally interleaved, so that if we define G' to be the set of good points for $\langle f_i' : i < \aleph_{\omega+1} \rangle$ the sets G and G' are equal modulo clubs. The set G is a kind of "canonical" or "invariant" object.

Suppose for simplicity that $2^{\aleph_\omega} = \aleph_{\omega+1}$. Then as we saw already there is a stationary set $A \subseteq \aleph_{\omega+1}$ which is maximal in $I[\aleph_{\omega+1}]$. This set is well defined modulo the club filter and is another "canonical" or "invariant" object. We can compare this set A with G as follows. For almost all $\gamma \in A$ of uncountable cofinality $\aleph_k$, we may find an IA chain of structures $\langle N_i : i < \aleph_k \rangle$ each of size less than $\aleph_k$ such that if $h_i : n \mapsto \sup(N_i \cap \aleph_n)$ then $\langle h_i : i < \aleph_k \rangle$ is cofinally interleaved with $\langle f_\alpha : \alpha < \gamma \rangle$. So $A \cap \mathrm{cof}(> \omega)$ is contained in G modulo clubs.

11.3 From squares to scales   We have already seen how to use a squarelike principle provable in ZFC (the nontriviality of $I[\lambda]$) to build scales which have stationarily many good points and therefore have an eub. In the rest of these notes we show how to derive nicer scales from stronger forms of square and then how to use these scales in resolving various combinatorial problems involving reflection.

To be a bit more precise we have the three squarelike principles:

$$\square_\mu \longrightarrow \square_\mu^* \longrightarrow \mu^+ \in I[\mu^+].$$

In the following sections we will see how for μ singular we can use each of these principles to produce scales of length μ^+ with additional "goodness" properties, which can then be used to solve certain kinds of combinatorial problems. The general idea is to use the squarelike principles to generate examples of incompactness, using the scales as a tool.

12   Transversals

We will illustrate the power of the PCF-theoretic ideas we have discussed by applying them to the classical *transversal problem*. In this section we collect some basic facts about transversals, and in the next section we show how to apply PCF theory.

A *transversal* of a set of nonempty sets is a 1-1 choice function for that set. We say that $\mathrm{PT}(\kappa, \lambda)$ holds if whenever $|F| = \kappa$, $|X| < \lambda$ for all $X \in F$, and every subset of F with size less than κ has a transversal, then F has a transversal. $\mathrm{NPT}(\kappa, \lambda)$ is the negation of $\mathrm{PT}(\kappa, \lambda)$. $\mathrm{PT}(\kappa, \omega)$ holds for all κ by the compactness theorem for first-order logic.

Note 12.1   Motivated by his work on almost-free groups and the Whitehead problem, Shelah has given a very general axiomatic theory of "free and almost-free structures." The theory of transversals and the properties PT and NPT is a very concrete and approachable special case of this theory.

It is easy to see that $\mathrm{NPT}(\aleph_1, \aleph_1)$ holds. We assign to each limit $\delta < \omega_1$ a cofinal subset S_δ of order type ω. By Fodor's lemma there is no transversal, but easily for any nonstationary (in particular countable) set T the set $\{S_\delta : \delta \in T\}$ has a transversal.

12.1 The Milner-Shelah theorem   Milner and Shelah gave a rather general "stepping up" theorem for NPT. Recall that a stationary subset S of a regular uncountable cardinal κ is *nonreflecting* if and only if $S \cap \alpha$ is nonstationary for every ordinal

$\alpha < \kappa$ of uncountable cofinality. The following fact is key: if $S \subseteq \kappa$ is a nonreflecting stationary set of cofinality μ ordinals and S_δ is cofinal in δ of order type μ for all $\delta \in S$, then for every $\gamma < \kappa$ we can find $\langle \eta_\delta : \delta \in S \cap \gamma \rangle$ such that the sets $S_\delta \setminus \eta_\delta$ are disjoint.

The proof is by induction on γ: for limit γ we choose C club in γ with $C \cap S = \varnothing$. If α and β are successive points in C then we use induction to choose η_δ so that $S_\delta \setminus \eta_\delta$ are disjoint for $\delta \in S \cap (\alpha, \beta)$. Arranging that $\eta_\delta > \alpha$ for all such δ we avoid overlap between different intervals, so we can "glue together" to choose η_δ for all $\delta \in S \cap \gamma$. If $\gamma \in S$ we can also choose $\eta_\delta > \sup(S_\delta \cap S_\gamma)$ in order to proceed past γ.

Note 12.2 By the same argument, if S is a set such that no initial segment of S (including S itself) is stationary then for any $\langle S_\delta : \delta \in S \rangle$ with S_δ cofinal in δ we can choose disjoint tails as above. In particular, this is true for S any nonstationary subset of $\aleph_1$.

We are now ready for the Milner-Shelah theorem.

Theorem 12.3 (Milner and Shelah [13]) *If* $\text{NPT}(\mu, \lambda)$ *holds and there is a nonreflecting stationary set of cofinality μ ordinals in κ then* $\text{NPT}(\kappa, \lambda)$ *holds.*

Proof Let A_σ for $\sigma < \mu$ be an example for $\text{NPT}(\mu, \lambda)$. Let S be a nonreflecting stationary set as above and fix for each $\delta \in S$ a cofinal μ-sequence $\langle \rho(\delta, \sigma) : \sigma < \mu \rangle$. Define $B(\delta, \sigma) = (\{\delta\} \times A_\sigma) \cup \{\rho(\delta, \sigma)\}$.

Claim 1: There is no transversal. Suppose that $T(\delta, \sigma) \in B(\delta, \sigma)$ and T is 1-1. For every $\delta \in S$ there is σ such that $T(\delta, \sigma) = \rho(\delta, \sigma)$, because $\{A_\sigma\}$ has no transversal. A contradiction is immediate by Fodor's lemma.

Claim 2: Every subset of size less than κ has a transversal. It is enough to show this for

$$\{B(\delta, \sigma) : \sigma < \mu, \delta \in S \cap \gamma\}$$

where $\gamma < \kappa$. Choose $\eta_\delta < \mu$ so that the sequences $\langle \rho(\delta, \sigma) : \eta_\delta < \sigma < \mu \rangle$ are disjoint for $\delta \in S \cap \gamma$. Choose $T(\delta, \sigma)$ so that $T(\delta, \sigma) = \rho(\delta, \sigma)$ for $\eta_\delta < \sigma < \mu$, and $\langle T(\delta, \sigma) : \sigma \leq \eta_\delta \rangle$ forms a transversal of $\{\delta\} \times \{A_\sigma : \sigma \leq \eta_\delta\}$. $\qquad \square$

For every regular κ, $\kappa^+ \cap \text{cof}(\kappa)$ is a nonreflecting stationary subset of κ. So by an easy induction $\text{NPT}(\aleph_n, \aleph_1)$ for all n such that $1 \leq n < \omega$. At this point things become more complex. For example, it is consistent that every stationary subset of $\aleph_{\omega+1}$ reflects.

12.2 Singular compactness In this section we digress to sketch a proof of Shelah's singular compactness theorem, in the special case of the transversal problem for families of countable sets.

Theorem 12.4 (Shelah [15]) *For every singular cardinal λ,* $\text{PT}(\lambda, \aleph_1)$ *holds.*

Proof Let λ be a singular cardinal, with $\mu = \text{cf}(\lambda) < \lambda$. We fix $\langle \lambda_i : i < \mu \rangle$ an increasing sequence of regular cardinals which is cofinal in λ, with $\mu < \lambda_0$. Let F be a family of countable sets such that every subfamily of size less than λ has a transversal; we show how to produce a transversal for F.

Some motivation for the proof: We will imagine that there are μ many workers, where worker i is responsible for building a transversal of a subset of F with size λ_i; the subsets of F claimed by different workers may overlap, but every element

of F will be claimed by at least one worker. We aim to put together the various workers' transversals to get a transversal for F. There are two main problems: two workers may both claim the same element A of F and their transversals may assign distinct elements of A to it, or dually two workers' transversals may assign the same element to distinct elements of F. In each case we will resolve the issue using an idea familiar from recursion theory, namely, if $i < j$ then worker i will win out over worker j; in the jargon of recursion theory "worker i has higher priority than worker j".

Worker i will build up her partial transversal in ω steps. In general when $A \subseteq B$ there is no guarantee that a transversal of A can be extended to a transversal of B; we will leave until the end of the proof the argument that it is possible for worker i to extend transversals as required.

More precisely we will construct for each $i < \mu$ an increasing ω-sequence of sets $\langle F_n^i : n < \omega \rangle$, together with functions f_n^i, such that

(i) $F_n^i \subseteq F$, $|F_n^i| = \lambda_i$, $F = \bigcup_{i < \mu} F_0^i$;
(ii) f_n^i is a transversal of F_n^i, $f_n^i \subseteq f_{n+1}^i$;
(iii) if $i < j < \mu$, $A \in F_n^i$, $B \in F_n^j$ with $A \neq B$, and $f_n^i(A) = f_n^j(B)$ then $B \in F_{n+1}^i$.

For each $A \in F_n^i$ and $j > i$ there is at most one $B \in F_n^j$ with $f_n^i(A) = f_n^j(B)$, since f_n^j is 1-1 on F_n^j; so we are required to add at most $\lambda_i \times \mu = \lambda_i$ many elements to F_n^i. In terms of our story about the workers, worker i is attempting to assume responsibility for all elements $B \in F$ to which some worker of lower priority has assigned an element which worker i had reserved for an element A in her domain.

Let $F_\infty^i = \bigcup_n F_n^i$, and $f_\infty^i = \bigcup_n f_n^i$. Clearly f_∞^i is a transversal of F_∞^i. Since $F_0^i \subseteq F_\infty^i$, $\bigcup_i F_\infty^i = F$. For each $A \in F$ let $i(A)$ be least such that $A \in F_\infty^{i(A)}$ and define $h(A) = f_\infty^{i(A)}(A)$. In terms of the story about the workers, this means that the highest-priority worker among those who have claimed responsibility for A is allowed to determine the value of the function h at A.

We claim that h is a transversal of F. To see this let A, B be distinct elements of F. If $i(A) = i(B) = i$ then $h(A) = f_\infty^i(A) \neq f_\infty^i(B) = h(B)$ because f_∞^i is a transversal of F_∞^i. Suppose for a contradiction that $i(A) = i < i(B) = j$, and $h(A) = h(B)$. Find an n so large that $A \in F_n^i$ and $B \in F_n^j$. Then by definition $f_n^i(A) = f_\infty^i(A) = h(A) = h(B) = f_\infty^j(B) = f_n^j(B)$. So by construction $B \in F_{n+1}^i \subseteq F_\infty^i$ and thus $i(B) \leq i$, contradiction! So h is a transversal as required.

To finish the proof we need to show that we can do the construction in such a way that worker i can always extend f_n^i to a transversal of F_{n+1}^i. We use the following key idea: if $A \subseteq B$ we say that B is free over A if there is a transversal of $B - A$ which does not use any elements of $\bigcup A$. If B is free over A then trivially any transversal of A can be extended to a transversal of B.

Consider the following game $G(\kappa)$ where $\kappa < \lambda$ is a cardinal. Two players I and II collaborate to build a chain of sets $E_0 \subseteq F_0 \subseteq E_1 \cdots$ with E_i, F_i subsets of F of cardinality κ. At round n player I plays E_n and II responds with F_n. II wins if and only F_{n+1} is free over F_n for all n.

We claim that II has a winning strategy for κ for every $\kappa < \lambda$. Once we establish this we are done with the proof of the theorem, because in the main construction

worker i can build the chain of sets $F_0^i \subseteq F_1^i \cdots$ using a strategy for II in $G(\lambda_i)$, so that F_{n+1}^i is free over F_n^i and there is no problem in extending f_n^i to get f_{n+1}^i.

The game $G(\kappa)$ is open for player I, so by the Gale-Stewart theorem it is determined. Suppose for a contradiction that player I has a winning strategy σ and build an increasing and continuous chain $\langle F_\alpha : \alpha < \kappa^+ \rangle$ of subsets of F with the following properties:

(i) $|F_\alpha| = \kappa$;
(ii) for any $\alpha_0 < \alpha_1 < \cdots \alpha_{n-1} < \kappa^+$, if E is the move by player I at round n which σ dictates when player II plays F_{α_i} at round i for every $i < n$ then $E \subseteq F_\alpha$ for some $\alpha < \kappa^+$.

Since $|\bigcup_{\alpha<\kappa^+} F_\alpha| = \kappa^+ < \lambda$, there is some transversal h of $\bigcup_{\alpha<\kappa^+} F_\alpha$. We fix $C \subseteq \kappa^+$ a club set such that if $\alpha \in C$ then

(i) for any $\alpha_0 < \alpha_1 < \cdots \alpha_{n-1} < \alpha$, if E is the move by player I at round n which σ dictates when player II plays F_{α_i} at round i for every $i < n$ then $E \subseteq F_\alpha$;
(ii) for all $x \in \bigcup F_\alpha$, if $x \in \mathrm{rge}(h)$ then $x = h(A)$ for some $A \in F_\alpha$.

Let α_i for $i < \omega$ be the first ω elements of C and consider a run of the game $E_0 \subseteq F_{\alpha_0} \subseteq E_1 \subseteq F_{\alpha_1} \cdots$ in which player I plays according to σ. It is easy to see that $F_{\alpha_{n+1}}$ is witnessed by h to be free over F_{α_n}, so that player II wins this run contradicting the assumption that σ is winning for player I. $\square$

13 Good Scales and Transversals

13.1 Good scales   Let $\langle f_i : i < \aleph_{\omega+1} \rangle$ be a scale in $\prod_{n \in A} \aleph_n / finite$ for some $A \subseteq \omega$ (A is not necessarily the maximal set B discussed above). It is helpful to characterize the good points of this scale in an alternative way. The following conditions are equivalent for γ in $\aleph_{\omega+1} \cap \mathrm{cof}(> \omega)$:

1. γ is good;
2. there is $Y \subseteq \gamma$ unbounded and $m < \omega$ such that $\langle f_\alpha(n) : \alpha \in Y \rangle$ is strictly increasing for all $n > m$;
3. for every unbounded $Z \subseteq \gamma$ there exist $Y \subseteq Z$ and m as in (2).

Clearly (3) implies (2). If (2) holds and we form the pointwise supremum of the f_α for $\alpha \in Y$ we get an eub which has cofinality $\mathrm{cf}(\gamma)$ past m, so (2) implies (1). To see (1) implies (3) we fix $\langle h_i : i < \mathrm{cf}(\gamma) \rangle$ pointwise increasing and cofinally interleaved with $\langle f_j : j < \gamma \rangle$. Find $Z_0 \subseteq Y$ and α_j, β_j for $j \in Z_0$ such that $h_{\alpha_j} <^* g_j <^* h_{\beta_j}$. Thinning out we get $Z \subseteq Z_0$ such that $h_{\alpha_j}(n) < g_j(n) < h_{\beta_j}(n)$ for n past some fixed m and $j \in Z$ and also $\beta_j < \alpha_k$ when $j, k \in Z$ with $j < k$. We are done since the h_i are pointwise increasing.

We say that the scale $\langle f_i : i < \aleph_{\omega+1} \rangle$ is *good* if it is good at every point in $\aleph_{\omega+1} \cap \mathrm{cof}(> \omega)$. The existence of a good scale is a rather weak consequence of square, but is still useful as a construction principle.

We claim that if $\aleph_{\omega+1} \in I[\aleph_{\omega+1}]$ there is a good scale. The construction of Section 11.1 gives a scale $\langle f_\alpha : \alpha < \aleph_{\omega+1} \rangle$ which is good at almost every point. Fix a club set C such that every uncountable limit point of C is good, enumerate C as $\langle \alpha_i : i < \aleph_{\omega+1} \rangle$, and consider a new scale $\langle g_i : i < \aleph_{\omega+1} \rangle$ given by $g_i = f_{\alpha_i}$. If i has uncountable cofinality then α_i is good and we may fix an eub h for $\langle f_\alpha : \alpha < \alpha_i \rangle$ such

that $cf(h(n)) = cf(i)$ for all n; the sequence $\langle g_j : j < i \rangle$ is cofinal in $\langle f_\alpha : \alpha < \alpha_i \rangle$ so h is also an eub for $\langle g_j : j < i \rangle$, and thus i is a good point for $\langle g_j : j < \aleph_{\omega+1} \rangle$.

We illustrate the utility of good scales with an extended example.

13.2 Transversals   We show how ideas from PCF theory can be relevant to the transversal problem by proving the following result of Magidor and Shelah.

Theorem 13.1 (Magidor and Shelah [12])   *If* $cf(\kappa) = \omega$ *and there exist a sequence* $\langle \kappa_i : i < \kappa \rangle$ *cofinal in* κ *and a good scale* $\langle f_\alpha : \alpha < \kappa^+ \rangle$ *in the reduced product* $\prod_i \kappa_i / finite$, *then* $NPT(\kappa^+, \aleph_1)$.

Proof   Start by noting the trivial fact that $A \times B$ is disjoint from $C \times D$ if and only if either A is disjoint from C or B is disjoint from D. Let $S = \kappa^+ \cap cof(\omega_1)$ and for each $\alpha \in S$ let $A_\alpha = \{(m, f_\alpha(m)) : m < \omega\}$. Clearly A_α meets A_β if and only if f_α and f_β agree at some n. Intuitively, the goodness of the scale implies that the A_α are quite "well spread out," which is helpful in building transversals.

The main point is to establish the following slightly technical result.

Claim 13.2 (Main Claim)   *For every* $\gamma < \kappa^+$ *there exist*

$$\langle B_\alpha, D_\alpha : \alpha \in S \cap \gamma \rangle$$

such that B_α *is a cofinite subset of* A_α, D_α *is club in* α, *and the sets* $\{B_\alpha \times D_\alpha : \alpha \in S \cap \gamma\}$ *are pairwise disjoint.*

The idea here is that while we are not in a position to choose disjoint tails of the sets A_α, we may use the extra "slack" provided by the club sets D_α to get a satisfactory substitute.

Suppose for the moment that we have established the Claim 13.2. We choose $E_\alpha \subseteq \alpha$ club of order type ω_1 for each $\alpha \in S$. It is routine to check that $\langle A_\alpha \times E_\alpha : \alpha \in S \rangle$ witnesses $NPT(\kappa^+, \aleph_2)$. Before proving the theorem, we show how to improve this to $NPT(\kappa,^+, \aleph_1)$ by an argument reminiscent of the Milner-Shelah theorem (Theorem 12.3).

Fix for each limit $\gamma < \aleph_1$ a set S_γ cofinal in γ with order type ω. Enumerate E_α as $\langle e_{\alpha,\gamma} : \gamma < \omega_1 \rangle$. Define

$$B_{\alpha,\gamma} = (S_\gamma \times \{\alpha\}) \cup (A_\alpha \times \{e_{\alpha,\gamma}\}).$$

Claim 1:   There is no transversal. Suppose for a contradiction that F is one. Then for all α there is γ such that F chooses from the second component, but this is impossible by Fodor's theorem.

Claim 2:   For $\eta < \kappa^+$, $\{B_{\alpha,\gamma} : \alpha < \eta, \gamma < \aleph_1\}$ has a transversal. To see this use Claim 13.2 to choose $B_\alpha \subseteq \alpha$ and $D_\alpha \subseteq E_\alpha$ so that the sets $B_\alpha \times D_\alpha$ are disjoint. If $e_{\alpha,\gamma} \in D_\alpha$ then we associate to (α, γ) a point in $B_\alpha \times \{e_{\alpha,\gamma}\}$. The set of γ such that $e_{\alpha,\gamma} \notin D_\alpha$ is nonstationary, so the corresponding set of S_γ has a transversal and we can associate to (α, γ) a point of $S_\gamma \times \{\alpha\}$.

To prove Claim 13.2 we show by induction on γ that for all $\delta < \gamma$ we may choose suitable D_α and B_α for $\alpha \in S \cap (\delta, \gamma]$, with the additional property that $D_\alpha \cap (\delta + 1) = \varnothing$.

If γ is a successor ordinal there is nothing to do. If γ is limit of cofinality ω or ω_1 then we may choose C club in γ and disjoint from S, use C to break up $S \cap \gamma$ into intervals, appeal to induction on each interval and combine the results. This gives a suitable choice of D_α and B_α for $S \cap (\delta, \gamma)$; if $cf(\gamma) = \aleph_1$ we may simply choose D_γ

to be some club set in γ, and then throw away initial segments of D_α for $\alpha \in S \cap \gamma$ to arrange that $D_\alpha \cap D_\gamma = \varnothing$.

Now suppose that $\mathrm{cf}(\gamma) > \omega_1$ and appeal to the goodness of the scale to find $A \subseteq \gamma$ unbounded and $n < \omega$ such that $\langle f_\alpha(m) : \alpha \in A \rangle$ is strictly increasing for all $m > n$. Let C be the set of α with $\sup(C \cap \alpha) = \alpha$, and note that γ can be decomposed into limit points of C and points which lie in some interval $(\delta, \gamma]$ where δ and γ are successive points in C.

We appeal to the induction hypothesis to choose suitable D_α and B_α for α in each such interval $(\delta, \gamma]$, making sure that $D_\alpha \cap (\delta + 1) = \varnothing$. Clearly there is no problem between αs from different intervals.

For $\alpha \in S$ which is also a limit point of C we choose $D_\alpha = C \cap \alpha$. Note that automatically $D_\alpha \cap D_\beta = \varnothing$ for $\beta \neq \alpha$ unless β is also a limit of C; the point is that for β not limit in C we have arranged that D_β starts past the largest point of C below.

The key point is the choice of B_α for $\alpha \in S$ which is limit in C. Let $\eta(\alpha)$ be the least point of A above α. Appealing to the choice of A and n and the fact that we have a scale, we will choose $n^*(\alpha) > n$ such that $f_\beta(m) < f_\alpha(m) < f_\eta(m)$ for all $\beta \in A \cap \alpha$ and all $m > n^*(\alpha)$. We then let $B_\alpha = \{(m, f_\alpha(m)) : n^*(\alpha) < m < \omega\}$.

To finish let $\beta < \alpha$ where β is also a limit point of C lying in S. For $m > \max\{n^*(\alpha), n^*(\beta)\}$ we have

$$f_\beta(m) < f_{\eta(\beta)}(m) < f_\alpha(m)$$

so that B_α and B_β are disjoint. $\qquad\square$

Actually NPT($\aleph_{\omega+1}, \aleph_1$) can be proved without recourse to a good scale. This requires a heavier dose of PCF than the rest of these notes, so we defer the proof until Section 19.

However, Magidor and Shelah [12] also showed that NPT($\aleph_{\omega^2+1}, \aleph_1$) is independent of ZFC. What is the difference? Essentially this: in a scale of length $\aleph_{\omega^2+1}$ it is possible that for unboundedly many n, there are many points of cofinality $\aleph_{\omega \cdot n + 1}$ where an eub of nonconstant cofinality exists.

14 Very Good Scales

We now consider some stronger properties of scales which follow from stronger versions of square. By analogy with the version of goodness we just discussed, let us say that a scale $\langle f_\alpha : \alpha < \aleph_{\omega+1} \rangle$ is *very good* if for every limit $\alpha < \aleph_{\omega+1}$ of uncountable cofinality there is $C \subseteq \alpha$ club in α such that $\langle f_\alpha(m) : \alpha \in C \rangle$ is strictly increasing for all large m.

It is fairly routine to construct such a scale given a square sequence. Let $\langle C_\alpha : \alpha < \aleph_{\omega+1} \rangle$ be a square and let $\langle g_\alpha : \alpha < \aleph_{\omega+1} \rangle$ be an arbitrary scale. We may assume that $\mathrm{ot}(C_\alpha) < \aleph_\omega$ for all α. Now we construct f_α so that it dominates g_α pointwise, and arrange that for α limit $f_\alpha(m) > f_\beta(m)$ for all $\beta \in \lim(C_\alpha)$ and all m with $\mathrm{ot}(C_\alpha) < \aleph_m$.

Remark 14.1 A very similar construction works from the assumption that $\square_{\aleph_\omega, \aleph_n}$ holds for some $n < \omega$.

Very good scales encapsulate some of the power of square sequences to construct noncompact objects. To illustrate this we relate very good scales to stationary reflection. We start by noting that typically in models where stationary reflection holds,

it holds in a form which asserts that several stationary sets reflect simultaneously: for example, when a weakly compact cardinal is collapsed to become $\aleph_2$, any $\aleph_1$ stationary subsets of $\aleph_2 \cap \text{cof}(\omega)$ reflect simultaneously to a point of cofinality $\aleph_1$. Magidor's model in which every stationary subset of $\aleph_{\omega+1}$ reflects exhibits a similar phenomenon.

Remark 14.2   The point is that the reflection phenomena in these models arise from a single "generic elementary embedding."

In our joint work with Foreman and Magidor [5] we showed that very good scales impose a barrier to simultaneous reflection. Let $\langle f_\alpha : \alpha < \aleph_{\omega+1} \rangle$ be a very good scale and let $S \subseteq \aleph_{\omega+1}$ be stationary. For each n let $T_n \subseteq S$ be a stationary set such that there is $\beta_n < \aleph_n$ with $f_\alpha(n) = \beta_n$ for all $\alpha \in T_n$. Suppose for a contradiction that all the T_n reflect simultaneously at some γ. Since the scale is very good there is $C \subseteq \gamma$ and m such that $\langle f_\alpha(m) : \alpha \in C \rangle$ is strictly increasing, and so C can meet T_m at most once.

   We actually showed that no infinite subfamily of the T_n can reflect simultaneously.

15 Better Scales

From weak square we cannot do quite as well, but can still derive a useful scale principle.

Definition 15.1   Aiming for a concept between "good" and "very good" we define a scale $\langle f_\alpha : \alpha < \aleph_{\omega+1} \rangle$ as *better* if for every limit $\alpha < \aleph_{\omega+1}$ there is a club subset $C \subseteq \alpha$, such that for every $\beta \in C$ there is $m < \omega$ such that $f_\gamma(n) < f_\beta(n)$ for all $\gamma \in C \cap \beta$ and $n > m$.

Note 15.2   Such a club set C always exists when $\text{cf}(\alpha) = \omega$, so that a very good scale is better. Also when $\text{cf}(\alpha) > \omega$ there is a single m which works for unboundedly many $\beta \in C$, so that a better scale is good.

To construct a better scale, we start with a weak square sequence $\langle \mathcal{C}_\alpha : \alpha < \aleph_{\omega+1} \rangle$ such that $\text{ot}(C) < \aleph_\omega$ for every club in $\mathcal{C}_\alpha$. At stage α we form f_C for each $C \in \mathcal{C}_\alpha$ by defining $f_C(m) = \sup\{f_\beta(m) : \beta \in C\}$ for $\aleph_m > \text{ot}(C)$, and then choose f_α so that $f_\alpha >^* f_C$ for all $C \in \mathcal{C}_\alpha$.

   It is interesting to note that if $\aleph_\omega$ is strong limit and $2^{\aleph_\omega} > \aleph_{\omega+1}$ there is a better scale. This is proved in Section 19.

15.1 From a better scale to an ADS sequence   As an example of the use of better scales, we construct a special kind of almost disjoint family and use it to show that stationary reflection fails for stationary sets of countable sets of ordinals.

Definition 15.3   Let κ be a cardinal. A sequence $\langle A_\alpha : \alpha < \kappa^+ \rangle$ is an *ADS-sequence* if each A_α is an unbounded subset of κ, and for every $\beta < \kappa^+$ it is possible to find a function $F : \beta \to \kappa$ such that the sets $\langle A_\alpha \setminus F(\alpha) : \alpha < \beta \rangle$ are pairwise disjoint.

Such sequences always exist for κ regular, but may or may not for κ singular. They were introduced by Shelah, who showed that if there is an ADS-sequence then $\text{cf}(\kappa) = \text{cf}(|\kappa|)$ in any extension of the universe in which κ^+ remains a cardinal.

Theorem 15.4 (Cummings, Foreman, and Magidor [5])   *If* $\langle f_\alpha : \alpha < \aleph_{\omega+1} \rangle$ *is a better scale, there is an* ADS-*sequence for* $\aleph_\omega$.

Proof This follows immediately from the assertion that for every $\beta < \aleph_{\omega+1}$ there is a function $F : \beta \rightarrow \omega$ such that for $\gamma < \delta < \beta$, $f_\gamma(n) < f_\delta(n)$ for all $n > \max\{F(\gamma), F(\delta)\}$. We will establish this by induction on β.

The successor case is easy, so assume that β is limit and fix $C \subseteq \beta$ as in the definition of better scale. Let us enumerate C in increasing order as $\{\beta_i : i < \gamma\}$ and suppose that $f_{\beta_i}(n) > f_{\beta_j}(n)$ for $j < i$ and $n > n_i$. Choose by induction functions F_i from $[\beta_i, \beta_{i+1})$ to ω which work for the functions with indices in this interval.

Increasing the values of the F_i if necessary we may assume that for all γ in $[\beta_i, \beta_{i+1})$ the following properties hold: $F_i(\gamma) \geq n_i$, and also $f_{\beta_i}(n) \leq f_\gamma(n) < f_{\beta_{i+1}}(n)$ for $n > F_i(\gamma)$. We claim taking the union of the F_i gives us a function which works for β. This is clear when γ and δ are in the same half-open interval, so suppose that $\gamma < \beta_{i+1} \leq \beta_j \leq \delta$. If $n > \max\{F(\gamma), F(\delta)\}$ then by construction $F_\gamma(n) < F_{\beta_{i+1}}(n) \leq F_{\beta_j}(n) \leq F_\delta(n)$, so we are done. $\square$

15.2 ADS and reflection Returning to our theme of comparing scales and squares with reflection, we showed in our joint work with Foreman and Magidor [5] that ADS-sequences are inconsistent with a standard stationary reflection hypothesis. To be more precise consider the reflection principle which states that for every stationary $S \subseteq [\kappa]^{\aleph_0}$ there is $X \in [\kappa^+]^{\aleph_1}$ such that $\aleph_1 \subseteq X$, $\mathrm{cf}(X \cap \kappa^+) = \aleph_1$ and $S \cap [X]^{\aleph_0}$ is stationary. This holds if a supercompact cardinal is collapsed to $\aleph_2$ or under the assumption of MM.

Suppose for a contradiction that this reflection principle holds and $\langle A_\alpha : \alpha < \kappa^+ \rangle$ is an ADS-sequence for κ some cardinal of cofinality ω; without loss of generality $\mathrm{ot}(A_\alpha) = \omega$. Consider the stationary set of countable $x \subseteq \kappa^+$ such that $A_{\sup(x)} \subseteq x$, and suppose that the stationarity of S reflects to X as above.

Let $F : \sup(X) \rightarrow \kappa$ be such that the $A_i \setminus F(i)$ are disjoint, and use Fodor's theorem to find $T \subseteq S \cap [X]^{\aleph_0}$ stationary and $a \in X$ such that $\min(A_{\sup(x)} \setminus F(\sup(x))) = a$ for all $x \in T$. This is impossible because since $\mathrm{cf}(X \cap \kappa^+) = \aleph_1$ we may find x and y in T with distinct suprema.

Remark 15.5 It follows from some of our joint work with Foreman and Magidor [5] that the various results we have shown comparing square principles with reflection are sharp. To be more precise from large enough cardinals it is consistent that

(i) $\aleph_{\omega+1} \in I[\aleph_{\omega+1}]$ and every stationary subset of $[\aleph_{\omega+1}]^{\aleph_0}$ reflects to some $X \in [\aleph_{\omega+1}]^{\aleph_1}$ with $\mathrm{cf}(X) = \aleph_1$;

(ii) $\square^*_{\aleph_\omega}$ holds and for every m and n with $m < n < \omega$, any $\aleph_n$ many stationary subsets of $\aleph_{\omega+1} \cap \mathrm{cof}(\aleph_m)$ reflect simultaneously at some point of cofinality $\aleph_n$;

(iii) $\square_{\aleph_\omega, \omega}$ holds and any finite family of stationary subsets of $\aleph_{\omega+1}$ reflects simultaneously.

16 PFA Versus Square

Todorčević [19] showed that PFA is incompatible with square and Magidor observed that by the same argument PFA is incompatible with some weaker squarelike principles.

Theorem 16.1 (Magidor [11]) *Let* PFA *hold. Then* $\square_{\kappa, \aleph_1}$ *fails for every uncountable* κ.

Proof Suppose that $\langle \mathcal{C}_\alpha : \alpha < \kappa^+ \rangle$ is a $\square_{\kappa,\aleph_1}$-sequence. If D is club in κ^+ and $D \cap \alpha \in \mathcal{C}_\alpha$ for all $\alpha \in \lim(D)$ we will say that D *threads* the square sequence. Of course, no such club set D can exist in V, but such sets may exist in extensions of V.

Let $\mathbb{P}$ be the poset whose conditions are countable closed bounded subsets of κ^+. This is equivalent to a countably closed Lévy collapse making $|\kappa| = \aleph_1$. In the extension by $\mathbb{P}$ there is no club set D which threads the square sequence. To see this suppose that $\dot{D}$ names a suitable club set. Since $\dot{D}$ is forced not to lie in V we may build a downward branching tree of conditions in $\mathbb{P}$, $\{p_s : s \in {}^{<\omega}2\}$ and an increasing sequence of ordinals $\langle \alpha_n : n < \omega \rangle$ such that $\max(p_s) = \alpha_{\mathrm{lh}(s)}$, p_t forces $\dot{D} \cap [\alpha_n, \alpha_{n+1}) \neq \varnothing$ for $t \in {}^{n+1}2$, and for $t_1 \neq t_2$ in ${}^n 2$ the conditions p_{t_1} and p_{t_2} force inconsistent information about $\dot{D}$.

Now choose for each $x \in {}^\omega 2$ a lower bound p_x for $\langle p_{x \restriction i} : i < \omega \rangle$. Each p_x forces that $\alpha \in \lim(\dot{D})$, where $\alpha = \sup_n \alpha_n$. So each p_x forces that $\dot{D} \cap \alpha \in \mathcal{C}_\alpha$, and extending p_x if necessary we may assume that p_x forces $\dot{D} \cap \alpha = D_x$ for some $D_x \in \mathcal{C}_\alpha$. The D_x must all be distinct, but by PFA $2^{\aleph_0} = \aleph_2$ while $|\mathcal{C}_\alpha| \leq \aleph_1$.

Let C be the club set in κ^+ introduced by $\mathbb{P}$, say $C = \{\gamma_i : i < \aleph_1\}$. Working in the extension by $\mathbb{P}$ we define a tree T of cardinality $\aleph_1$. The elements are pairs (γ_j, E) where $E \in \mathcal{C}_{\gamma_j}$ and $(\gamma_j, E) <_T (\gamma_k, F)$ if and only if $\gamma_j \in \lim(F)$ and $F \cap \gamma_j = E$. We have just seen that T has no branch of length $\aleph_1$.

Let $\mathbb{Q}$ be Baumgartner's forcing [3] to specialize T using finite partial specializing functions. $\mathbb{Q}$ is c.c.c. and so $\mathbb{P} * \mathbb{Q}$ is proper. Meeting a suitable collection of $\aleph_1$ many dense sets we may produce in V a continuous increasing sequence $\langle \gamma_i : i < \aleph_1 \rangle$ and a specializing function N from $\{(\gamma_i, E) : i < \omega_1, E \in \mathcal{C}_{\gamma_i}\}$ to ω, that is, a function N such that $(\gamma_j, E) <_T (\gamma_k, F) \Rightarrow N(\gamma_j, E) \neq N(\gamma_k, F)$.

If we now let $\gamma = \sup_i \gamma_i$ and choose $C \in \mathcal{C}_\gamma$, then $\gamma_i \in \lim(C)$ and hence $C \cap \gamma_i \in \mathcal{C}_{\gamma_i}$ for a club set A of $i < \aleph_1$. The sequence $\langle (\gamma_i, C \cap \gamma_i) : i \in A \rangle$ is increasing in the $<_T$-ordering, but this is impossible since N is 1-1 on this sequence. $\square$

Magidor [11] showed that PFA is consistent with the assertion that $\square_{\kappa,\aleph_2}$ holds for all uncountable cardinals κ. As we see in the next section, this is in very sharp contrast to the situation for MM.

17 MM Versus Good Scales

In contrast with PFA, Magidor showed that MM is inconsistent with even an extremely weak form of square principle.

Theorem 17.1 (Magidor [11]) *If* MM *holds, there is no good scale (in fact no scale which is good at every point of cofinality $\aleph_1$).*

Proof We consider a forcing poset $\mathbb{P}$ which resembles Namba forcing and adds a new function in $\prod_n \aleph_n$. A condition is a tree T such that each $t \in T$ is a finite sequence with $t(n) \in \aleph_{n+2} \cap \mathrm{cof}(\omega)$ for $n < \mathrm{lh}(t)$. A condition is required to have a *stem* s such that every $t \in T$ is comparable with s, and if t extends s then $\{\alpha : t \frown \alpha \in T\}$ is stationary in $\aleph_{\mathrm{lh}(t)+2}$. The ordering on $\mathbb{P}$ is inclusion.

If T_1 and T_2 are conditions we say that T_1 is a *direct extension* of T_2 and write $T_1 \leq^* T_2$ if $T_1 \leq T_2$ and the conditions have the same stem. A key fact is that if $S \leq T$ then $S \leq^* T_s$, where s is the stem of S and T_s is the subtree of T consisting of elements comparable with s.

Claim 1: If $\dot{\tau}$ is a name for an element of $\aleph_1$ and $S \in \mathbb{P}$, there is $T \leq^* S$ which decides $\dot{\tau}$.

Proof Let s be the stem of S, $\mathrm{lh}(s) = n$. If the claim fails, then for stationarily many $\alpha \in \aleph_{n+2}$ we will have $s \frown \alpha \in S$ and no direct extension of $S_{s\frown\alpha}$ decides $\dot{\tau}$. Otherwise by $\aleph_2$-completeness we may build a direct extension of S deciding $\dot{\tau}$. Repeating this argument, we may work up the tree and build $U \leq^* S$ such that for every $t \in U$, there is no direct extension of U_t deciding $\dot{\tau}$; but this is impossible because some extension of U decides $\dot{\tau}$.

Similarly if S has a stem of length n and $\dot{\tau}$ is a name for an element of $\aleph_{n+1}$ there is a direct extension of S deciding $\dot{\tau}$. $\qquad\square$

Let $f \in \prod_n \aleph_{n+2}$ be the generic function added by $\mathbb{P}$.

Claim 2: If $S \Vdash \dot{g} < \dot{f}$ then there exist $T \leq^* S$ and $h \in V$ such that $T \Vdash \dot{g} < \check{h}$.

Proof For simplicity assume that S has empty stem. For each α such that $\langle \alpha \rangle \in S$, we may find a direct extension of $S_{\langle\alpha\rangle}$ deciding $g(0)$. Appealing to Fodor's lemma we may build a direct extension of S deciding $g(0)$. Now working up S level by level we build $T \leq^* S$ such that for every $t \in T$, T_t decides $g(\mathrm{lh}(t))$. Easily T bounds g. $\qquad\square$

To apply MM we need to see that $\mathbb{P}$ is stationary preserving. For this we use a game-theoretic argument.

Claim 3: $\mathbb{P}$ is stationary preserving.

Proof Fix A a stationary subset of ω_1, $\dot{C}$ a name for a club subset, and S a condition in $\mathbb{P}$. We show how to find $U \leq S$ and $\delta \in A$ with $U \Vdash \delta \in \dot{C}$. For simplicity we assume that S is the trivial condition.

We assign to each $\langle \alpha \rangle \in S$ an ordinal $\gamma_{\langle\alpha\rangle}$ in such a way that for every $i < \omega_1$ there are stationarily many $\alpha \in \aleph_2 \cap \mathrm{cof}(\omega)$ such that $\gamma_{\langle\alpha\rangle} = i$. Using Claim 1 we find $S^1 \leq^* S$ with the same first level such that $S^1_{\langle\alpha\rangle}$ decides $\min(\dot{C} \setminus \gamma_{\langle\alpha\rangle})$ as some $\delta_{\langle\alpha\rangle}$.

Repeating we thin out level by level to find $T \leq^* S$ together with an assignment of γ_t and δ_t to $t \in T$ such that $T_t \Vdash \min(\dot{C} \setminus \gamma_t) = \delta_t$, and for every t and i there are stationarily many α with $\gamma_{t\frown\alpha} = i$.

We consider for each $\delta < \aleph_1$ a game G_δ in which the players collaborate to build a branch through T; at round n player I chooses a nonstationary set $A_n \subseteq \aleph_{n+2}$ and a countable ordinal $\beta_n < \delta$. Player II responds with $\alpha_n \notin A_n$. Player II loses immediately if $\langle \alpha_0, \ldots, \alpha_n \rangle \notin T$ or $\gamma_{\alpha_0,\ldots,\alpha_n} \leq \beta_n$ or $\delta_{\alpha_0,\ldots,\alpha_n} \geq \delta$.

Clearly G_δ is open, so it is determined by the Gale-Stewart theorem. Let S be the set of δ where I wins and fix $\vec{\tau} = \langle \tau_\delta : \delta \in S \rangle$ winning strategies for each such δ.

We claim that S is nonstationary. If not we choose a countable elementary $N \prec H_\theta$ such that N contains everything relevant, and $\delta = N \cap \omega_1 \in S$. We will build a run in which I plays according to τ_δ while II plays ordinals from N and never loses. The only problem is that $\tau_\delta \notin N$.

To build the run we work as follows. If II has played $\alpha_0, \ldots, \alpha_{i-1}$ then let $\beta_i < \delta$ be the ordinal part of τ_δ's response. We compute the *union* over all $\gamma \in S$ of the nonstationary sets dictated by the various τ_γs in response to $\alpha_0, \ldots, \alpha_{i-1}$; this union is a nonstationary subset of $\aleph_{i+2}$ lying in N, and $\beta_i \in N$ so we may choose a

suitable $\alpha_i \in N$. Key point: the map $s \mapsto \delta_s$ is in N so the condition $\delta_{\alpha_0,\dots,\alpha_n} < \delta$ is automatic.

Now we choose $\delta \in A$ such that II wins G_δ. Fixing a winning strategy ρ and a sequence $\langle \delta_n : n < \omega \rangle$ increasing and cofinal in δ, we use ρ to thin out T to $U \leq^* T$ such that for all $u \in U$ of length n we have $\delta_n < \gamma_u \leq \delta_u < \delta$. Clearly U forces that δ is a limit point of $\dot{C}$. $\square$

Claim 4: If MM holds then there is no good scale of length $\aleph_{\omega+1}$ in $\prod_{n \in A} /finite$ for any $A \subseteq \omega$.

Proof Suppose that $\langle f_\alpha : \alpha < \aleph_{\omega+1} \rangle$ is such a scale. By a standard application of MM to the stationary preserving poset $\mathbb{P}$ we may produce an ordinal δ of cofinality $\aleph_1$ and a function $f \in \prod_{n \in A} \aleph_n \cap \mathrm{cof}(\omega)$ such that f is an eub for $\langle f_\alpha : \alpha < \delta \rangle$. This is impossible because MM implies that there is no cofinal increasing sequence of length ω_1 in $^\omega \omega /finite$. Contradiction. $\square$

                                                                              $\square$

18   Another Model with No Good Scales

We give another (slightly simpler) proof that it is consistent for no good scale of length $\aleph_{\omega+1}$ to exist. This proof comes from our joint work with Foreman and Magidor [4]. In the course of the proof we will give another argument that $\square_\mu^*$ fails when $\mathrm{cf}(\mu) < \kappa < \mu$ and κ is μ^+-supercompact.

We start by showing that large cardinals put limits on the existence of good scales.

Theorem 18.1 (Shelah [17]) *Let κ and μ be cardinals such that $\mathrm{cf}(\mu) < \kappa < \mu$ and κ is μ^+-supercompact. Let $\langle \mu_i : i < \mathrm{cf}(\mu) \rangle$ be a sequence of regular cardinals which is increasing and cofinal in μ with $\mu_0 > \kappa$, and let $\langle f_\alpha : \alpha < \mu^+ \rangle$ be a scale in $\prod_i \mu_i$. Then this scale fails to be good at stationarily many points in μ^+.*

Proof Let $j : V \longrightarrow M$ witness that κ is μ^+-supercompact. That is to say $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \mu^+$ and $^{\mu^+}M \subseteq M$.

Let $\gamma = \sup j``\mu^+$, and consider the image of the scale under j, where we note that since $\mathrm{cf}(\mu) < \kappa = \mathrm{crit}(j)$ this image consists of a $j(\mu^+)$-sequence $\langle j(f)_\alpha : \alpha < j(\mu^+) \rangle$ in $\prod_{i < \mathrm{cf}(\mu)} j(\mu_i)$. We claim that if we define a function h with domain $\mathrm{cf}(\mu)$ by $h : i \mapsto \sup j``\mu_i$, then $h \in \prod_{i < \mathrm{cf}(\kappa)} j(\mu_i)$, and in $M$ the function $h$ is an eub for $\langle j(f)_\alpha : \alpha < \gamma \rangle$. The first point is immediate because in $M$ the cofinality of $\sup j``\mu_i$ is μ_i, while $\mathrm{cf}(j(\mu_i)) > j(\kappa) > \mu_i$. If $g < h$ then $g < j \circ h_0$ for some $h_0 \in \prod_i \mu_i$, $h_0 <^* f_\alpha$ for some α and so $g < j \circ h_0 <^* j(f)_{j(\alpha)}$ where $j(\alpha) < \gamma$. The point γ cannot be good because in M we have $\mathrm{cf}(h(i)) = \mu_i$, whereas if γ were good $\mathrm{cf}(h(i))$ would be constant for large i. $\square$

Now we specialize to the case where $\mu = \kappa^{+\omega}$. The same argument shows that there are stationarily many $\delta < \kappa$ such that the set $S_\delta = \{\alpha \in \kappa^{+\omega+1} \cap \mathrm{cof}(\delta^{+\omega+1}) : \alpha \text{ is not good for } \vec{f}\}$ is stationary.

Let us fix such a δ and force with $\mathrm{Coll}(\omega, \delta^{+\omega}) \times \mathrm{Coll}(\delta^{+\omega+2}, < \kappa)$. Since this is small forcing, the stationarity of S_δ is preserved and $\vec{f}$ is still a scale. We claim that every point of S_δ is now a nongood point of cofinality $\aleph_1$.

To see this let $\gamma \in S_\delta$ and suppose for a contradiction that γ is good in the generic extension. We may therefore fix $X \subseteq \gamma$ unbounded of order type $\delta_V^{+\omega+1}$ such that

$\langle f_\alpha(i) : \alpha \in X \rangle$ is increasing for all large i. However, it is easy to see that there is $Y \in V$ with $Y \subseteq X$ unbounded. This is impossible since γ is not good in V.

19 More PCF

In this final section of these notes we collect some results which require a little more sophistication in PCF theory than those of the preceding sections. All the results in this section are due to Shelah [17].

19.1 Analysis of eubs in a scale of length $\aleph_{\omega+1}$ Let $\langle f_\alpha : \alpha < \aleph_{\omega+1} \rangle$ be a scale of length $\aleph_{\omega+1}$ in $\prod_{n \in A} \aleph_n / finite$. Let $\beta < \aleph_{\omega+1}$ be such that $\mathrm{cf}(\beta)$ is uncountable and there is an eub g for $\langle f_\alpha : \alpha < \beta \rangle$ such that $\mathrm{cf}(g(n)) > \omega$ for all n. Then we claim that $\mathrm{cf}(g(n)) = \mathrm{cf}(\beta)$ for all large n, so, in particular, β is good. To see that $\mathrm{cf}(g(n)) = \mathrm{cf}(\beta)$ for all large n, suppose for a contradiction that $\mathrm{cf}(g(n)) \neq \mathrm{cf}(\beta)$ for an unbounded set of n.

Case 1: $\mathrm{cf}(g(n)) > \mathrm{cf}(\beta)$ for $n \in C$ with C unbounded. Fix $\langle \beta_i : i < \mathrm{cf}(\beta) \rangle$ increasing and cofinal in β; thinning out this sequence if necessary we may find a fixed M such that $f_{\beta_i}(n) < g(n)$ for all i and all $n \in A$ with $n > M$. Define a function h by setting $h(n) = \sup\{f_{\beta_i}(n) : i < \mathrm{cf}(\beta)\}$ if $n \in C$ and $n > M$, and $h(n) = 0$ otherwise. Clearly $h < g$ and so since g is an eub $h <^* f_{\beta_i}$ for some i, which is impossible as $h(n) \geq f_{\beta_i}(n)$ for cofinally many n.

Case 2: $\omega < \mathrm{cf}(g(n)) < \mathrm{cf}(\beta)$ for $n \in D$ with D unbounded. Since there are only finitely many regular cardinals below $\mathrm{cf}(\beta)$, we may thin out D to arrange that $\mathrm{cf}(g(n))) = \gamma$ for all $n \in D$, where γ is a regular uncountable cardinal with $\gamma < \mathrm{cf}(\beta)$. Choose functions $\langle g_j : j < \gamma \rangle$ with $g_j \in \prod_{n \in C} \aleph_n$ and $\langle g_j(n) : j < \gamma \rangle$ increasing and cofinal in $g(n)$. It is routine to check that $\langle g_j : j < \gamma \rangle$ is cofinally interleaved with $\langle f_\alpha \upharpoonright D : \alpha < \beta \rangle$, but this is impossible as $\gamma \neq \mathrm{cf}(\beta)$.

Remark 19.1   The stipulation that $\mathrm{cf}(g(n)) > \omega$ for all n is necessary here. The point is that consistently ${}^\omega\omega/finite$ can have a well-ordered cofinal subset of length some uncountable regular cardinal, giving rise to a situation where an eub exists at a nongood point.

19.2 The trichotomy theorem   The idea of Theorem 10.1 gives us a more general statement of the same sort, Shelah's Trichotomy Theorem.

Theorem 19.2 (Trichotomy)   *Let $|X|^+ < \lambda = \mathrm{cf}(\lambda)$ and let $\langle f_i : i < \lambda \rangle$ be a $<_I$-increasing sequence. Then one of the following possibilities holds:*

1. *there is an eub h such that $\mathrm{cf}(h(x)) > |X|$ for all x;*
2. *there is an ultrafilter U on X disjoint from I and a sequence $\langle S_x : x \in X \rangle$ such that $|S_x| \leq |X|$ for all x, and some subfamily of $\prod_x S_x$ is cofinally interleaved with $\langle f_i : i < \lambda \rangle$ modulo U;*
3. *there is a function h such that the sequence $\langle \{x : f_\alpha(x) < h(x)\} : \alpha < \lambda \rangle$ does not stabilize modulo I for large α.*

The proof is very similar to that of Theorem 10.1, using failure of alternative 2 in Phase I and failure of alternative 3 in Phase II.

Notice that if $\mathrm{cf}(\lambda) > 2^{|X|}$ then alternatives 2 and 3 must fail, so that an eub as in alternative 1 exists. In particular, in a scale of length $\aleph_{\omega+1}$ on some product $\prod_{n \in A} \aleph_n / finite$, an eub exists at every point of cofinality greater than the continuum: by the analysis we did in the first part of this section, every such point is good!

It is also easy to see that if alternatives 2 or 3 hold at λ then they hold at almost every $\lambda^* \in \lambda \cap \mathrm{cof}(> |X|)$. In particular, if we are given a scale of length $\aleph_{\omega+1}$ on some product $\prod_{n \in A} \aleph_n / \mathit{finite}$, and λ is an nongood point of cofinality greater than $\aleph_1$, then there is a club subset C of λ such that all points in $C \cap \mathrm{cof}(> \omega)$ are nongood.

We can use this last remark to prove $\mathrm{NPT}(\aleph_{\omega+1}, \aleph_1)$ outright in ZFC; we let S be the set of good points in $\aleph_{\omega_1} \cap \mathrm{cof}(\omega_1)$ and then establish the Claim 13.2 as in Section 13. The difference is that not all limit stages of cofinality greater than ω_1 are good; however, at the nongood limit stages λ we can choose a club set C such that $C \cap \mathrm{cof}(\omega_1)$ avoids S, and there is no problem, while at the good limit stages we use exactly the same argument as we did in Section 13.

Finally we use the Trichotomy Theorem 19.2 to establish a connection between the classical Singular Cardinal Problem and the combinatorial principles we have discussed here.

Theorem 19.3   *If $\aleph_\omega$ is strong limit and $2^{\aleph_\omega} > \aleph_{\omega+1}$ there is a better scale of length $\aleph_{\omega+1}$ on some product $\prod_{n \in D} \aleph_n / \mathit{finite}$.*

Proof   Let $A = \{\aleph_n : n < \omega\}$. We appeal to the theory of PCF generators for A. Easily $J_{<\aleph_\omega} = J_{<\aleph_\omega + 1} = [A]^{<\omega}$. Let B generate $J_{<\aleph_{\omega+2}}$ over $J_{<\aleph_{\omega+1}}$, and let C generate $J_{<\aleph_{\omega+3}}$ over $J_{<\aleph_{\omega+2}}$.

Since $C \notin J_{<\aleph_{\omega+2}}$, and B generates $J_{<\aleph_{\omega+2}}$ over $J_{<\aleph_{\omega+1}}$, we see that $C \setminus B \notin J_{<\aleph_{\omega+1}}$, that is to say, $C \setminus B$ is infinite. Of course $C \cap B \in J_{<\aleph_{\omega+1}}$, so we may as well replace C by $C \setminus B$ and assume that the generators C and B are disjoint.

We know on general grounds that $\prod C / J_{<\aleph_{\omega+2}}$ is $\aleph_{\omega+2}$-*directed*, that is, any subset of $\prod C$ with size at most $\aleph_{\omega+1}$ is bounded modulo $J_{<\aleph_{\omega+2}}$. For any $X \subseteq C$ we have $X = X \setminus B$, so $X \in J_{<\aleph_{\omega+2}}$ if and only if X is finite. It follows that $\prod C / \mathit{finite}$ is $\aleph_{\omega+2}$-directed.

We now fix a "silly square" sequence, that is, a $\square_{\aleph_\omega, \aleph_{\omega+1}}$-sequence. We may assume that every club set which appears has order type less than $\aleph_\omega$. Imitating the construction of a better scale from a $\square_{\aleph_\omega, \aleph_\omega}$-sequence in Section 15, we build an increasing sequence $\langle f_\alpha : \alpha < \aleph_{\omega+1} \rangle$ such that for every $C \in \mathcal{C}_\alpha$ the function f_α eventually dominates the pointwise supremum $\sup_{\beta \in C_\alpha} f_\beta$; this is possible by the directedness of $\prod C / \mathit{finite}$.

It is routine to check, using Trichotomy, that there is an eub h for $\langle f_\alpha : \alpha < \aleph_{\omega+1} \rangle$ with the property that for every $n < \omega$ the set

$$A_n = \{m : \mathrm{cf}(f_\alpha(\aleph_m)) = \aleph_n\}$$

is finite. If we let $D = \{\aleph_n : A_n \neq \varnothing\}$ then we can "collapse" the sequence $\langle f_\alpha : \alpha < \aleph_{\omega+1} \rangle$ as in Section 11.1 to get a scale of length $\aleph_{\omega+1}$ in $\prod D / \mathit{finite}$. It is routine to check that this scale is better. $\square$

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