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Investigations into Quantified Modal Logic

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Abstract The paper discusses several first-order modal logics that extend the classical predicate calculus. The model theory involves possible worlds with world-variable domains. The logics rely on the philosophical tenet known as serious actualism in that within modal contexts they allow existential generalization from atomic formulas. The language may or may not have a sign of identity, includes no primitive existence predicate, and has individual constants. Some logics correspond to various standard constraints on the accessibility relation, whereas others correspond to various constraints on the domains of the worlds. Soundness and strong completeness are proved in every case; a novel method is used for proving completeness.

1 Introduction

This paper presents a number of first-order modal logics. To justify discussion of a logical system, we may show what requirements the system fulfils. The main requirements to be fulfilled by the logics which will be presented here are that the model theory should involve possible worlds with world-variable domains, the non-modal fragment of each logic should be a version of the classical predicate calculus with or without identity, and all logics should conform to serious actualism.

The model theory that involves possible worlds is more familiar than other ways of defining models for modal logic, and at least in that respect it is more convenient and so preferable. Its version adopted here is characterized by world-variable domains: each world is assigned a set of objects which may vary between worlds, and a formula $(\forall \mathbf{x})\mathbf{A}[\mathbf{x}]$ counts as true at a world w if and only if $\mathbf{A}[\mathbf{x}]$ is true at w of all the objects in the set assigned to w. I adopted that version because I did not wish to include all the instances of the Barcan Formula, $(\forall \mathbf{x})\Box\mathbf{A} \to \Box(\forall \mathbf{x})\mathbf{A}$, and the instances of the Converse Barcan Formula, $\Box(\forall \mathbf{x})\mathbf{A} \to (\forall \mathbf{x})\Box\mathbf{A}$, in all the logics we shall discuss.

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There are well-known intuitive counterexamples to the Barcan Formula. For example, let us use the demonstrative 'these' to refer to all the entities there are, without any restriction. It seems clear that every being x is such that, necessarily, if these exist then x is one of them, but it also seems clear that there could have been something x such that these existed but x was not one of them. For there could have been something that does not in fact exist. There are also well-known intuitive counterexamples to the Converse Barcan Formula. Let us consider any ordinary material object x. It seems clear that x might not have existed, although necessarily every being exists. Our intuitions strongly support the idea that there are many contingent beings and the idea that there could have been things which do not actually exist. If our intuitions have not gone awry here, we should not admit all the instances of the Barcan Formula and the Converse Barcan Formula in our preferred first-order modal logic.

The model theory I chose is put forward as just a set-theoretic way of defining some logics. There is no suggestion that it reflects the structure of modal reality or the relation between modal reality and interpreted language. The term 'worlds' is used here because it has become standard and not because the model theory is intended to be about possible worlds in a philosophically substantial sense of the word. So the model theory should not be judged by how well it captures the relations between language and modal reality.

The model theory might be justified by the logics it defines, although this paper will not provide such a justification. A logic is a class of formulas which arguably express the logical principles governing some concepts (or at least is similar, from a formal point of view, to such a class of formulas). A model theory introduces one or more notions of a valid formula, and the extension of any such notion is a logic. The best way of justifying a definition of validity is to find some concepts whose logical properties may interest us and to show that the formulas that come under the definition express just the logical principles governing the interaction of those concepts. The model theory itself could then be justified because it included such definitions.

The second requirement to be fulfilled by the logics discussed in this paper is that their nonmodal fragment should be classical. Even if one has philosophical objections to the classical predicate calculus, one cannot deny that it has been central to both mathematics and philosophy. Most mathematical theories can be couched in a framework provided by classical first-order logic, and when a philosophical discussion presupposes a first-order logic it is normally classical. Thus, however interesting the first-order modal logics may be whose nonmodal fragment is not classical, there is clearly a need for logics whose nonmodal fragment is classical.

That need has not yet been fully met. When the model theory involves possible worlds with a constant domain the nonmodal fragment of the resulting logics is classical, but when the domains vary from world to world the resulting logics are usually based on some free logic or other. Two exceptions are Kripke [6] and Menzel [7]. The nonmodal fragment of any system in [6] is not free, but it is not classical either. It is an inclusive logic, a logic that admits the empty domain; so the systems exclude the classical theorem $(x)Px \rightarrow (\exists x)Px$. Moreover, there is a double restriction on Kripke's logics: the language has no individual constants, and open formulas are not accepted as theorems. As long as that double restriction applies, the peculiarities of free logic cannot appear. And if we admit either open formulas as theorems or individual constants, an inclusive logic based on a classical propositional (nonmodal) system is inevitably free. For the inclusive logic, in order to avoid $(x)Px \rightarrow (\exists x)Px$, must exclude at least one of the classical theorems $(x)Px \rightarrow Pa$ and $Pa \rightarrow (\exists x)Px$, as well as at least one of $(x)Px \rightarrow Px$ and $Px \rightarrow (\exists x)Px$. On the other hand, the nonmodal fragment of system **A** in the appendix to [7] is a formulation of the classical predicate calculus with identity. **A** is based on **S5**, which will play a peripheral part in this paper.[1]

The requirement that every one of the logics to be discussed should have a classical nonmodal fragment suggests that the language should have no primitive existence predicate. Of course, we can formulate a nonmodal logic that employs such a predicate and otherwise does not differ from a usual version of the classical predicate calculus. That logic would presumably be classical. Yet standard formulations of the classical predicate calculus have no primitive existence predicate, and for this reason no such predicate will be used here. Also, our language will have individual constants (which are treated as rigid designators because they are intended as schematic letters for proper names) but not function symbols. It seems that any expression that we may be inclined to render through a function symbol can be accommodated, when we formalize modal discourse, in terms of definite descriptions analyzable in Russell's style.

Serious actualism is the principle that, for every being x and every property φ, it is necessarily the case that if x has φ then x exists. Its main proponent is Plantinga (see [8], pp. 11–15 and [9], pp. 316–23 and pp. 344–49). The logics we shall discuss conform to this principle in that within modal contexts they allow existential generalization from atomic formulas. More specifically, the formulas of the form $(\forall\mathbf{x})\Box[\mathbf{F}\ldots\mathbf{x}\ldots \rightarrow (\exists\mathbf{x})\mathbf{F}\ldots\mathbf{x}\ldots]$ are theorems (or rather, by definition abbreviate theorems) in all those logics; and the formulas of the form $(\forall\mathbf{x})\Box[\mathbf{F}\ldots\mathbf{x}\ldots \rightarrow (\exists\mathbf{y})[\mathbf{x} = \mathbf{y}]]$ abbreviate theorems in all the logics with identity. Every predicate letter **F** is a schematic letter for predicates, and a predicate expresses a property or relation. Thus, if we accept serious actualism and the analogous principle about relations, then we should endorse those theorems. Since existential generalization from atomic formulas is allowed even within modal contexts, the model theory has local predicates: for every world w in any model and for every k-place predicate letter **F**, the value of **F** at w contains only k-tuples of members of the domain of w.

Serious actualism is controversial. Pollock ([10], pp. 126–29) and Fine ([1], pp. 160–71) have objected to it; Forbes ([3], chap. 3) argued for a version restricted to unstructured properties and relations. I will not defend serious actualism here. I hope to show elsewhere that it is right and that formulas of the form $(\forall\mathbf{x})\Box[\mathbf{F}\ldots\mathbf{x}\ldots \rightarrow (\exists\mathbf{x})\mathbf{F}\ldots\mathbf{x}\ldots]$ or the form $(\forall\mathbf{x})\Box[\mathbf{F}\ldots\mathbf{x}\ldots \rightarrow (\exists\mathbf{y})[\mathbf{x} = \mathbf{y}]]$ should be admitted in our preferred first-order modal logic. This paper must be seen as articulating a number of formal systems which conform to a philosophical position that is under debate.

As usual, each logic will be specified both model-theoretically and axiomatically, and completeness will be proved. When we prove completeness in first-order modal logic, we usually specify a property of sets of well-formed formulas, show how a consistent set of such formulas can be extended to a maximal consistent set with that property, and consider the class of all the maximal consistent sets that have the property. The property is normally to do with quantifiers, and the class we consider is used as the class of worlds in a canonical model. The two methods for proving completeness which are presented both in Hughes and Cresswell ([5], chap. 16) and in Garson [4] are of that kind. However, rather than take the class of all the maximal consistent sets having a specified property, we can start with a consistent set of well-formed formulas and construct a class of maximal consistent sets which can then be used as the class of worlds in a canonical model. Constructing the class of worlds may render the proof of completeness more complicated but should allow us greater flexibility in giving the class the desirable features. Both the method I shall employ here and the method used in Fine [2] are of this second kind. When we construct the class of maximal consistent sets according to the method in [2], at each stage we form a class W that effectively has the following feature: $\{\Diamond[\mathbf{A}_1 \wedge \cdots \wedge \mathbf{A}_i] :$ there is a set $w \in W$ that contains all the formulas $\mathbf{A}_1, \ldots, \mathbf{A}_i\}$ is consistent. In the context of **S5**, that feature contributes greatly to making the class of maximal consistent sets suitable for use in a canonical model. When we construct the class according to the method I shall employ here, at no stage do we try to form a class with that feature. At the end of the construction the class is, however, suitable for use in a canonical model; in particular, it is such that if one of its members, w, contains a well-formed formula $\neg\Box\mathbf{A}$, then either w or another one of its members contains $\neg\mathbf{A}$ and includes $\{\mathbf{B} : \Box\mathbf{B} \in w\}$.

There are two sections in the paper after the introduction. Section 2 contains the claims that are proved in Section 3.

2 Models and Axioms

The language $\mathcal{L}$ has the following symbols:

$$
\begin{array}{ll}
x_1, x_2, \ldots & \text{(variables)} \\
a_1, a_2, \ldots & \text{(individual constants)} \\
F_1^1, F_2^1, \ldots,\ F_1^2, F_2^2, \ldots,\ \ldots & \text{(predicate letters)}
\end{array}
$$

and

$$\to,\quad \neg,\quad \Box,\quad \forall,\quad [,\quad],\quad (,\quad).$$

The order of variables which is indicated here will be called *alphabetical*, as will the above order of the predicate letters of degree 1 (i.e., those whose superscript is 1).

Bold letters will generally be used as metalinguistic variables. The letters **A**, **B**, **C**, and **D** (with or without a prime or subscript) are variables ranging over the well-formed formulas (wffs) of $\mathcal{L}$; **x** and **y** range over the variables of $\mathcal{L}$, and **a** over the individual constants; **b**, **c**, and **d** range over both the variables and the individual constants; **F** and **G** range over the predicate letters. As usual, the letters i, j, k, and h will be variables ranging over the positive integers, while m, n, l, and r will range over the natural numbers.

A sequence of symbols is an atomic wff just in case it has the form $F_j^i\mathbf{b}_1\ldots\mathbf{b}_i$. For any **A**, **B**, and **x**, the formulas $[\mathbf{A} \to \mathbf{B}]$, $\neg\mathbf{A}$, $\Box\mathbf{A}$, and $(\forall\mathbf{x})\mathbf{A}$ are well formed. Nothing else is well formed. $\Diamond\mathbf{A}$ is defined as $\neg\Box\neg\mathbf{A}$, while other connectives and the existential quantifier are introduced by standard abbreviatory definitions.[2] Brackets will be omitted according to standard conventions.[3]

We shall say that **b** is *free for* **x** *in* **A** if and only if in **A** no free occurrence of **x** is in the scope of an occurrence of $(\forall\mathbf{b})$. $\dot{\mathbf{S}}^{\mathbf{x}}_{\mathbf{b}}\mathbf{A}$ is the wff that will result from **A** if we replace every free occurrence of **x** with **b**, while $\bar{\mathbf{S}}^{\mathbf{x}}_{\mathbf{y}}\mathbf{A}$ is the wff that will result from **A** if we replace every bound occurrence of $\mathbf{x}$ with $\mathbf{y}$. If $\mathbf{a}_1, \ldots, \mathbf{a}_i$ are distinct individual constants, $\mathbf{S}^{\mathbf{a}_1,\ldots,\mathbf{a}_i}_{\mathbf{x}_1,\ldots,\mathbf{x}_i}\mathbf{A}$ is the wff that results from $\mathbf{A}$ when we simultaneously substitute the variables $\mathbf{x}_1, \ldots, \mathbf{x}_i$ for $\mathbf{a}_1, \ldots, \mathbf{a}_i$, respectively. Finally, if $\mathbf{x}_1, \ldots, \mathbf{x}_i$ are distinct variables arranged in alphabetical order and they are all the variables that have a free occurrence in $\mathbf{A}$, the *closure of* $\mathbf{A}$, clo($\mathbf{A}$), will be $(\forall \mathbf{x}_1) \cdots (\forall \mathbf{x}_i)\mathbf{A}$; if no variable has a free occurrence in $\mathbf{A}$, clo($\mathbf{A}$) will be $\mathbf{A}$ itself.

A *structure* is a quintuple $\langle W, R, D, Q, w^*\rangle$. W and D are nonempty sets, while R and Q are relations: R (the accessibility relation) is a subset of $W \times W$, and Q is a subset of $W \times D$. The elements of W and D will be called 'worlds' and 'individuals', respectively. For every $w \in W$, D_w (the *domain of* w) will be the range of the restriction of Q to $\{w\}$. Finally, w^* (the *actual world*) is a member of W such that D_{w^*} is nonempty. Any such quintuple is a structure.

A *model* is a sextuple $\langle W, R, D, Q, w^*, V\rangle$ where $\langle W, R, D, Q, w^*\rangle$ is a structure while V is a function that assigns an element of D_{w^*} to each individual constant and a subset of $(D_w)^i$ to each pair $\langle F^i_j, w\rangle$ of a predicate letter and a world belonging to W. Any such sextuple is a model. We shall say that the model $\langle W, R, D, Q, w^*, V\rangle$ is *based on* the structure $\langle W, R, D, Q, w^*\rangle$.

Given a model, we can define what it means for a (denumerable) sequence s of individuals to satisfy a wff at a world. We first define the function s^*: $s^*(\mathbf{a}) = V(\mathbf{a})$; $s^*(x_i) = s_i$.

1. s satisfies $F^i_j\mathbf{b}_1 \ldots \mathbf{b}_i$ at w if and only if $\langle s^*(\mathbf{b}_1), \ldots, s^*(\mathbf{b}_i)\rangle \in V(F^i_j, w)$.
2. s satisfies $\mathbf{A} \rightarrow \mathbf{B}$ at w if and only if either s does not satisfy $\mathbf{A}$ at w or s satisfies $\mathbf{B}$ at w.
3. s satisfies $\neg\mathbf{A}$ at w if and only if s does not satisfy $\mathbf{A}$ at w.
4. s satisfies $\Box\mathbf{A}$ at w if and only if, for every world w' such that wRw', s satisfies $\mathbf{A}$ at w'.
5. s satisfies $(\forall x_i)\mathbf{A}$ at w if and only if every sequence s' of individuals which differs from s in at most the ith position and is such that $s'_i \in D_w$ satisfies $\mathbf{A}$ at w.

A wff is *true in* the model if and only if it is satisfied at w^* by every sequence of elements of D_{w^*}.

A wff $\mathbf{A}$ is *valid in* a structure $\langle W, R, D, Q, w^*\rangle$ if and only if it is true in every model based on the structure. A set Λ of wffs is *satisfiable in* $\langle W, R, D, Q, w^*\rangle$ if and only if, in some model based on the structure, there is a sequence of elements of D_{w^*} which satisfies all members of Λ at w^*. $\mathbf{A}$ is a *consequence of* Λ *in* $\langle W, R, D, Q, w^*\rangle$ if and only if, in every model based on the structure, every sequence of elements of D_{w^*} that satisfies all members of Λ at w^* satisfies $\mathbf{A}$, too, at w^*. Thus a wff is valid in a structure just in case it is a consequence of the empty set in that structure.

Our basic axiomatization will be QK. Apart from axiom schemata and primitive rules of inference, QK also has a rule of proof which places a restriction on how those schemata and rules can be used in the course of a proof. The axiomatization runs as follows.

QKA1 Every substitution instance of a classical propositional tautology.

QKA2 $\Box[\mathbf{A} \rightarrow \mathbf{B}] \rightarrow [\Box\mathbf{A} \rightarrow \Box\mathbf{B}]$.

QKA3 $\mathbf{B} \rightarrow [(\forall\mathbf{x})\mathbf{A} \rightarrow \dot{\mathbf{S}}^{\mathbf{x}}_{\mathbf{b}}\mathbf{A}]$, where $\mathbf{b}$ is free for $\mathbf{x}$ in $\mathbf{A}$, and $\mathbf{B}$ is an atomic wff in which $\mathbf{b}$ occurs.

QKA4 $(\forall \mathbf{x})[\mathbf{A} \rightarrow \mathbf{B}] \rightarrow [(\forall \mathbf{x})\mathbf{A} \rightarrow (\forall \mathbf{x})\mathbf{B}]$.

QKA5 $\mathbf{A} \rightarrow (\forall \mathbf{x})\mathbf{A}$ where $\mathbf{x}$ has no free occurrence in $\mathbf{A}$.

QKA6 $(\forall \mathbf{x})\mathbf{A} \rightarrow \dot{\mathbf{S}}^{\mathbf{x}}_{\mathbf{b}}\mathbf{A}$ where $\mathbf{b}$ is free for $\mathbf{x}$ in $\mathbf{A}$.

QKR1 From $\mathbf{A} \rightarrow \mathbf{B}$ and $\mathbf{A}$ we may infer $\mathbf{B}$.

QKR2 From $\mathbf{A}$ we may infer $\Box\mathbf{A}$.

QKR3 From $\mathbf{A}$ we may infer $(\forall \mathbf{x})\mathbf{A}$.

QKR4 From $\mathbf{A}_1 \rightarrow \Box[\cdots \rightarrow \Box[\mathbf{A}_i \rightarrow \Box\mathbf{B}]\ldots]$, we may infer $\mathbf{A}_1 \rightarrow \Box[\cdots \rightarrow \Box[\mathbf{A}_i \rightarrow \Box(\forall \mathbf{x})\mathbf{B}]\ldots]$ if $\mathbf{x}$ has no free occurrence in $\mathbf{A}_1, \ldots, \mathbf{A}_i$.

QKR5 From $\Box^n(\forall \mathbf{x})\mathbf{F}\mathbf{x} \rightarrow \mathbf{A}$ we may infer $\mathbf{A}$ if $\mathbf{F}$ has no occurrence in $\mathbf{A}$.

QKP Once QKA6 has been used in the course of a proof, no rule of inference or axiom schema may be used except QKR1 and QKA6.

$\Box^n$ means $\underbrace{\Box\cdots\Box}_{n\text{ times}}$. So the premise in an application of QKR5 may begin with zero, one, or more boxes. If the schema QKA6 is used in a proof in QK, the proof divides into two parts. The first part, which ends just before the first time QKA6 is used, relies at most on QKA1–5 and QKR1–5. The second part, which begins with the first time QKA6 is invoked, relies at most on QKA6, QKR1, and everything that has been proved in the first part. Of course, it may be that the first time QKA6 is used in the course of a proof is other than the first time an instance of that schema appears as a step in the proof; it may be that an instance of QKA6 appears at some point in a proof but there either it is supported by QKA1 because it is an instance of that schema too or it is inferred from previous wffs in accordance with one of QKR1–5, without any axiom schema being used.

Each rule of inference is a permission to draw inferences of some kind, but in the context of QK our primitive rules, except modus ponens, are not unconditional permissions; they are subject to the proviso that QKP should not be violated. It is because of QKP that we need the axiom schema QKA3 along with QKA6: the way in which a proof may develop after QKA3 has been invoked is not subject to any such restriction as QKP imposes on the way in which a proof may develop after QKA6 has been invoked.[4] We shall see that the theorems of QK are the wffs that are valid in every structure.

There are at least two ways in which we can specify conditions that a structure may meet. First, we can put conditions on the accessibility relation and characterize structures accordingly. For instance, a structure $\langle W, R, D, Q, w^*\rangle$ is serial if and only if R is serial, and it is reflexive if and only if R is reflexive. Second, we can put various conditions on the relation Q. We shall here discuss the following conditions.

1. For any worlds w and w' and for any individual d, if wRw' and wQd then $w'Qd$.
2. If wRw' and $w'Qd$ then wQd.
3. If wRw', $w'Rw''$, wQd, and $w''Qd$, then $w'Qd$.
4. For every world w, there is an individual d such that wQd.
5. For every individual d, there is a world w such that wQd.
6. For any worlds w and w' and for any individual d, wQd if and only if $w'Qd$.

(1) is met only by structures in which all instances of the schema $\Box(\forall\mathbf{x})\mathbf{A} \to (\forall\mathbf{x})\Box\mathbf{A}$ (the Converse Barcan Formula) are valid. (2) is met only by structures in which all instances of the schema $(\forall\mathbf{x})\Box\mathbf{A} \to \Box(\forall\mathbf{x})\mathbf{A}$ (the Barcan Formula) are valid. (3) is a condition weaker than (1) and weaker than (2). It is a model-theoretic analogue of the view that, for every being x, it is necessarily the case that if x does not exist then x could not have existed. This view was tentatively suggested in Stephanou [11]. (4) corresponds to the position that necessarily there is something, while (5) means that D is just the union of the domains of the various worlds. (6) means that all worlds have the same domain.

In order to capture some conditions on structures, we need additional axiom schemata:

QDA $\Box\mathbf{A} \to \Diamond\mathbf{A}$.

QTA $\Box\mathbf{A} \to \mathbf{A}$.

QS4A $\Box\mathbf{A} \to \Box\Box\mathbf{A}$.

QBA $\mathbf{A} \to \Box\Diamond\mathbf{A}$.

QS5A $\Diamond\mathbf{A} \to \Box\Diamond\mathbf{A}$.

Q1A $\mathbf{B} \to \Box[(\forall\mathbf{x})\mathbf{A} \to \dot{\mathbf{S}}^{\mathbf{x}}_{\mathbf{b}}\mathbf{A}]$, where $\mathbf{b}$ is free for $\mathbf{x}$ in $\mathbf{A}$, and $\mathbf{B}$ is an atomic wff in which $\mathbf{b}$ occurs.

Q2A $\Diamond\mathbf{B} \to [(\forall\mathbf{x})\mathbf{A} \to \dot{\mathbf{S}}^{\mathbf{x}}_{\mathbf{b}}\mathbf{A}]$, where $\mathbf{b}$ is free for $\mathbf{x}$ in $\mathbf{A}$, and $\mathbf{B}$ is an atomic wff in which $\mathbf{b}$ occurs.

Q3A $\mathbf{B} \to \Box[\Diamond\mathbf{C} \to [(\forall\mathbf{x})\mathbf{A} \to \dot{\mathbf{S}}^{\mathbf{x}}_{\mathbf{b}}\mathbf{A}]]$ where $\mathbf{b}$ is free for $\mathbf{x}$ in $\mathbf{A}$ while $\mathbf{B}$ and $\mathbf{C}$ are atomic wffs in both of which $\mathbf{b}$ occurs.

Q4A $(\exists\mathbf{x})[\mathbf{A} \to \mathbf{A}]$.

The axiomatic systems QD, QT, QS4, QB, and QS5 will be QK + QDA, QK + QTA, QT + QS4A, QT + QBA, and QT + QS5A, respectively. The axiomatic systems QK1, QK2, QK3, and QK4 will be QK + Q1A, QK + Q2A, QK + Q3A, and QK + Q4A. Finally, QK1/2 will be QK + Q1A + Q2A.

As expected, QD, QT, QS4, QB, and QS5 are sound and (weakly) complete regarding validity in all serial structures, all reflexive structures, all transitive reflexive structures, all symmetrical reflexive structures, and all equivalence structures, respectively. QS5 is also sound and complete regarding validity in all universal structures, that is, the structures in which, for any worlds w and w', wRw'. QK1, QK2, QK3, and QK4 are sound and complete regarding validity in the structures that meet condition (1), in those that meet (2), in the structures meeting (3), and in those meeting (4), respectively. QK, without any additional axiom schema, is sound and complete regarding validity in the structures that satisfy condition (5). QK1/2 is sound and complete regarding validity in the structures that satisfy (6).

We shall see that satisfiability in a structure is compact: it is the case for any set Λ of wffs that if, for each finite subset of Λ, there is a structure in which that subset is satisfiable, then Λ itself is satisfiable in a structure. And QK is strongly complete regarding consequence in all structures: it is the case for any $\mathbf{A}$ and any set Λ of wffs that if $\mathbf{A}$ is a consequence of Λ in every structure, then for some wffs $\mathbf{B}_1, \ldots, \mathbf{B}_n \in \Lambda (n \geq 0) \vdash_{\mathrm{QK}} \mathbf{B}_1 \wedge \cdots \wedge \mathbf{B}_n \to \mathbf{A}$. Actually, if any one of the axiomatizations QD, QT, QS4, QB, QS5, QK1, QK2, QK3, QK4, and QK1/2 was reported in the preceding paragraph to be sound and (weakly) complete regarding validity in some kind of structures, then satisfiability in a structure of that kind is compact, and the axiomatization is strongly complete regarding consequence in the structures of that kind. For instance, satisfiability in a serial structure is compact, and QD is strongly complete regarding consequence in serial structures.

Turning to quantified modal logic with identity, we shall use the letter F_1^2 as a sign of identity. $[\mathbf{b} = \mathbf{c}]$ is defined as $F_1^2\mathbf{bc}$. The brackets will be omitted when it is clear what they would enclose. A model is an *identity model* if and only if, for every world w and all individuals d and d' such that wQd and wQd', $\langle d, d' \rangle \in V(F_1^2, w)$ just in case d is d'. A wff $\mathbf{A}$ is *I-valid in* a structure $\langle W, R, D, Q, w^* \rangle$ if and only if it is true in every identity model based on the structure. A set Λ of wffs is *I-satisfiable in* $\langle W, R, D, Q, w^* \rangle$ if and only if, in some identity model based on the structure, there is a sequence of elements of D_{w^*} which satisfies all members of Λ at w^*. And $\mathbf{A}$ is an *I-consequence of* Λ *in* $\langle W, R, D, Q, w^* \rangle$ just in case, in every identity model based on the structure, every sequence of elements of D_{w^*} that satisfies all members of Λ at w^* satisfies $\mathbf{A}$, too, at w^*.

Our basic axiomatic system with identity, QIK, results from QK by adding the following axiom schemata:

QIKA1 $\mathbf{A} \to \mathbf{x} = \mathbf{x}$ where $\mathbf{A}$ is an atomic wff in which $\mathbf{x}$ occurs.

QIKA2 $\mathbf{x} = \mathbf{y} \to [\mathbf{A} \to \mathbf{B}]$, where $\mathbf{A}$ is an atomic wff in which $\mathbf{x}$ occurs, and $\mathbf{B}$ results from $\mathbf{A}$ by substituting $\mathbf{y}$ for an occurrence of $\mathbf{x}$.

QIKA3 $\Diamond^n[\mathbf{b} = \mathbf{c}] \to \Box^m[\mathbf{A} \to \mathbf{b} = \mathbf{c}]$ where $\mathbf{A}$ is an atomic wff in which $\mathbf{b}$ occurs or $\mathbf{c}$ occurs.

It will be proved that the theorems of QIK are the wffs that are I-valid in every structure.

The axiomatizations QID, QIT, QIS4, QIB, and QIS5 will be QIK + QDA, QIK + QTA, QIT + QS4A, QIT + QBA, and QIT + QS5A, respectively. The axiomatizations QIK1, QIK2, QIK3, and QIK4 will be QIK + Q1A, QIK + Q2A, QIK + Q3A, and QIK + Q4A. QIK1/2 will be QIK + Q1A + Q2A. QID, QIT, QIS4, QIB, and QIS5 are sound and (weakly) complete regarding I-validity in all serial structures, all reflexive structures, all transitive reflexive structures, all symmetrical reflexive structures, and all equivalence structures, respectively. QIS5 is also sound and complete regarding I-validity in all universal structures. QIK1, QIK2, QIK3, and QIK4 are sound and complete regarding I-validity in the structures that satisfy condition (1), in those that satisfy (2), in the structures satisfying (3), and in those satisfying (4). QIK is also sound and complete regarding I-validity in the structures that meet condition (5). QIK1/2 is sound and complete regarding I-validity in the structures that meet (6).

Moreover, I-satisfiability in a structure is compact: it is the case for any set Λ of wffs that if, for each finite subset of Λ, there is a structure in which that subset is I-satisfiable, then Λ itself is I-satisfiable in a structure. And QIK is strongly complete regarding I-consequence in all structures: it is the case for any $\mathbf{A}$ and any set Λ of wffs that if $\mathbf{A}$ is an I-consequence of Λ in every structure, then for some wffs $\mathbf{B}_1, \ldots, \mathbf{B}_n \in \Lambda (n \geq 0) \vdash_{\mathrm{QIK}} \mathbf{B}_1 \wedge \cdots \wedge \mathbf{B}_n \to \mathbf{A}$. Actually, if any one of the axiomatizations QID, QIT, QIS4, QIB, QIS5, QIK1, QIK2, QIK3, QIK4, and QIK1/2 has just been reported to be sound and (weakly) complete regarding I-validity in some kind of structures, then I-satisfiability in a structure of that kind is compact, and the axiomatization is strongly complete regarding I-consequence in the structures of that kind.

Let $\mathcal{L}'$ be the language whose wffs are the wffs of $\mathcal{L}$ in which there is no occurrence of $\Box$; $\mathcal{L}'$ is a language for nonmodal first-order logic. It will be shown that the nonmodal fragment of any one of the axiomatic systems QK, QD, QT, QS4, QB, QS5, QK1, QK2, QK3, QK4, and QK1/2 is classical; in other words, the theorems that are wffs of $\mathcal{L}'$ are just the formulas that make up the classical predicate calculus without identity when this calculus is formulated in $\mathcal{L}'$. It will also be shown that if we take any one of the systems QIK, QID, QIT, QIS4, QIB, QIS5, QIK1, QIK2, QIK3, QIK4, and QIK1/2, its theorems that are wffs of $\mathcal{L}'$ are just the formulas that make up the classical predicate calculus with identity when this calculus is formulated in $\mathcal{L}'$ and the sign of identity is F_1^2.

It is left to the reader to discuss axiomatizations that extend QK or QIK by adding axiom schemata both from among QDA–QS5A and from among Q1A–Q4A.

3 Proofs

We shall begin with some metatheorems that are useful at several places below.

Metatheorem 3.1 *It is the case in every model that for any* $\mathbf{A}$*, any world* w*, and any sequences* s *and* s' *of individuals such that, for every variable* x_i *that has a free occurrence in* $\mathbf{A}$*,* $s_i = s'_i$ *:* s *satisfies* $\mathbf{A}$ *at* w *if and only if* s' *satisfies* $\mathbf{A}$ *at* w.

Metatheorem 3.2 *It is the case in every model that for any* $\mathbf{A}$*,* $\mathbf{x}$*, and* $\mathbf{b}$ *such that* $\mathbf{b}$ *is free for* $\mathbf{x}$ *in* $\mathbf{A}$*, for any world* w*, and for any sequences* s *and* s' *of individuals such that* $s^*(\mathbf{b}) = s'^*(\mathbf{x})$ *while, for every variable* x_i *that has a free occurrence in* $\mathbf{A}$ *but is other than* $\mathbf{x}$*,* $s_i = s'_i$ *:* s' *satisfies* $\mathbf{A}$ *at* w *if and only if* s *satisfies* $\dot{\mathbf{S}}^{\mathbf{x}}_{\mathbf{b}}\mathbf{A}$ *at* w.

Metatheorem 3.3 *It is the case in every model that for any* $\mathbf{A}$*, any* $\mathbf{x}$*, any* $\mathbf{y}$ *that does not occur in* $\mathbf{A}$*, any world* w*, and any sequence* s *of individuals :* s *satisfies* $\mathbf{A}$ *at* w *if and only if* s *satisfies* $\bar{\mathbf{S}}^{\mathbf{x}}_{\mathbf{y}}\mathbf{A}$ *at* w.

Metatheorem 3.4 *For any models* $M = \langle W, R, D, Q, w^*, V\rangle$ *and* $M' = \langle W, R, D, Q, w^*, V'\rangle$ *where* V *and* V' *differ in at most what they assign to a predicate letter* $\mathbf{F}$*, for any* $\mathbf{A}$ *in which* $\mathbf{F}$ *does not occur, for any world* $w \in W$*, and for any sequence* s *of members of* D *:* s *satisfies* $\mathbf{A}$ *at* w *in* M *if and only if* s *satisfies* $\mathbf{A}$ *at* w *in* M'.

Metatheorem 3.5 *For any models* $M = \langle W, R, D, Q, w^*, V\rangle$ *and* $M' = \langle W, R, D', Q, w^*, V\rangle$ *where* $D \supseteq D'$*, for any* $\mathbf{A}$*, for any world* $w \in W$*, and for any sequence* s *of members of* D' *:* s *satisfies* $\mathbf{A}$ *at* w *in* M *if and only if* s *satisfies* $\mathbf{A}$ *at* w *in* M'.

(3.1)–(3.5) can be proved by induction on the number of the occurrences of $\rightarrow$, $\neg$, $\Box$, and $\forall$ in $\mathbf{A}$. The proof of (3.2) presupposes (3.1), and the proof of (3.3) presupposes (3.2).

For any set Λ of axiom schemata, QK + Λ will be the axiomatic system that results from QK by adding those schemata. A wff is *W-true in* a model if and only if it is satisfied at every world by every sequence of individuals. (Note that if $\mathbf{A}$ is W-true in every model based on a given structure then it is valid in the structure, but the converse does not hold. For example, the wff $(\forall x_1)F_1^1x_1 \rightarrow F_1^1x_1$ is valid in every structure, but it is not W-true in every model.)

Metatheorem 3.6 *If, for any structure,* Λ *is a set of axiom schemata whose instances are W-true in every model based on that structure, the theorems of* QK + Λ *are valid in the structure.*

Proof A proof in QK + Λ will consist of two parts, although one of them may be empty: the first part will rely at most on QKA1–5, Λ, and QKR1–5 while the second part will rely at most on QKA6, QKR1, and everything that has been proved in the first part. Now, the instances of QKA1–5 are W-true in every model based on the given structure, for they are W-true in all models. (To demonstrate that, use (3.2) in the case of QKA3 and (3.1) in the case of QKA5.) It can be shown that for any model the rules QKR1–4 preserve W-truth in that model. It can also be shown that for any structure the rule QKR5 preserves the property of being W-true in every model based on that structure. (For if a wff $\Box^n(\forall\mathbf{x})\mathbf{Fx} \rightarrow \mathbf{A}$, where $\mathbf{F}$ does not occur in $\mathbf{A}$, is W-true in every model based on the structure, so is $\mathbf{A}$. To see that, let $\langle W, R, D, Q, w^*, V\rangle$ be a model in which a sequence s of individuals does not satisfy $\mathbf{A}$ at a world w. Consider the model $\langle W, R, D, Q, w^*, V'\rangle$ where, for every world w' that is n R-steps away from w, $V'(\mathbf{F}, w') = D_{w'}$ but otherwise V' does not differ from V.[5] In this model, which is based on the same structure, by (3.4) s does not satisfy $\mathbf{A}$ at w, whereas it satisfies $\Box^n(\forall\mathbf{x})\mathbf{Fx}$ at w.) Therefore, everything proved in the first part of the proof in QK + Λ will be W-true in every model based on the given structure, and so it will be valid in the structure. Finally, the instances of QKA6 are valid in the given structure (for they are valid in all structures) and QKR1 preserves validity in it. $\Box$

(3.6) will be crucial to establishing the soundness of the axiomatizations without identity. The following is a corollary of (3.6).

Metatheorem 3.7 *If* $\vdash_{\mathrm{QK}} \mathbf{A}$*,* $\mathbf{A}$ *is valid in every structure.*

To establish completeness, we need some proof-theoretic development. S will be an arbitrary axiomatic system which differs from QK in at most having additional axiom schemata such that if $\mathbf{A}$ is an instance of one of those schemata, $\mathbf{a}_1, \ldots, \mathbf{a}_i$ are distinct individual constants, and $\mathbf{y}_1, \ldots, \mathbf{y}_i$ are variables that do not occur in $\mathbf{A}$, then $\mathbf{S}^{\mathbf{a}_1,\ldots,\mathbf{a}_i}_{\mathbf{y}_1,\ldots,\mathbf{y}_i}\mathbf{A}$ is also an instance of the relevant schema. This restriction on additional axiom schemata is presupposed in the proof of (3.9) below; all the extensions of QK that were named in Section 2 comply with it. While the notation $\vdash_S \mathbf{A}$ means, as usual, that $\mathbf{A}$ is a theorem of S, the notation $\vdash^*_S \mathbf{A}$ will mean that $\mathbf{A}$ can be proved in S without using QKA6; in other words, there is a proof of $\mathbf{A}$ which relies at most on QKA1–5, QKR1–5, and any axiom schemata that S has in addition to those of QK. When the name of a theorem schema ends with a star, for example, ST1*, the star will indicate that QKA6 has not been used in the proof. Most easy or standard proofs will be omitted.

If we have proved $\mathbf{A}$ without using QKA6, and we are then sketching a proof for a further theorem, we may begin with $\mathbf{A}$ and subsequently employ any one of the rules QKR1–5 and the schemata QKA1–5; if, on the other hand, we have used QKA6 in our proof of $\mathbf{A}$, then in sketching a further proof we may begin with $\mathbf{A}$ but we must not go on to employ any one of QKR2–5 and QKA1–5.

ST1* $\Box[\mathbf{A} \leftrightarrow \mathbf{B}] \rightarrow [\Box\mathbf{A} \leftrightarrow \Box\mathbf{B}]$.
ST2* $\Box\mathbf{A} \wedge \Box\mathbf{B} \rightarrow \Box[\mathbf{A} \wedge \mathbf{B}]$.

$\mathcal{S}$T3* $\Diamond\mathbf{A} \to \Box\mathbf{B} \to \Box[\mathbf{A} \to \mathbf{B}]$.
$\mathcal{S}$T4* $\Diamond[\mathbf{A} \wedge \mathbf{B}] \to \Diamond\mathbf{A}$.
$\mathcal{S}$T5* $(\forall\mathbf{x})[\mathbf{A} \to \mathbf{B}] \to [\mathbf{A} \to (\forall\mathbf{x})\mathbf{B}]$ where **x** has no free occurrence in **A**.

$\mathcal{S}$T6* $(\forall\mathbf{x})(\forall\mathbf{y})\mathbf{A} \to (\forall\mathbf{y})(\forall\mathbf{x})\mathbf{A}$.

Proof Suppose that **x** and **y** are distinct variables, for otherwise the schema is trivial. Let **F** be a predicate letter of degree 1 which does not occur in **A**. Then $\mathbf{Fy} \to [(\forall\mathbf{y})\mathbf{A} \to \mathbf{A}]$ is an axiom. Hence, by QKR3 and $\mathcal{S}$T5*, $\mathbf{Fy} \to (\forall\mathbf{x})[(\forall\mathbf{y})\mathbf{A} \to \mathbf{A}]$ and so $\mathbf{Fy} \to [(\forall\mathbf{x})(\forall\mathbf{y})\mathbf{A} \to (\forall\mathbf{x})\mathbf{A}]$. Thus, by QKR3 and QKA4, $(\forall\mathbf{y})\mathbf{Fy} \to (\forall\mathbf{y})[(\forall\mathbf{x})(\forall\mathbf{y})\mathbf{A} \to (\forall\mathbf{x})\mathbf{A}]$ and so $(\forall\mathbf{y})\mathbf{Fy} \to [(\forall\mathbf{x})(\forall\mathbf{y})\mathbf{A} \to (\forall\mathbf{y})(\forall\mathbf{x})\mathbf{A}]$. Then apply QKR5. □

$\mathcal{S}$T7* $(\forall\mathbf{x})[(\forall\mathbf{y})\mathbf{A} \to \dot{\mathbf{S}}^{\mathbf{y}}_{\mathbf{x}}\mathbf{A}]$ where **x** is free for **y** in **A**.

Proof Let **F** be a predicate letter of degree 1 which does not occur in **A**. Then $\mathbf{Fx} \to [(\forall\mathbf{y})\mathbf{A} \to \dot{\mathbf{S}}^{\mathbf{y}}_{\mathbf{x}}\mathbf{A}]$ is an axiom. Hence, by QKR3 and QKA4, $(\forall\mathbf{x})\mathbf{Fx} \to (\forall\mathbf{x})[(\forall\mathbf{y})\mathbf{A} \to \dot{\mathbf{S}}^{\mathbf{y}}_{\mathbf{x}}\mathbf{A}]$. Then apply QKR5. □

$\mathcal{S}$T8 $(\forall\mathbf{x})[(\forall\mathbf{y})\mathbf{A} \to \dot{\mathbf{S}}^{\mathbf{y}}_{\mathbf{b}}\mathbf{A}]$ where **b** is free for **y** in **A**.

Proof If **b** and **x** are the same variable, we have $\mathcal{S}$T7*. Suppose that **b** is other than **x**, and let **z** be a variable that does not occur in **A** and is other than **x**. By $\mathcal{S}$T7* $\vdash^{*}_{\mathcal{S}} (\forall\mathbf{z})[(\forall\mathbf{y})\mathbf{A} \to \dot{\mathbf{S}}^{\mathbf{y}}_{\mathbf{z}}\mathbf{A}]$ and so, by QKR3 and $\mathcal{S}$T6*, $\vdash^{*}_{\mathcal{S}} (\forall\mathbf{z})(\forall\mathbf{x})[(\forall\mathbf{y})\mathbf{A} \to \dot{\mathbf{S}}^{\mathbf{y}}_{\mathbf{z}}\mathbf{A}]$. But $(\forall\mathbf{z})(\forall\mathbf{x})[(\forall\mathbf{y})\mathbf{A} \to \dot{\mathbf{S}}^{\mathbf{y}}_{\mathbf{z}}\mathbf{A}] \to (\forall\mathbf{x})[(\forall\mathbf{y})\mathbf{A} \to \dot{\mathbf{S}}^{\mathbf{y}}_{\mathbf{b}}\mathbf{A}]$ is an instance of QKA6. □

$\mathcal{S}$T9* $(\forall\mathbf{x})[\mathbf{A} \leftrightarrow \mathbf{B}] \to [(\forall\mathbf{x})\mathbf{A} \leftrightarrow (\forall\mathbf{x})\mathbf{B}]$.

$\mathcal{S}$R6 If **B** results from **A** by substituting **D** for an occurrence of **C** and $\vdash^{*}_{\mathcal{S}} \mathbf{C} \leftrightarrow \mathbf{D}$, then $\vdash^{*}_{\mathcal{S}} \mathbf{A} \leftrightarrow \mathbf{B}$.

Proof By induction on the number n of the occurrences of $\Box$ and $\forall$ in whose scope lies the relevant occurrence of **C**. If $n = 0$ then $[\mathbf{C} \leftrightarrow \mathbf{D}] \to [\mathbf{A} \leftrightarrow \mathbf{B}]$ is a substitution instance of a propositional tautology. Now let $n > 0$ and assume that what we are trying to prove holds for numbers smaller than n. Then **B** results from **A** by substituting $O \ldots \mathbf{D} \ldots$, where O is either $\Box$ or $\forall$, for an occurrence of $O \ldots \mathbf{C} \ldots$ which is in the scope of $n - 1$ occurrences of $\Box$ and $\forall$. $[\mathbf{C} \leftrightarrow \mathbf{D}] \to [\ldots \mathbf{C} \ldots \leftrightarrow \ldots \mathbf{D} \ldots]$ is again an instance of a tautology. Since $\vdash^{*}_{\mathcal{S}} \mathbf{C} \leftrightarrow \mathbf{D}$, $\vdash^{*}_{\mathcal{S}} \ldots \mathbf{C} \ldots \leftrightarrow \ldots \mathbf{D} \ldots$. So, by QKR2 and $\mathcal{S}$T1* or by QKR3 and $\mathcal{S}$T9*, $\vdash^{*}_{\mathcal{S}} O \ldots \mathbf{C} \ldots \leftrightarrow O \ldots \mathbf{D} \ldots$. Hence, by the inductive hypothesis, $\vdash^{*}_{\mathcal{S}} \mathbf{A} \leftrightarrow \mathbf{B}$. □

$\mathcal{S}$T10* $(\forall\mathbf{x})\mathbf{A} \leftrightarrow (\forall\mathbf{y})\dot{\mathbf{S}}^{\mathbf{x}}_{\mathbf{y}}\mathbf{A}$ where **y** is free for **x** in **A** and has no free occurrence in **A**.

Proof By $\mathcal{S}$T7* and $\mathcal{S}$T5* we have $(\forall\mathbf{x})\mathbf{A} \to (\forall\mathbf{y})\dot{\mathbf{S}}^{\mathbf{x}}_{\mathbf{y}}\mathbf{A}$. The converse is an instance of the same schema — unless **x** is **y**, in which case the proof is trivial. □

$\mathcal{S}$T11* $\mathbf{A} \leftrightarrow \bar{\mathbf{S}}^{\mathbf{x}}_{\mathbf{y}}\mathbf{A}$ where **y** does not occur in **A**.
$\mathcal{S}$T12* $(\forall\mathbf{x})[\mathbf{A} \wedge \mathbf{B}] \to (\forall\mathbf{x})\mathbf{A} \wedge (\forall\mathbf{x})\mathbf{B}$.

ST13* $(\forall \mathbf{x})\mathbf{A} \to \mathbf{A} \vee (\forall \mathbf{x})\neg[\mathbf{B} \to \mathbf{B}]$ where $\mathbf{x}$ has no free occurrence in $\mathbf{A}$.

Proof Begin with $\mathbf{A} \to [\neg\mathbf{A} \to \neg[\mathbf{B} \to \mathbf{B}]]$. By QKR3, QKA4, and ST5* we get $(\forall \mathbf{x})\mathbf{A} \to [\neg\mathbf{A} \to (\forall \mathbf{x})\neg[\mathbf{B} \to \mathbf{B}]]$. □

ST14* $(\forall \mathbf{x})\neg[\mathbf{A} \to \mathbf{A}] \to (\forall \mathbf{y})\neg[\mathbf{A} \to \mathbf{A}]$.

Proof Let $\mathbf{B}$ be a wff in which neither $\mathbf{x}$ nor $\mathbf{y}$ occurs. We can prove $(\forall \mathbf{x})\neg[\mathbf{A} \to \mathbf{A}] \to (\forall \mathbf{x})\neg[\mathbf{B} \to \mathbf{B}]$, by ST10* $(\forall \mathbf{x})\neg[\mathbf{B} \to \mathbf{B}] \to (\forall \mathbf{y})\neg[\mathbf{B} \to \mathbf{B}]$, and $(\forall \mathbf{y})\neg[\mathbf{B} \to \mathbf{B}] \to (\forall \mathbf{y})\neg[\mathbf{A} \to \mathbf{A}]$. □

ST15* $(\forall \mathbf{x})\mathbf{A} \to \mathbf{A} \vee (\forall \mathbf{y})\neg[\mathbf{B} \to \mathbf{B}]$ where $\mathbf{x}$ has no free occurrence in $\mathbf{A}$.

As usual, a set Λ of wffs is *S-consistent* if and only if there are no wffs $\mathbf{A}_1, \ldots, \mathbf{A}_i \in \Lambda$ ($i \geq 1$) such that $\vdash_S \neg[\mathbf{A}_1 \wedge \cdots \wedge \mathbf{A}_i]$. It is *S-consistent** if and only if there are no wffs $\mathbf{A}_1, \ldots, \mathbf{A}_i \in \Lambda$ ($i \geq 1$) such that $\vDash^*_S \neg[\mathbf{A}_1 \wedge \cdots \wedge \mathbf{A}_i]$. Λ is *maximal* if and only if, for every $\mathbf{A}$, $\mathbf{A} \in \Lambda$ or $\neg\mathbf{A} \in \Lambda$.

Metatheorem 3.8 *If Γ is a maximal S-consistent* set of wffs, then every* $\mathbf{A}$ *such that* $\vDash^*_S \mathbf{A}$ *belongs to* Γ*, and if* Γ *contains both* $\mathbf{A} \to \mathbf{B}$ *and* $\mathbf{A}$ *it also contains* $\mathbf{B}$.

Λ is *finite** just in case there are infinitely many variables that have no free occurrence in its members, as well as infinitely many predicate letters of degree 1 that do not occur in its members. It has the $\forall$*-property* if and only if, for every wff $(\forall \mathbf{x})\mathbf{A}$, it contains $(\forall \mathbf{x})\neg[\mathbf{A} \to \mathbf{A}]$ or else there are a variable $\mathbf{y}$, not occurring in $\mathbf{A}$, and a predicate letter $\mathbf{F}$ of degree 1 such that Λ contains both $\dot{\mathbf{S}}^{\mathbf{x}}_{\mathbf{y}}\mathbf{A} \to (\forall \mathbf{x})\mathbf{A}$ and $\mathbf{F}\mathbf{y}$.

A set W is *S-full* if and only if its members are maximal S-consistent* sets of wffs, they have the $\forall$-property, they each contain a wff of the form $(\forall \mathbf{x})\mathbf{F}\mathbf{x}$, and, for every set $\Gamma \in W$ and each wff $\neg\Box\mathbf{A} \in \Gamma$, there is a set $\Gamma' \in W$ which contains $\neg\mathbf{A}$ and includes $\{\mathbf{B} : \Box\mathbf{B} \in \Gamma\}$.[6] We can also define what it means to say that, in an S-full set W, a member Γ' is *n steps away from* a member Γ: Γ' is 0 steps away if and only if it is Γ itself, and it is $n+1$ steps away just in case a set $\Gamma'' \in W$ is n steps away from Γ and $\{\mathbf{A} : \Box\mathbf{A} \in \Gamma''\} \subseteq \Gamma'$. An S-full set W *starts at* one of its members if and only if, for each set $\Gamma \in W$, there is a number n such that Γ is n steps away from that member.

Assuming that we have specified an enumeration of all the wffs, we can also define the function con. For each finite and nonempty set Λ of wffs, con(Λ) will be the wff that we abbreviate when we form the conjunction of all the members of Λ in order of appearance in that enumeration.

Metatheorem 3.9 *If Λ is an S-consistent, finite*, and nonempty set of wffs, there is an S-consistent* and finite* set of wffs Λ' that is a superset of Λ and contains, for each individual constant* $\mathbf{a}$*, an atomic wff in which* $\mathbf{a}$ *occurs.*

Proof Let $\mathbf{F}_1, \mathbf{F}_2, \ldots$ be infinitely many distinct predicate letters of degree 1, excluding those that occur in members of Λ and also excluding infinitely many others. Λ' will be $\Lambda \cup \{\mathbf{F}_1 a_1, \mathbf{F}_2 a_2, \ldots\}$. Of the properties of Λ' that we must demonstrate, the only one that is not obvious is S-consistency*. So assume that Λ' is not S-consistent*. Then

$$\vDash^*_S \neg[\mathbf{G}_1\mathbf{a}_1 \wedge \cdots \wedge \mathbf{G}_i\mathbf{a}_i \wedge \mathbf{A}_1 \wedge \cdots \wedge \mathbf{A}_j]$$

where $\mathbf{G}_1\mathbf{a}_1, \ldots, \mathbf{G}_i\mathbf{a}_i$ are distinct wffs belonging to $\{\mathbf{F}_1 a_1, \mathbf{F}_2 a_2, \ldots\}$ while $\mathbf{A}_1, \ldots, \mathbf{A}_j \in \Lambda$. In other words,

$$\Vdash^*_S \mathbf{G}_1\mathbf{a}_1 \rightarrow [\cdots \rightarrow [\mathbf{G}_i\mathbf{a}_i \rightarrow \neg[\mathbf{A}_1 \wedge \cdots \wedge \mathbf{A}_j]] \ldots].$$

So this theorem has a proof in which QKA6 is not used. Let $\mathbf{y}_1, \ldots, \mathbf{y}_i$ be distinct variables that do not occur in the proof. If throughout that proof we replace $\mathbf{a}_1, \ldots, \mathbf{a}_i$ with $\mathbf{y}_1, \ldots, \mathbf{y}_i$, the result will be a proof of

$$\mathbf{G}_1\mathbf{y}_1 \rightarrow [\cdots \rightarrow [\mathbf{G}_i\mathbf{y}_i \rightarrow \neg\mathbf{S}^{\mathbf{a}_1,\ldots,\mathbf{a}_i}_{\mathbf{y}_1,\ldots,\mathbf{y}_i}[\mathbf{A}_1 \wedge \cdots \wedge \mathbf{A}_j]] \ldots]$$

in which QKA6 is not used. Hence,

$$\Vdash^*_S (\forall\mathbf{y}_1)\cdots(\forall\mathbf{y}_i)[\mathbf{G}_1\mathbf{y}_1 \rightarrow [\cdots \rightarrow [\mathbf{G}_i\mathbf{y}_i \rightarrow \neg\mathbf{S}^{\mathbf{a}_1,\ldots,\mathbf{a}_i}_{\mathbf{y}_1,\ldots,\mathbf{y}_i}[\mathbf{A}_1 \wedge \cdots \wedge \mathbf{A}_j]] \ldots]].$$

So, if $i > 1$, then by ST5*, QKA4, and QKR3

$$\begin{aligned}&\Vdash^*_S (\forall\mathbf{y}_1)\cdots(\forall\mathbf{y}_{i-1})[\mathbf{G}_1\mathbf{y}_1 \rightarrow [\cdots \\ &\qquad \rightarrow [\mathbf{G}_{i-1}\mathbf{y}_{i-1} \rightarrow [(\forall\mathbf{y}_i)\mathbf{G}_i\mathbf{y}_i \rightarrow (\forall\mathbf{y}_i)\neg\mathbf{S}^{\mathbf{a}_1,\ldots,\mathbf{a}_i}_{\mathbf{y}_1,\ldots,\mathbf{y}_i}[\mathbf{A}_1 \wedge \cdots \wedge \mathbf{A}_j]]] \ldots]].\end{aligned}$$

Proceeding similarly, we infer that

$$\Vdash^*_S (\forall\mathbf{y}_1)\mathbf{G}_1\mathbf{y}_1 \rightarrow [\cdots \rightarrow [(\forall\mathbf{y}_i)\mathbf{G}_i\mathbf{y}_i \rightarrow (\forall\mathbf{y}_1)\cdots(\forall\mathbf{y}_i)\neg\mathbf{S}^{\mathbf{a}_1,\ldots,\mathbf{a}_i}_{\mathbf{y}_1,\ldots,\mathbf{y}_i}[\mathbf{A}_1 \wedge \cdots \wedge \mathbf{A}_j]] \ldots].$$

Hence, by QKR5, $\Vdash^*_S (\forall\mathbf{y}_1)\cdots(\forall\mathbf{y}_i)\neg\mathbf{S}^{\mathbf{a}_1,\ldots,\mathbf{a}_i}_{\mathbf{y}_1,\ldots,\mathbf{y}_i}[\mathbf{A}_1 \wedge \cdots \wedge \mathbf{A}_j]$. So, if $i > 1$, then by QKA6 $\vdash_S (\forall\mathbf{y}_2)\cdots(\forall\mathbf{y}_i)\neg\mathbf{S}^{\mathbf{a}_2,\ldots,\mathbf{a}_i}_{\mathbf{y}_2,\ldots,\mathbf{y}_i}[\mathbf{A}_1 \wedge \cdots \wedge \mathbf{A}_j]$. Continuing likewise, we conclude that $\vdash_S \neg[\mathbf{A}_1 \wedge \cdots \wedge \mathbf{A}_j]$, which contradicts the S-consistency of Λ. □

We can now establish (3.10), which is pivotal for completeness. (3.10) can also be proved by modifying and adapting the first of the two methods presented in [5], chap. 16. That method is due to R. Thomason. The proof I shall offer is different. It is somewhat longer than the proof by Thomason's method, but it is more interesting because it displays a new way of proving completeness in quantified modal logic.

Metatheorem 3.10 *If Λ is an S-consistent*, finite*, and nonempty set of wffs, there is an S-full set W that contains, and starts at, a superset of Λ.*

Proof Let $\varphi_1, \varphi_2, \ldots$ be an enumeration of all the ordered k-tuples of wffs of the form $\Box\mathbf{A}$, as well as all the ordered k-tuples whose first $k-1$ positions (if $k > 1$) are occupied by wffs of the form $\Box\mathbf{A}$, but whose last position is occupied by a wff that begins with a universal quantifier. The enumeration must be such that each k-tuple $\langle\Box\mathbf{A}_1, \ldots, \Box\mathbf{A}_k\rangle$ appears before all the $(k+1)$-tuples $\langle\Box\mathbf{A}_1, \ldots, \Box\mathbf{A}_k, \mathbf{B}\rangle$ which coincide with it in their first k positions.[7] On the basis of that enumeration, I will recursively define the matrix

$$\begin{array}{cccc}\Delta^0_0, & \Delta^0_1, & \Delta^0_2, & \ldots \\ \psi^1, & \Delta^1_1, & \Delta^1_2, & \ldots \\ \psi^2, & \Delta^2_1, & \Delta^2_2, & \ldots \\ \vdots & \vdots & \vdots & \vdots\end{array}$$

where each Δ^0_n is a finite* and nonempty set of wffs, each Δ^i_j is a finite and nonempty set of wffs, and each ψ^i is a k-tuple of wffs of the form $\Box\mathbf{A}$. There will not be any i and j such that $i \neq j$ but ψ^i is ψ^j. A matrix here is a function from a set of ordered pairs of natural numbers. There will not be a last column, but there may be a last row, in the matrix that is defined.

The idea of the proof is that each row represents the gradual construction of a world while $\psi^1, \psi^2, \ldots$ record the accessibility relation: if either $m = 0$ and $\psi^{m'}$ is $\langle \Box \mathbf{A} \rangle$ or ψ^m and $\psi^{m'}$ are $\langle \Box \mathbf{A}_1, \ldots, \Box \mathbf{A}_k \rangle$ and $\langle \Box \mathbf{A}_1, \ldots, \Box \mathbf{A}_k, \Box \mathbf{B} \rangle$, respectively, then the world constructed in the row whose superscript is m' will be accessible from the world constructed in the row whose superscript is m.

Δ_0^0 will be $\Lambda \cup \{(\forall x_1)\mathbf{F}x_1\}$ where $\mathbf{F}$ is the first predicate letter of degree 1 (in alphabetical order) that does not occur in the members of Λ. Owing to QKR5, Δ_0^0 is S-consistent*. When we reach φ_{n+1}, what has already been defined will be, for some m,

$$\begin{array}{ccc} \Delta_0^0, & \ldots, & \Delta_n^0 \\ \vdots & \vdots & \vdots \\ \psi^m, & \ldots, & \Delta_n^m \end{array}$$

Then, at the stage of φ_{n+1}, a new column will be added and possibly a new row: a new row will be added only if we have either the second subcase of Case 1 or Subcase 4b.

Case 1 φ_{n+1} is $\langle \Box \mathbf{A} \rangle$. Then $\Delta_{n+1}^1, \ldots, \Delta_{n+1}^m$ will be the same as $\Delta_n^1, \ldots, \Delta_n^m$, respectively. If there are wffs $\mathbf{B}_1, \ldots, \mathbf{B}_i \in \Delta_n^0$ such that $\vDash^*_S \mathbf{B}_1 \wedge \cdots \wedge \mathbf{B}_i \to \Box \mathbf{A}$, Δ_{n+1}^0 will be $\Delta_n^0 \cup \{\Box \mathbf{A}\}$. Otherwise, a new row will be introduced: ψ^{m+1} will be φ_{n+1}, and $\Delta_1^{m+1} = \cdots = \Delta_{n+1}^{m+1} = \{\neg \mathbf{A}, (\forall x_1)\mathbf{F}x_1\}$ where $\mathbf{F}$ is the first predicate letter of degree 1 that occurs neither in the members of Δ_n^0 nor in $\mathbf{A}$. Then Δ_{n+1}^0 will be $\Delta_n^0 \cup \{\neg\Box\neg\mathrm{con}(\Delta_{n+1}^{m+1})\}$.

Case 2 φ_{n+1} is $\langle (\forall \mathbf{x})\mathbf{A} \rangle$. Then $\Delta_{n+1}^1, \ldots, \Delta_{n+1}^m$ will be the same as $\Delta_n^1, \ldots, \Delta_n^m$. If there are wffs $\mathbf{B}_1, \ldots, \mathbf{B}_i \in \Delta_n^0$ such that $\vDash^*_S \mathbf{B}_1 \wedge \cdots \wedge \mathbf{B}_i \to (\forall \mathbf{x})\neg[\mathbf{A} \to \mathbf{A}]$, Δ_{n+1}^0 will be $\Delta_n^0 \cup \{(\forall \mathbf{x})\neg[\mathbf{A} \to \mathbf{A}]\}$. Otherwise, Δ_{n+1}^0 will be $\Delta_n^0 \cup \{\dot{\mathbf{S}}^{\mathbf{x}}_{\mathbf{y}}\mathbf{A} \to (\forall \mathbf{x})\mathbf{A},\ \mathbf{F}\mathbf{y}\}$, where $\mathbf{y}$ is the first variable (in alphabetical order) that has no free occurrence in any member of Δ_n^0 and does not occur in $\mathbf{A}$, and $\mathbf{F}$ is the first predicate letter of degree 1 that occurs neither in the members of Δ_n^0 nor in $\mathbf{A}$.

Case 3 φ_{n+1} is $\langle \Box \mathbf{A}_1, \ldots, \Box \mathbf{A}_k, \mathbf{B} \rangle$ and, in what has already been defined, either there is no row beginning with $\langle \Box \mathbf{A}_1 \rangle$ or there is no row beginning with $\langle \Box \mathbf{A}_1, \Box \mathbf{A}_2 \rangle$ or $\cdots$ or there is no row beginning with $\langle \Box \mathbf{A}_1, \ldots, \Box \mathbf{A}_k \rangle$. Then $\Delta_{n+1}^0, \ldots, \Delta_{n+1}^m$ will be the same as $\Delta_n^0, \ldots, \Delta_n^m$.

Case 4 φ_{n+1} is $\langle \Box \mathbf{A}_1, \ldots, \Box \mathbf{A}_k, \mathbf{B} \rangle$ and there is a row beginning with $\langle \Box \mathbf{A}_1 \rangle, \ldots,$ there is a row beginning with $\langle \Box \mathbf{A}_1, \ldots, \Box \mathbf{A}_k \rangle$. Let $m_1, \ldots, m_k$ be the superscripts in the rows that begin with $\langle \Box \mathbf{A}_1 \rangle, \ldots, \langle \Box \mathbf{A}_1, \ldots, \Box \mathbf{A}_k \rangle$, respectively. Then, except for $\Delta_{n+1}^{m_1}, \ldots, \Delta_{n+1}^{m_k}$, the sets $\Delta_{n+1}^1, \ldots, \Delta_{n+1}^m$ will be the same as $\Delta_n^1, \ldots, \Delta_n^m$. Δ_{n+1}^0 will be $\Delta_n^0 \cup \{\neg\Box\neg\mathrm{con}(\Delta_{n+1}^{m_1})\}$, $\Delta_{n+1}^{m_1}$ will be $\Delta_n^{m_1} \cup \{\neg\Box\neg\mathrm{con}(\Delta_{n+1}^{m_2})\}, \ldots,$ $\Delta_{n+1}^{m_{k-1}}$ will be $\Delta_n^{m_{k-1}} \cup \{\neg\Box\neg\mathrm{con}(\Delta_{n+1}^{m_k})\}$. It remains to specify $\Delta_{n+1}^{m_k}$.

Subcase 4a **B** is $\Box$**C**, and there are wffs $\mathbf{D}_1, \ldots, \mathbf{D}_i \in \Delta_n^0$ such that

$$\vDash^*_S \mathbf{D}_1 \wedge \cdots \wedge \mathbf{D}_i \rightarrow \Box[\mathrm{con}(\Delta_n^{m_1}) \rightarrow \Box[\cdots \rightarrow \Box[\mathrm{con}(\Delta_n^{m_k}) \rightarrow \Box\mathbf{C}]\ldots]].$$

Then $\Delta_{n+1}^{m_k}$ will be $\Delta_n^{m_k} \cup \{\Box\mathbf{C}\}$.

Subcase 4b **B** is $\Box$**C**, and there are no wffs $\mathbf{D}_1, \ldots, \mathbf{D}_i \in \Delta_n^0$ such that

$$\vDash^*_S \mathbf{D}_1 \wedge \cdots \wedge \mathbf{D}_i \rightarrow \Box[\mathrm{con}(\Delta_n^{m_1}) \rightarrow \Box[\cdots \rightarrow \Box[\mathrm{con}(\Delta_n^{m_k}) \rightarrow \Box\mathbf{C}]\ldots]].$$

Then a new row will be introduced: ψ^{m+1} will be φ_{n+1}, and

$$\Delta_1^{m+1} = \cdots = \Delta_{n+1}^{m+1} = \{\neg\mathbf{C}, (\forall x_1)\mathbf{F}x_1\}$$

where **F** is the first predicate letter of degree 1 that occurs neither in the members of Δ_n^0 nor in **C**. $\Delta_{n+1}^{m_k}$ will be $\Delta_n^{m_k} \cup \{\neg\Box\neg\mathrm{con}(\Delta_{n+1}^{m+1})\}$.

Subcase 4c **B** is $(\forall\mathbf{x})\mathbf{C}$, and there are wffs $\mathbf{D}_1, \ldots, \mathbf{D}_i \in \Delta_n^0$ such that

$$\vDash^*_S \mathbf{D}_1 \wedge \cdots \wedge \mathbf{D}_i \rightarrow \Box[\mathrm{con}(\Delta_n^{m_1}) \rightarrow \Box[\cdots \rightarrow \Box[\mathrm{con}(\Delta_n^{m_k}) \rightarrow (\forall\mathbf{x})\neg[\mathbf{C} \rightarrow \mathbf{C}]]\ldots]].$$

Then $\Delta_{n+1}^{m_k}$ will be $\Delta_n^{m_k} \cup \{(\forall\mathbf{x})\neg[\mathbf{C} \rightarrow \mathbf{C}]\}$.

Subcase 4d **B** is $(\forall\mathbf{x})\mathbf{C}$, and there are no wffs $\mathbf{D}_1, \ldots, \mathbf{D}_i \in \Delta_n^0$ such that

$$\vDash^*_S \mathbf{D}_1 \wedge \cdots \wedge \mathbf{D}_i \rightarrow \Box[\mathrm{con}(\Delta_n^{m_1}) \rightarrow \Box[\cdots \rightarrow \Box[\mathrm{con}(\Delta_n^{m_k}) \rightarrow (\forall\mathbf{x})\neg[\mathbf{C} \rightarrow \mathbf{C}]]\ldots]].$$

Then $\Delta_{n+1}^{m_k}$ will be $\Delta_n^{m_k} \cup \{\dot{\mathbf{S}}^{\mathbf{x}}_{\mathbf{y}}\mathbf{C} \rightarrow (\forall\mathbf{x})\mathbf{C},\ \mathbf{F}\mathbf{y}\}$, where **y** is the first variable that has no free occurrence in any member of Δ_n^0 and does not occur in **C**, and **F** is the first predicate letter of degree 1 that occurs neither in the members of Δ_n^0 nor in **C**.[8]

In Case 4 we have that $\neg\Box\neg\mathrm{con}(\Delta_n^{m_k}) \in \Delta_n^{m_{k-1}}, \ldots, \neg\Box\neg\mathrm{con}(\Delta_n^{m_2}) \in \Delta_n^{m_1}$, and $\neg\Box\neg\mathrm{con}(\Delta_n^{m_1}) \in \Delta_n^0$. To see that, let $\langle\Box\mathbf{A}_1, \ldots, \Box\mathbf{A}_k\rangle$ have been $\varphi_{n'}$. Then the row whose superscript is m_k was introduced at the stage of $\varphi_{n'}$. So $\Delta_{n'}^{m_k}$ contained $\neg\mathbf{A}_k$, $\neg\Box\neg\mathrm{con}(\Delta_{n'}^{m_k}) \in \Delta_{n'}^{m_{k-1}}, \ldots, \neg\Box\neg\mathrm{con}(\Delta_{n'}^{m_2}) \in \Delta_{n'}^{m_1}$, and $\neg\Box\neg\mathrm{con}(\Delta_{n'}^{m_1}) \in \Delta_{n'}^0$. Now let n'' be any number such that $n' < n'' \leq n$. If a wff is added, when we reach $\varphi_{n''}$, to any one of the sets $\Delta_{n''-1}^{m_1}, \ldots, \Delta_{n''-1}^{m_k}$, it is added through an application of Case 4. Let $\Delta_{n''-1}^{m_i}$ be the set whose superscript is greatest among those to which a wff is added. Then $\neg\Box\neg\mathrm{con}(\Delta_{n''}^{m_i}) \in \Delta_{n''}^{m_{i-1}}, \ldots, \neg\Box\neg\mathrm{con}(\Delta_{n''}^{m_2}) \in \Delta_{n''}^{m_1}$, and $\neg\Box\neg\mathrm{con}(\Delta_{n''}^{m_1}) \in \Delta_{n''}^0$.

Assume that $\Delta_n^0, \ldots, \Delta_n^m$ are all S-consistent* but one of $\Delta_{n+1}^0, \ldots, \Delta_{n+1}^m$ (or one of $\Delta_{n+1}^0, \ldots, \Delta_{n+1}^{m+1}$ if we have the second subcase of Case 1 or Subcase 4b) is not. I will only consider the nontrivial cases.

Case 1 Consider the subcase in which there are no wffs $\mathbf{B}_1, \ldots, \mathbf{B}_i \in \Delta_n^0$ such that $\vDash^*_S \mathbf{B}_1 \wedge \cdots \wedge \mathbf{B}_i \rightarrow \Box\mathbf{A}$. Then Δ_{n+1}^0 or Δ_{n+1}^{m+1} is S-inconsistent*. In either event, there are wffs $\mathbf{B}_1, \ldots, \mathbf{B}_i \in \Delta_n^0$ such that

$$\vDash^*_S \mathbf{B}_1 \wedge \cdots \wedge \mathbf{B}_i \rightarrow \Box[(\forall x_1)\mathbf{F}x_1 \rightarrow \mathbf{A}].$$

Hence $\vDash^*_S \Box(\forall x_1)\mathbf{F}x_1 \rightarrow [\mathbf{B}_1 \wedge \cdots \wedge \mathbf{B}_i \rightarrow \Box\mathbf{A}]$. Thus, by QKR5, $\vDash^*_S \mathbf{B}_1 \wedge \cdots \wedge \mathbf{B}_i \rightarrow \Box\mathbf{A}$, contrary to the characteristics of the subcase.

Case 2 Consider the subcase in which there are no wffs $\mathbf{B}_1, \ldots, \mathbf{B}_i \in \Delta_n^0$ such that $\vDash^*_S \mathbf{B}_1 \wedge \cdots \wedge \mathbf{B}_i \rightarrow (\forall \mathbf{x})\neg[\mathbf{A} \rightarrow \mathbf{A}]$. Then Δ_{n+1}^0 is not S-consistent*, so there are wffs $\mathbf{B}_1, \ldots, \mathbf{B}_i \in \Delta_n^0$ such that

$$\vDash^*_S \mathbf{B}_1 \wedge \cdots \wedge \mathbf{B}_i \rightarrow [\mathbf{Fy} \rightarrow \dot{\mathbf{S}}^{\mathbf{x}}_{\mathbf{y}}\mathbf{A} \wedge \neg(\forall \mathbf{x})\mathbf{A}].$$

Hence, by QKR3, ST5*, and QKA4,

$$\vDash^*_S \mathbf{B}_1 \wedge \cdots \wedge \mathbf{B}_i \rightarrow [(\forall \mathbf{y})\mathbf{Fy} \rightarrow (\forall \mathbf{y})[\dot{\mathbf{S}}^{\mathbf{x}}_{\mathbf{y}}\mathbf{A} \wedge \neg(\forall \mathbf{x})\mathbf{A}]].$$

Thus

$$\vDash^*_S (\forall \mathbf{y})\mathbf{Fy} \rightarrow [\mathbf{B}_1 \wedge \cdots \wedge \mathbf{B}_i \rightarrow (\forall \mathbf{y})\dot{\mathbf{S}}^{\mathbf{x}}_{\mathbf{y}}\mathbf{A} \wedge (\forall \mathbf{y})\neg(\forall \mathbf{x})\mathbf{A}].$$

So, by QKR5, ST10*, and ST15*,

$$\vDash^*_S \mathbf{B}_1 \wedge \cdots \wedge \mathbf{B}_i \rightarrow (\forall \mathbf{x})\mathbf{A} \wedge [\neg(\forall \mathbf{x})\mathbf{A} \vee (\forall \mathbf{x})\neg[\mathbf{A} \rightarrow \mathbf{A}]].$$

Therefore, $\vDash^*_S \mathbf{B}_1 \wedge \cdots \wedge \mathbf{B}_i \rightarrow (\forall \mathbf{x})\neg[\mathbf{A} \rightarrow \mathbf{A}]$, contrary to the characteristics of the subcase.

Subcase 4a At least one of $\Delta_{n+1}^0, \Delta_{n+1}^{m_1}, \ldots, \Delta_{n+1}^{m_k}$ is not S-consistent*. Let $\Delta_{n+1}^{m_1}$ be S-inconsistent*, for example. Then

$$\vDash^*_S \operatorname{con}(\Delta_n^{m_1}) \rightarrow \Box[\cdots \rightarrow \Box[\operatorname{con}(\Delta_n^{m_k}) \rightarrow \neg\Box\mathbf{C}]\ldots].$$

So

$$\vDash^*_S \mathbf{D}_1 \wedge \cdots \wedge \mathbf{D}_i \rightarrow \Box[\operatorname{con}(\Delta_n^{m_1}) \rightarrow \Box[\cdots \rightarrow \Box[\operatorname{con}(\Delta_n^{m_k}) \rightarrow \neg\Box\mathbf{C}]\ldots]].$$

Hence, if $k > 1$, then by repeated use of QKA1, QKR2, and QKA2

$$\vDash^*_S \mathbf{D}_1 \wedge \cdots \wedge \mathbf{D}_i \rightarrow \Box[\operatorname{con}(\Delta_n^{m_1}) \rightarrow \Box[\cdots \rightarrow \Box[\operatorname{con}(\Delta_n^{m_{k-1}}) \rightarrow \Box\neg\operatorname{con}(\Delta_n^{m_k})]\ldots]].$$

But, since $\neg\Box\neg\operatorname{con}(\Delta_n^{m_k}) \in \Delta_n^{m_{k-1}}$,

$$\vDash^*_S \mathbf{D}_1 \wedge \cdots \wedge \mathbf{D}_i \rightarrow \Box[\operatorname{con}(\Delta_n^{m_1}) \rightarrow \Box[\cdots \rightarrow \Box[\operatorname{con}(\Delta_n^{m_{k-1}}) \rightarrow \neg\Box\neg\operatorname{con}(\Delta_n^{m_k})]\ldots]].$$

Going on in the same way, we conclude that $\vDash^*_S \mathbf{D}_1 \wedge \cdots \wedge \mathbf{D}_i \rightarrow \Box\neg\operatorname{con}(\Delta_n^{m_1})$, contrary to the S-consistency* of Δ_n^0.

Subcase 4b At least one of $\Delta_{n+1}^0, \Delta_{n+1}^{m_1}, \ldots, \Delta_{n+1}^{m_k}, \Delta_{n+1}^{m+1}$ is not S-consistent*. In any event, there are wffs $\mathbf{D}_1, \ldots, \mathbf{D}_i \in \Delta_n^0$ such that

$$\vDash^*_S \mathbf{D}_1 \wedge \cdots \wedge \mathbf{D}_i \rightarrow \Box[\operatorname{con}(\Delta_n^{m_1}) \rightarrow \Box[\cdots \rightarrow \Box[\operatorname{con}(\Delta_n^{m_k}) \rightarrow \Box[(\forall x_1)\mathbf{F}x_1 \rightarrow \mathbf{C}]]\ldots]].$$

So, if $k > 1$,

$$\vDash^*_S \mathbf{D}_1 \wedge \cdots \wedge \mathbf{D}_i \rightarrow \Box[\operatorname{con}(\Delta_n^{m_1}) \rightarrow \Box[\cdots \rightarrow \Box[\operatorname{con}(\Delta_n^{m_{k-1}}) \rightarrow \Box[\Box(\forall x_1)\mathbf{F}x_1 \rightarrow [\operatorname{con}(\Delta_n^{m_k}) \rightarrow \Box\mathbf{C}]]]\ldots]].$$

Proceeding in the same manner, we infer that

$$\vDash^*_S \underbrace{\Box\cdots\Box}_{k+1\text{ times}}(\forall x_1)\mathbf{F}x_1 \rightarrow [\mathbf{D}_1 \wedge \cdots \wedge \mathbf{D}_i \rightarrow \Box[\operatorname{con}(\Delta_n^{m_1}) \rightarrow \Box[\cdots \rightarrow \Box[\operatorname{con}(\Delta_n^{m_k}) \rightarrow \Box\mathbf{C}]\ldots]]].$$

Therefore,

$$\vdash^*_S \mathbf{D}_1 \wedge \cdots \wedge \mathbf{D}_i \to \Box[\mathrm{con}(\Delta_n^{m_1}) \to \Box[\cdots \to \Box[\mathrm{con}(\Delta_n^{m_k}) \to \Box\mathbf{C}]\ldots]],$$

contrary to the characteristics of the subcase.

Subcase 4c This subcase is similar to (4a).

Subcase 4d Since at least one of $\Delta^0_{n+1}, \Delta^{m_1}_{n+1}, \ldots, \Delta^{m_k}_{n+1}$ will not be S-consistent*, we can say that, for some wffs $\mathbf{D}_1, \ldots, \mathbf{D}_i \in \Delta^0_n$,

$$\vdash^*_S \mathbf{D}_1 \wedge \cdots \wedge \mathbf{D}_i \to \Box[\mathrm{con}(\Delta_n^{m_1}) \to \Box[\cdots \to \Box[\mathrm{con}(\Delta_n^{m_k}) \to [\mathbf{Fy} \to \dot{\mathbf{S}}^{\mathbf{x}}_{\mathbf{y}}\mathbf{C} \wedge \neg(\forall\mathbf{x})\mathbf{C}]]\ldots]].$$

Hence, by QKR4 and ST5*,

$$\vdash^*_S \mathbf{D}_1 \wedge \cdots \wedge \mathbf{D}_i \to \Box[\mathrm{con}(\Delta_n^{m_1}) \to \Box[\cdots \to \Box[\mathrm{con}(\Delta_n^{m_k}) \to (\forall\mathbf{y})[\mathbf{Fy} \to \dot{\mathbf{S}}^{\mathbf{x}}_{\mathbf{y}}\mathbf{C} \wedge \neg(\forall\mathbf{x})\mathbf{C}]]\ldots]]$$

and so, if $k > 1$,

$$\vdash^*_S \mathbf{D}_1 \wedge \cdots \wedge \mathbf{D}_i \to \Box[\mathrm{con}(\Delta_n^{m_1}) \to \Box[\cdots \to \Box[\mathrm{con}(\Delta_n^{m_{k-1}}) \to \Box[(\forall\mathbf{y})\mathbf{Fy} \to [\mathrm{con}(\Delta_n^{m_k}) \to (\forall\mathbf{y})\dot{\mathbf{S}}^{\mathbf{x}}_{\mathbf{y}}\mathbf{C} \wedge (\forall\mathbf{y})\neg(\forall\mathbf{x})\mathbf{C}]]]\ldots]].$$

Proceeding as we did at a similar point in Subcase 4b, we infer that

$$\vdash^*_S \underbrace{\Box\cdots\Box}_{k\text{ times}}(\forall\mathbf{y})\mathbf{Fy} \to [\mathbf{D}_1 \wedge \cdots \wedge \mathbf{D}_i \to \Box[\mathrm{con}(\Delta_n^{m_1}) \to \Box[\cdots \to \Box[\mathrm{con}(\Delta_n^{m_k}) \to (\forall\mathbf{y})\dot{\mathbf{S}}^{\mathbf{x}}_{\mathbf{y}}\mathbf{C} \wedge (\forall\mathbf{y})\neg(\forall\mathbf{x})\mathbf{C}]\ldots]]].$$

Therefore,

$$\vdash^*_S \mathbf{D}_1 \wedge \cdots \wedge \mathbf{D}_i \to \Box[\mathrm{con}(\Delta_n^{m_1}) \to \Box[\cdots \to \Box[\mathrm{con}(\Delta_n^{m_k}) \to (\forall\mathbf{y})\dot{\mathbf{S}}^{\mathbf{x}}_{\mathbf{y}}\mathbf{C} \wedge (\forall\mathbf{y})\neg(\forall\mathbf{x})\mathbf{C}]\ldots]]$$

and so,

$$\vdash^*_S \mathbf{D}_1 \wedge \cdots \wedge \mathbf{D}_i \to \Box[\mathrm{con}(\Delta_n^{m_1}) \to \Box[\cdots \to \Box[\mathrm{con}(\Delta_n^{m_k}) \to (\forall\mathbf{x})\mathbf{C} \wedge [\neg(\forall\mathbf{x})\mathbf{C} \vee (\forall\mathbf{x})\neg[\mathbf{C} \to \mathbf{C}]]]\ldots]].$$

Hence,

$$\vdash^*_S \mathbf{D}_1 \wedge \cdots \wedge \mathbf{D}_i \to \Box[\mathrm{con}(\Delta_n^{m_1}) \to \Box[\cdots \to \Box[\mathrm{con}(\Delta_n^{m_k}) \to (\forall\mathbf{x})\neg[\mathbf{C} \to \mathbf{C}]]\ldots]],$$

contrary to the characteristics of the subcase.

We have thus shown that all Δ_n^m are S-consistent*. I will now define the three sequences

$$\begin{array}{ccccc} \Delta^0 & & \Theta^0 & & \Gamma^0 \\ \Delta^1 & & \Theta^1 & & \Gamma^1 \\ \Delta^2 & \text{and} & \Theta^2 & \text{and} & \Gamma^2 \\ \vdots & & \vdots & & \vdots \end{array}$$

(The sequences here may be finite ones.)

Δ^0 will be the union of $\Delta^0_0, \Delta^0_1, \Delta^0_2, \ldots$; for every i such that the matrix previously defined has a row whose superscript is i, Δ^i will be the union of $\Delta^i_1, \Delta^i_2, \ldots$.

Each one of $\Delta^0, \Delta^1, \Delta^2, \ldots$ is S-consistent*, has the $\forall$-property, and contains a wff of the form $(\forall\mathbf{x})\mathbf{Fx}$.

$\Theta^0 = \Delta^0$. If ψ^i is $\langle\Box\mathbf{A}\rangle$, Θ^i will be $\Delta^i \cup \{\mathbf{B} : \Box\mathbf{B} \in \Delta^0\}$. If ψ^i is $\langle\Box\mathbf{A}_1, \ldots, \Box\mathbf{A}_k\rangle$ where $k > 1$, let $\langle\Box\mathbf{A}_1, \ldots, \Box\mathbf{A}_{k-1}\rangle$ have been ψ^j; then Θ^i will be $\Delta^i \cup \{\mathbf{B} : \Box\mathbf{B} \in \Delta^j\}$.

Assume that some Θ^i is not S-consistent*, and consider the case in which ψ^i is $\langle\Box\mathbf{A}_1, \ldots, \Box\mathbf{A}_k\rangle$ where $k > 1$. Then, for some wffs $\mathbf{B}_1, \ldots, \mathbf{B}_l$ such that $\Box\mathbf{B}_1, \ldots, \Box\mathbf{B}_l \in \Delta^j$ and some wffs $\mathbf{C}_1, \ldots, \mathbf{C}_h \in \Delta^i$, $\vdash^*_S \mathbf{B}_1 \wedge \cdots \wedge \mathbf{B}_l \to \neg[\mathbf{C}_1 \wedge \cdots \wedge \mathbf{C}_h]$. Hence

$$\vdash^*_S \Box\mathbf{B}_1 \wedge \cdots \wedge \Box\mathbf{B}_l \to \Box\neg[\mathbf{C}_1 \wedge \cdots \wedge \mathbf{C}_h].$$

Now, let $\langle\Box\mathbf{A}_1, \ldots, \Box\mathbf{A}_k\rangle$ have been φ_r. Then, in the matrix defined previously, the row whose superscript is i was introduced at the stage of φ_r. Among $\Delta^i_r, \Delta^i_{r+1}, \ldots$ there will be a set Δ^i_n that contains all of $\mathbf{C}_1, \ldots, \mathbf{C}_h$. (Actually, there are infinitely many such sets.) $\neg\Box\neg\text{con}(\Delta^i_n)$ will belong to Δ^j_n. For $\neg\Box\neg\text{con}(\Delta^i_r) \in \Delta^j_r$ and, for any r' such that $r < r' \leq n$, even if $\Delta^i_{r'}$ is not the same as $\Delta^i_{r'-1}$ still $\neg\Box\neg\text{con}(\Delta^i_{r'}) \in \Delta^j_{r'}$. But we have both

$$\vdash^*_S \Box\mathbf{B}_1 \wedge \cdots \wedge \Box\mathbf{B}_l \wedge \neg\Box\neg\text{con}(\Delta^i_n) \to \neg\Box\neg[\mathbf{C}_1 \wedge \cdots \wedge \mathbf{C}_h]$$

and

$$\vdash^*_S \Box\mathbf{B}_1 \wedge \cdots \wedge \Box\mathbf{B}_l \wedge \neg\Box\neg\text{con}(\Delta^i_n) \to \Box\neg[\mathbf{C}_1 \wedge \cdots \wedge \mathbf{C}_h].$$

That contradicts the S-consistency* of Δ^j. Things are similar if ψ^i is $\langle\Box\mathbf{A}\rangle$. Thus $\Theta^0, \Theta^1, \Theta^2, \ldots$ are all S-consistent*.

A standard method extends every Θ^m to the maximal S-consistent* set Γ^m. Note that, for any $\mathbf{B}$, if $\Box\mathbf{B} \in \Gamma^m$ then $\Box\mathbf{B} \in \Delta^m$. For if $\Box\mathbf{B} \notin \Delta^m$ then Δ^m will contain, for some $\mathbf{F}$, the wff abbreviated as $\Diamond[\neg\mathbf{B} \wedge (\forall x_1)\mathbf{F}x_1]$ or the wff abbreviated as $\Diamond[(\forall x_1)\mathbf{F}x_1 \wedge \neg\mathbf{B}]$ and so, by ST4*, Γ^m will contain the wff abbreviated as $\Diamond\neg\mathbf{B}$. The set $\{\Gamma^0, \Gamma^1, \Gamma^2, \ldots\}$ is S-full and starts at Γ^0. $\Box$

A model $\langle W, R, D, Q, w^*, V\rangle$ is *S-canonical* just in case the following conditions are satisfied: W is an S-full set starting at w^*; for all worlds w and w', wRw' if and only if $\{\mathbf{A} : \Box\mathbf{A} \in w\} \subseteq w'$; D is the set of variables and individual constants; for every world w and every $\mathbf{b}$, $wQ\mathbf{b}$ if and only if w contains an atomic wff in which $\mathbf{b}$ occurs; for every $\mathbf{a}$, $V(\mathbf{a}) = \mathbf{a}$; finally, for every predicate letter F^i_j, every world w, and all $\mathbf{b}_1, \ldots, \mathbf{b}_i$ that belong to the domain of w, $\langle\mathbf{b}_1, \ldots, \mathbf{b}_i\rangle \in V(F^i_j, w)$ if and only if $F^i_j\mathbf{b}_1 \ldots \mathbf{b}_i \in w$. The following is a corollary of (3.9) and (3.10).

Metatheorem 3.11 *If Λ is an S-consistent, finite*, and nonempty set of wffs, there is an S-canonical model $\langle W, R, D, Q, w^*, V\rangle$ in which $w^* \supseteq \Lambda$.*

We can now demonstrate (3.12).

Metatheorem 3.12 *It is the case in any S-canonical model that, for every $\mathbf{A}$ and every world w, $(x_1, x_2, \ldots)$ satisfies $\mathbf{A}$ at w if and only if $\mathbf{A} \in w$.*

Proof As usual, the proof proceeds by induction on the number of the occurrences of $\neg$, $\to$, $\Box$, and $\forall$ in $\mathbf{A}$. Let s be $(x_1, x_2, \ldots)$.

If $\mathbf{A}$ is $F^i_j\mathbf{b}_1 \ldots \mathbf{b}_i$, then s satisfies $\mathbf{A}$ at w just in case $\langle\mathbf{b}_1, \ldots, \mathbf{b}_i\rangle \in V(F^i_j, w)$. If $\mathbf{b}_1, \ldots, \mathbf{b}_i \in D_w$, the case follows from the definition of V; otherwise,

$\langle \mathbf{b}_1, \ldots, \mathbf{b}_i \rangle \notin V(F^i_j, w)$, but also $\mathbf{A} \notin w$. If $\mathbf{A}$ is $\neg\mathbf{B}$, $\mathbf{B} \to \mathbf{C}$, or $\Box\mathbf{B}$, the case follows in a standard manner from the definition of an S-full set. There are two more hypotheses to consider.

(a) $\mathbf{A}$ is $(\forall x_i)\mathbf{B}$ and belongs to a world w. Let s' be a sequence that differs from s in at most the ith position and is such that wQs'_i. Then there is an atomic wff $\mathbf{C} \in w$ in which s'_i occurs.

Case 1 s'_i is free for x_i in $\mathbf{B}$. Then, by QKA3 and (3.8), $\dot{\mathbf{S}}^{x_i}_{s'_i}\mathbf{B} \in w$. Hence, by the inductive hypothesis, s satisfies $\dot{\mathbf{S}}^{x_i}_{s'_i}\mathbf{B}$ at w, so by (3.2) s' satisfies $\mathbf{B}$ at w.

Case 2 s'_i is not free for x_i in $\mathbf{B}$. Let $\mathbf{y}$ be a variable that does not occur in $\mathbf{A}$. Then, by ST11*, $\bar{\mathbf{S}}^{s'_i}_{\mathbf{y}}\mathbf{A}$, that is, $(\forall x_i)\bar{\mathbf{S}}^{s'_i}_{\mathbf{y}}\mathbf{B}$, belongs to w. So $\dot{\mathbf{S}}^{x_i}_{s'_i}\bar{\mathbf{S}}^{s'_i}_{\mathbf{y}}\mathbf{B} \in w$. Hence s satisfies $\dot{\mathbf{S}}^{x_i}_{s'_i}\bar{\mathbf{S}}^{s'_i}_{\mathbf{y}}\mathbf{B}$ at w, and thus s' satisfies $\bar{\mathbf{S}}^{s'_i}_{\mathbf{y}}\mathbf{B}$ at w. Therefore, by (3.3), s' satisfies $\mathbf{B}$ at w. Since, whichever of the two cases may obtain, s' satisfies $\mathbf{B}$ at w, s satisfies $(\forall x_i)\mathbf{B}$ at w.

(b) $\mathbf{A}$ is $(\forall x_i)\mathbf{B}$ and is satisfied by s at a world w. Then, since w has the $\forall$-property, we can distinguish the following two cases.

Case 1 $(\forall x_i)\neg[\mathbf{B} \to \mathbf{B}] \in w$. Then $(\forall x_i)\mathbf{B} \in w$, for $\vdash^*_S (\forall x_i)\neg[\mathbf{B} \to \mathbf{B}] \to (\forall x_i)\mathbf{B}$.

Case 2 There are a variable $\mathbf{y}$, not occurring in $\mathbf{B}$, and a predicate letter $\mathbf{F}$ of degree 1 such that w contains both $\dot{\mathbf{S}}^{x_i}_{\mathbf{y}}\mathbf{B} \to (\forall x_i)\mathbf{B}$ and $\mathbf{F}\mathbf{y}$. Then $wQ\mathbf{y}$. Let s' be a sequence which differs from s in at most the ith position and is such that s'_i is $\mathbf{y}$. As s' will satisfy $\mathbf{B}$ at w, s satisfies $\dot{\mathbf{S}}^{x_i}_{\mathbf{y}}\mathbf{B}$ at w. Hence $\dot{\mathbf{S}}^{x_i}_{\mathbf{y}}\mathbf{B} \in w$, and so $(\forall x_i)\mathbf{B} \in w$. □

(3.11) and (3.12) imply (3.13), which is the central result about completeness.

Metatheorem 3.13 *If* $\mathbf{A}$ *is true in every* S*-canonical model, then* $\vdash_S \mathbf{A}$.

Proof If $\mathbf{A}$ is true in every S-canonical model, so is clo($\mathbf{A}$). Now assume that $\nvdash_S \mathbf{A}$. Then, by QKA6, $\nvdash_S$ clo($\mathbf{A}$), so $\{\neg$clo($\mathbf{A}$)$\}$ is S-consistent. Thus, by (3.11), there is an S-canonical model whose actual world, w^*, contains $\neg$clo($\mathbf{A}$). Hence $(x_1, x_2, \ldots)$ satisfies $\neg$clo($\mathbf{A}$) at w^*, and so does not satisfy clo($\mathbf{A}$) at w^*. Take any individual constant $\mathbf{a}$. We know that $\mathbf{a} \in D_{w^*}$, and by (3.1) the sequence all of whose positions are occupied by $\mathbf{a}$ fails to satisfy clo($\mathbf{A}$) at w^*. Then clo($\mathbf{A}$) is not true in the S-canonical model we are considering. □

The completeness of QK regarding validity in all structures is a corollary of (3.13).

Metatheorem 3.14 *If* $\mathbf{A}$ *is valid in all structures,* $\vdash_{\mathrm{QK}} \mathbf{A}$.

As for the completeness of QK regarding validity in all the structures that meet condition (5), it follows from (3.14) and the next metatheorem.

Metatheorem 3.15 *If a wff* $\mathbf{A}$ *is true in every model that is based on a structure meeting (5), then it is true in every model.*

Proof Let $\langle W, R, D, Q, w^*, V \rangle$ be a model in which a sequence s of individuals from the domain of w^* does not satisfy $\mathbf{A}$ at w^*. Consider the model $\langle W, R, D', Q, w^*, V \rangle$ where $D' = D - \{d : d \in D$ but there is no $w \in W$ such that $wQd\}$. This model is based on a structure meeting condition (5), and in it, by (3.5), s does not satisfy $\mathbf{A}$ at w^*. □

Let us turn to the axiomatic systems that extend QK but do not introduce a sign of identity. It is now easy to establish the soundness and completeness of QD, QT, QS4, QB, and QS5. Their soundness can be derived from (3.6). For example, in the case of QT we must show that the instances of the schema QTA are W-true in every model that is based on a reflexive structure. The completeness can be derived from (3.13). In the case of QT, say, we must show that every QT-canonical model is based on a reflexive structure. The details are quite standard.

The soundness of QK1, QK2, QK3, QK4, and QK1/2 also follows from (3.6). For each QKi ($1 \leq i \leq 4$) we need only show that the instances of QiA are W-true in every model that is based on a structure meeting condition (i). Showing that suffices even for proving the soundness of QK1/2, since every structure that meets condition (6) also meets conditions (1) and (2). The completeness of QK1, QK2, QK3, QK4, and QK1/2 is a consequence of (3.13) and the next three metatheorems.

Metatheorem 3.16 *Each* QK1*-canonical model is based on a structure that satisfies condition (1). Each* QK2*-canonical model is based on a structure that satisfies condition (2). Each* QK3*-canonical model is based on a structure that satisfies condition (3).*

Proof For illustration I shall deal with the case of QK3. What we must show is that, in any QK3-canonical model, if $\{\mathbf{A} : \Box\mathbf{A} \in w\} \subseteq w'$ and $\{\mathbf{A} : \Box\mathbf{A} \in w'\} \subseteq w''$ while w contains an atomic wff $\mathbf{B}$ in which $\mathbf{b}$ occurs and w'' contains an atomic wff $\mathbf{C}$ in which $\mathbf{b}$ occurs, then w' contains such a wff too. $\neg\Box\neg\mathbf{C}$ will belong to w', for otherwise $\Box\neg\mathbf{C} \in w'$ and so $\neg\mathbf{C} \in w''$, contrary to the QK3-consistency* of w''. We know that w', like each world in an S-canonical model, contains a wff $(\forall\mathbf{x})\mathbf{Fx}$. By Q3A and (3.8), $\Box[\neg\Box\neg\mathbf{C} \to [(\forall\mathbf{x})\mathbf{Fx} \to \mathbf{Fb}]] \in w$. Therefore, $\mathbf{Fb} \in w'$. □

Metatheorem 3.17 *Each* QK4*-canonical model is based on a structure that satisfies condition (4).*

Proof We must show that, in any QK4-canonical model, each world w contains an atomic wff. We know that w has the $\forall$-property. Take any wff $(\forall\mathbf{x})\mathbf{A}$. $\neg(\forall\mathbf{x})\neg[\mathbf{A} \to \mathbf{A}]$ belongs to w, so $(\forall\mathbf{x})\neg[\mathbf{A} \to \mathbf{A}] \notin w$. Hence there are a variable $\mathbf{y}$ and a predicate letter $\mathbf{F}$ of degree 1 such that $\mathbf{Fy} \in w$. □

Metatheorem 3.18 *Each* QK1/2*-canonical model is based on a structure that satisfies condition (6).*

Proof Proceeding as in the proof of (3.16), we can show that each QK1/2-canonical model is based on a structure that meets conditions (1) and (2). Now, let w and w' be any worlds in a QK1/2-canonical model. We know that w, as well as w', is finitely many steps away from w^* (the actual world of the model). Thus, if wQd, then by condition (2), w^*Qd, and so by condition (1), $w'Qd$. □

It is worth seeing how the Converse Barcan Formula and the Barcan Formula can be derived in QK1 and QK2.

QK1T1* $\Box(\forall\mathbf{x})\mathbf{A} \to (\forall\mathbf{x})\Box\mathbf{A}$.

Proof Let $\mathbf{F}$ be a predicate letter of degree 1 which does not occur in $\mathbf{A}$. By Q1A we have $\mathbf{Fx} \to \Box[(\forall\mathbf{x})\mathbf{A} \to \mathbf{A}]$. Therefore $\mathbf{Fx} \to [\Box(\forall\mathbf{x})\mathbf{A} \to \Box\mathbf{A}]$, so

$(\forall \mathbf{x})\mathbf{Fx} \to (\forall \mathbf{x})[\Box(\forall \mathbf{x})\mathbf{A} \to \Box\mathbf{A}]$. Thus, by QKR5, $(\forall \mathbf{x})[\Box(\forall \mathbf{x})\mathbf{A} \to \Box\mathbf{A}]$. Hence, by ST5*, $\Box(\forall \mathbf{x})\mathbf{A} \to (\forall \mathbf{x})\Box\mathbf{A}$. □

QK2T1* $(\forall \mathbf{x})\Box\mathbf{A} \to \Box(\forall \mathbf{x})\mathbf{A}$.

Proof Let $\mathbf{F}$ be a predicate letter of degree 1 which does not occur in $\mathbf{A}$. By Q2A we get $(\forall \mathbf{x})\Box\mathbf{A} \to [\Diamond\mathbf{Fx} \to \Box\mathbf{A}]$. Thus, by ST3*, $(\forall \mathbf{x})\Box\mathbf{A} \to \Box[\mathbf{Fx} \to \mathbf{A}]$. Hence, by QKR4, $(\forall \mathbf{x})\Box\mathbf{A} \to \Box(\forall \mathbf{x})[\mathbf{Fx} \to \mathbf{A}]$, so $(\forall \mathbf{x})\Box\mathbf{A} \to [\Box(\forall \mathbf{x})\mathbf{Fx} \to \Box(\forall \mathbf{x})\mathbf{A}]$. Thus $\Box(\forall \mathbf{x})\mathbf{Fx} \to [(\forall \mathbf{x})\Box\mathbf{A} \to \Box(\forall \mathbf{x})\mathbf{A}]$. Finally, use QKR5. □

We should now see why the theorems of QK (or those of QD, QT, QS4, QB, QS5, QK1, QK2, QK3, QK4, or QK1/2) which belong to the language $\mathcal{L}'$ (the nonmodal part of $\mathcal{L}$) are just the wffs that make up the classical predicate calculus without identity when this calculus is formulated in $\mathcal{L}'$.

Metatheorem 3.19 *If* $\vdash_S \mathbf{A}$ *then* $\vdash_S (\forall \mathbf{x})\mathbf{A}$.

Proof As we know, a proof in S consists of two parts; the second part, which is empty if QKA6 is not used in the proof, but which otherwise begins with the first time QKA6 is used, relies at most on the first part, on QKA6, and on QKR1. Now, consider that we prefix $(\forall \mathbf{x})$ to each step in a proof of $\mathbf{A}$. Then the wffs in the first part become theorems of S. By ST8 the instances of QKA6 also become theorems of S. Finally, if $\vdash_S (\forall \mathbf{x})[\mathbf{B} \to \mathbf{C}]$ and $\vdash_S (\forall \mathbf{x})\mathbf{B}$, then $\vdash_S (\forall \mathbf{x})\mathbf{C}$: if we begin with $(\forall \mathbf{x})[\mathbf{B} \to \mathbf{C}] \to [(\forall \mathbf{x})\mathbf{B} \to (\forall \mathbf{x})\mathbf{C}]$, continue with the first part of a proof of $(\forall \mathbf{x})[\mathbf{B} \to \mathbf{C}]$ and the first part of a proof of $(\forall \mathbf{x})\mathbf{B}$, add the second part of the proof of $(\forall \mathbf{x})[\mathbf{B} \to \mathbf{C}]$ and the second part of the proof of $(\forall \mathbf{x})\mathbf{B}$, and conclude with two appropriate applications of modus ponens, we get a proof of $(\forall \mathbf{x})\mathbf{C}$. □

(3.19) is essential for demonstrating (3.20).

Metatheorem 3.20 *If* $\mathbf{A}$ *is one of the wffs that make up the classical predicate calculus without identity when this calculus is formulated in* $\mathcal{L}'$*, then* $\vdash_S \mathbf{A}$.

Proof We can axiomatize classical first-order logic without identity by restricting our metalinguistic variables to $\mathcal{L}'$ and using QKA1, QKA6, ST5*, QKR1, and QKR3. We can then see that all the axioms will be theorems of S and that the primitive rules of inference will preserve theoremhood in S. □

We know that a classical interpretation for the language $\mathcal{L}'$ will be a pair $\langle \mathcal{D}, \mathcal{I} \rangle$ where $\mathcal{D}$ is a nonempty set and $\mathcal{I}$ is a function that assigns to each individual constant $\mathbf{a}$ an element of $\mathcal{D}$, and to each predicate letter F^i_j a subset of $\mathcal{D}^i$. If $\langle W, R, D, Q, w^*, V \rangle$ is a model such that $D_{w^*} = \mathcal{D}$, $V(\mathbf{a}) = \mathcal{I}(\mathbf{a})$ for every $\mathbf{a}$, and $V(F^i_j, w^*) = \mathcal{I}(F^i_j)$ for every F^i_j, then it is easy to prove inductively the following theorem.

Metatheorem 3.21 *For any wff* $\mathbf{A}$ *of* $\mathcal{L}'$ *and any sequence* s *of elements of* $\mathcal{D}$*,* s *satisfies* $\mathbf{A}$ *at* w^* *in* $\langle W, R, D, Q, w^*, V \rangle$ *if and only if* s *satisfies* $\mathbf{A}$ *in* $\langle \mathcal{D}, \mathcal{I} \rangle$.

As a consequence, we have (3.22).

Metatheorem 3.22 *If* $\mathbf{A}$ *is a wff of* $\mathcal{L}'$ *and* $\vdash_{\mathrm{QK}} \mathbf{A}$ *then* $\mathbf{A}$ *is one of the wffs that make up the classical predicate calculus without identity when this calculus is formulated in* $\mathcal{L}'$.

Proof By (3.7), since $\vdash_{\mathrm{QK}}$ **A**, **A** is true in every model. Now, if **A** is not part of classical first-order logic without identity, there will be a classical interpretation $\langle \mathcal{D}, \mathcal{I} \rangle$ in which a sequence s of elements of $\mathcal{D}$ does not satisfy **A**. Then we can construct a model $\langle W, R, D, Q, w^*, V \rangle$ such that $D_{w^*} = \mathcal{D}$, $V(\mathbf{a}) = \mathcal{I}(\mathbf{a})$ for every **a**, and $V(F^i_j, w^*) = \mathcal{I}(F^i_j)$ for every F^i_j. By (3.21), **A** will not be true in that model. □

For each one of the axiomatic systems QD, QT, QS4, QB, QS5, QK1, QK2, QK3, QK4, and QK1/2, we can prove, in the same manner, an analogue of (3.22).

Turning to the axiomatizations that introduce a sign of identity, we can first prove an analogue of (3.6). For any set Λ of axiom schemata, QIK+Λ will be the axiomatic system that results from QIK by adding those schemata.

Metatheorem 3.23 *If, for any structure, Λ is a set of axiom schemata whose instances are W-true in every identity model based on that structure, the theorems of* QIK + Λ *are I-valid in the structure.*

The proof of (3.23) is similar to that of (3.6) and relies inter alia on the fact that the instances of the schemata QIKA1–3 are W-true in every identity model.

Once we have (3.23), it is easy to establish the soundness of QIK, QID, QIT, QIS4, QIB, QIS5, QIK1, QIK2, QIK3, QIK4, and QIK1/2. Before we demonstrate their completeness, we need to prove some theorems in those systems. Let SI be an arbitrary axiomatization which differs from QIK in at most having additional axiom schemata such that if **A** is an instance of one of those schemata, $\mathbf{a}_1, \ldots, \mathbf{a}_i$ are distinct individual constants, and $\mathbf{y}_1, \ldots, \mathbf{y}_i$ are variables that do not occur in **A**, then $\mathbf{S}^{\mathbf{a}_1,\ldots,\mathbf{a}_i}_{\mathbf{y}_1,\ldots,\mathbf{y}_i}\mathbf{A}$ is also an instance of the relevant schema.

SIT1* $(\forall \mathbf{x})[\mathbf{x} = \mathbf{x}]$.

Proof The proof is by QIKA1 and QKR5. □

SIT2* $\mathbf{A} \to \mathbf{b} = \mathbf{b}$ where **A** is an atomic wff in which **b** occurs.

Proof Let **x** be a variable that does not occur in **A**, and let $\mathbf{A}'$ result from **A** by replacing every occurrence of **b** with **x**. $\mathbf{A}' \to \mathbf{x} = \mathbf{x}$ is an axiom. Hence $(\forall \mathbf{x})[\mathbf{A}' \to \mathbf{x} = \mathbf{x}]$. But $\mathbf{A} \to [(\forall \mathbf{x})[\mathbf{A}' \to \mathbf{x} = \mathbf{x}] \to [\mathbf{A} \to \mathbf{b} = \mathbf{b}]]$ is an instance of QKA3. □

SIT3* $\mathbf{b} = \mathbf{c} \to [\mathbf{A} \to \mathbf{B}]$, where **A** is an atomic wff in which **b** occurs, and **B** results from **A** by substituting **c** for an occurrence of **b**.

Proof Suppose that **b** is other than **c**, for otherwise the theorem is trivial. Let **x** and **y** be distinct variables that do not occur in $\mathbf{b} = \mathbf{c} \to [\mathbf{A} \to \mathbf{B}]$. Let $\mathbf{A}'$ result from **A** by replacing the relevant occurrence of **b** with **x**, and let $\mathbf{B}'$ result from **B** by replacing the relevant occurrence of **c** with **y**. $\mathbf{x} = \mathbf{y} \to [\mathbf{A}' \to \mathbf{B}']$ is an axiom. Hence $(\forall \mathbf{x})(\forall \mathbf{y})[\mathbf{x} = \mathbf{y} \to [\mathbf{A}' \to \mathbf{B}']]$. Thus, by QKA3, $\mathbf{b} = \mathbf{c} \to (\forall \mathbf{y})[\mathbf{b} = \mathbf{y} \to [\mathbf{A} \to \mathbf{B}']]$. But by QKA3 we also have $\mathbf{b} = \mathbf{c} \to [(\forall \mathbf{y})[\mathbf{b} = \mathbf{y} \to [\mathbf{A} \to \mathbf{B}']] \to [\mathbf{b} = \mathbf{c} \to [\mathbf{A} \to \mathbf{B}]]]$. □

SIT4* $\mathbf{b} = \mathbf{c} \to \mathbf{c} = \mathbf{b}$.

Proof By the preceding two theorem schemata we have $\mathbf{b} = \mathbf{c} \to \mathbf{b} = \mathbf{b}$ and $\mathbf{b} = \mathbf{c} \to [\mathbf{b} = \mathbf{b} \to \mathbf{c} = \mathbf{b}]$. □

$SIT5^*$ $\quad \mathbf{b} = \mathbf{c} \to [\mathbf{c} = \mathbf{d} \to \mathbf{b} = \mathbf{d}]$.

$SIT6^*$ $\quad \mathbf{b}_1 = \mathbf{c}_1 \to [\cdots \to [\mathbf{b}_i = \mathbf{c}_i \to [\mathbf{F}\mathbf{b}_1 \ldots \mathbf{b}_i \to \mathbf{F}\mathbf{c}_1 \ldots \mathbf{c}_i]] \ldots]$.

$SIT7^*$ $\quad \Diamond^m[\mathbf{A} \wedge \neg[\mathbf{b} = \mathbf{c}]] \to \Box^n \neg[\mathbf{b} = \mathbf{c}]$ where $\mathbf{A}$ is an atomic wff in which $\mathbf{b}$ or $\mathbf{c}$ occurs.

The *standard order* of variables and individual constants will be $x_1, a_1, x_2, a_2, \ldots$. Given an SI-full set W, we can define the function f from variables and individual constants: for each $\mathbf{b}$, $f(\mathbf{b})$ is the first individual constant or variable $\mathbf{c}$ in the standard order such that, for every world $w \in W$ containing an atomic wff in which $\mathbf{b}$ or $\mathbf{c}$ occurs, $\mathbf{b} = \mathbf{c} \in w$.[9] Note that, for every world $w \in W$ containing an atomic wff in which $\mathbf{b}$ occurs, $\mathbf{b} = \mathbf{b} \in w$.

A model $\langle W, R, D, Q, w^*, V\rangle$ is *SI-identity-canonical* just in case it is an identity model and satisfies the following conditions: W is an SI-full set starting at w^*; for all worlds w and w', wRw' if and only if $\{\mathbf{A} : \Box\mathbf{A} \in w\} \subseteq w'$; D is the range of f; for every world w and every $\mathbf{b} \in D$, $wQ\mathbf{b}$ if and only if w contains an atomic wff in which $\mathbf{b}$ occurs; for every $\mathbf{a}$, $V(\mathbf{a})$ is $f(\mathbf{a})$; finally, for every predicate letter F^i_j other than F^2_1, for every world w, and for all $\mathbf{b}_1, \ldots, \mathbf{b}_i$ that belong to the domain of w, $\langle \mathbf{b}_1, \ldots, \mathbf{b}_i\rangle \in V(F^i_j, w)$ if and only if $F^i_j\mathbf{b}_1 \ldots \mathbf{b}_i \in w$. The following metatheorem can be deduced from (3.9) and (3.10).

Metatheorem 3.24 *If Λ is an SI-consistent, finite*, and nonempty set, there is an SI-identity-canonical model whose actual world is a superset of Λ.*[10]

Proceeding as in the proof of (3.12), we can establish (3.25).

Metatheorem 3.25 *It is the case in any SI-identity-canonical model that, for every* $\mathbf{A}$ *and every world w, $(f(x_1), f(x_2), \ldots)$ satisfies* $\mathbf{A}$ *at w if and only if* $\mathbf{A} \in w$.

Proof I shall only discuss the clauses that differ from their counterparts in the proof of (3.12). Let s be the sequence $(f(x_1), f(x_2), \ldots)$.

(1) $\mathbf{A}$ is $F^i_j\mathbf{b}_1 \ldots \mathbf{b}_i$. Then s satisfies $F^i_j\mathbf{b}_1 \ldots \mathbf{b}_i$ at w if and only if $\langle f(\mathbf{b}_1), \ldots, f(\mathbf{b}_i)\rangle \in V(F^i_j, w)$.

(1a) Either it is not true that $wQf(\mathbf{b}_1)$ or $\cdots$ or it is not true that $wQf(\mathbf{b}_i)$. Then $\langle f(\mathbf{b}_1), \ldots, f(\mathbf{b}_i)\rangle \notin V(F^i_j, w)$ but also $F^i_j\mathbf{b}_1 \ldots \mathbf{b}_i \notin w$. For if $F^i_j\mathbf{b}_1 \ldots \mathbf{b}_i \in w$, then w contains all of $\mathbf{b}_1 = f(\mathbf{b}_1), \ldots, \mathbf{b}_i = f(\mathbf{b}_i)$, in which case $wQf(\mathbf{b}_1)$ and $\cdots$ and $wQf(\mathbf{b}_i)$.

(1b) F^i_j is F^2_1, so $\mathbf{A}$ can be abbreviated as $\mathbf{b}_1 = \mathbf{b}_2$, while $wQf(\mathbf{b}_1)$ and $wQf(\mathbf{b}_2)$. Then $\mathbf{b}_1 = f(\mathbf{b}_1) \in w$ and $\mathbf{b}_2 = f(\mathbf{b}_2) \in w$. Also $\langle f(\mathbf{b}_1), f(\mathbf{b}_2)\rangle \in V(F^2_1, w)$ if and only if $f(\mathbf{b}_1)$ is $f(\mathbf{b}_2)$. Now, if $f(\mathbf{b}_1)$ is $f(\mathbf{b}_2)$, then $\mathbf{b}_1 = f(\mathbf{b}_2) \in w$ and so, since by $SIT4^*$ $f(\mathbf{b}_2) = \mathbf{b}_2 \in w$, by $SIT5^*$ $\mathbf{b}_1 = \mathbf{b}_2 \in w$. On the other hand, if $f(\mathbf{b}_1)$ is other than $f(\mathbf{b}_2)$, let, for example, $f(\mathbf{b}_1)$ appear before $f(\mathbf{b}_2)$ in the standard order. Then there is a world w' which does not contain $\mathbf{b}_2 = f(\mathbf{b}_1)$ but contains an atomic wff, $\mathbf{C}$, in which $\mathbf{b}_2$ or $f(\mathbf{b}_1)$ occurs. We know that w' is, for some m, m steps away from w^* and that w is, for some n, n steps away from w^*.

Thus w^* contains the wff abbreviated as $\Diamond^m[\mathbf{C} \wedge \neg[\mathbf{b}_2 = f(\mathbf{b}_1)]]$ — for if it does not, then it will contain $\Box^m[\mathbf{C} \rightarrow \mathbf{b}_2 = f(\mathbf{b}_1)]$ and so w' will contain $\mathbf{b}_2 = f(\mathbf{b}_1)$. Hence, by $SIT7^*$, w^* contains $\Box^n\neg[\mathbf{b}_2 = f(\mathbf{b}_1)]$. Thus $\neg[\mathbf{b}_2 = f(\mathbf{b}_1)]$ belongs to w and $\mathbf{b}_2 = f(\mathbf{b}_1)$ does not. Therefore, since $\mathbf{b}_1 = f(\mathbf{b}_1) \in w$, $\mathbf{b}_2 = \mathbf{b}_1 \notin w$ and so $\mathbf{b}_1 = \mathbf{b}_2 \notin w$.

(1c) F^i_j is other than F^2_1 while $wQf(\mathbf{b}_1)$ and $\cdots$ and $wQf(\mathbf{b}_i)$. Then $\langle f(\mathbf{b}_1), \ldots, f(\mathbf{b}_i)\rangle \in V(F^i_j, w)$ if and only if $F^i_j f(\mathbf{b}_1)\ldots f(\mathbf{b}_i) \in w$. But w will contain all of $\mathbf{b}_1 = f(\mathbf{b}_1)$, $\ldots$, $\mathbf{b}_i = f(\mathbf{b}_i)$. Hence, by $SIT4^*$ and $SIT6^*$, $F^i_j f(\mathbf{b}_1)\ldots f(\mathbf{b}_i) \in w$ if and only if $F^i_j \mathbf{b}_1\ldots\mathbf{b}_i \in w$.

(2a) $\mathbf{A}$ is $(\forall x_i)\mathbf{B}$ and belongs to w. Let s' be any sequence that differs from s in at most the ith position and is such that wQs'_i. Since $s'_i \in D$, s'_i is $f(\mathbf{b})$ for some $\mathbf{b}$. Thus s'_i is the first individual constant or variable $\mathbf{c}$ such that, for every world w' containing an atomic wff in which $\mathbf{b}$ or $\mathbf{c}$ occurs, $\mathbf{b} = \mathbf{c} \in w'$. Likewise, $f(s'_i)$ is the first individual constant or variable $\mathbf{c}$ such that, for every world w' containing an atomic wff in which s'_i or $\mathbf{c}$ occurs, $s'_i = \mathbf{c} \in w'$. Hence $\mathbf{b} = s'_i \in w$ and $s'_i = f(s'_i) \in w$. Now, if s'_i is other than $f(s'_i)$, then by $SIT2^*$ $f(s'_i)$ comes before s'_i in the standard order, so there is a world w' that does not contain $\mathbf{b} = f(s'_i)$ but contains an atomic wff, $\mathbf{C}$, in which $\mathbf{b}$ or $f(s'_i)$ occurs. Hence, by the same reasoning as we employed in Case 1b above, $\mathbf{b} = f(s'_i) \notin w$. As by $SIT5^*$ that conclusion is unacceptable, s'_i is $f(s'_i)$. The rest of Case (2a) proceeds like clause (a) in the proof of (3.12).

(2b) $\mathbf{A}$ is $(\forall x_i)\mathbf{B}$ and is satisfied at w by s.

Case 1 $(\forall x_i)\neg[\mathbf{B} \rightarrow \mathbf{B}] \in w$. Then $(\forall x_i)\mathbf{B} \in w$.

Case 2 There are a variable $\mathbf{y}$, not occurring in $\mathbf{B}$, and a predicate letter $\mathbf{F}$ of degree 1 such that w contains both $\dot{\mathbf{S}}^{x_i}_{\mathbf{y}}\mathbf{B} \rightarrow (\forall x_i)\mathbf{B}$ and $\mathbf{Fy}$. Then $\mathbf{y} = f(\mathbf{y}) \in w$, so $wQf(\mathbf{y})$. Let s' be the sequence which differs from s in at most the ith position, and in which s'_i is $f(\mathbf{y})$. Then s' satisfies $\mathbf{B}$ at w, so s satisfies $\dot{\mathbf{S}}^{x_i}_{\mathbf{y}}\mathbf{B}$ at w. Hence $\dot{\mathbf{S}}^{x_i}_{\mathbf{y}}\mathbf{B} \in w$, and thus $(\forall x_i)\mathbf{B} \in w$. □ □

From (3.24) and (3.25) we can deduce the following.

Metatheorem 3.26 *If* $\mathbf{A}$ *is true in every* SI*-identity-canonical model, then* $\vdash_{SI} \mathbf{A}$.

The proof of (3.26) is very similar to that of (3.13). Using (3.26), we can easily demonstrate the completeness of the axiomatic systems QIK, QID, QIT, QIS4, QIB, QIS5, QIK1, QIK2, QIK3, QIK4, and QIK1/2: the proof is similar to the completeness proof for their counterparts without identity.

We should now see why the theorems of QIK (or those of QID, QIT, QIS4, QIB, QIS5, QIK1, QIK2, QIK3, QIK4, or QIK1/2) which belong to the language $\mathcal{L}'$ are just the wffs that make up the classical predicate calculus with identity when this calculus is formulated in $\mathcal{L}'$ and the sign of identity is F^2_1.

Metatheorem 3.27 *If* $\mathbf{A}$ *is one of the wffs that make up the classical predicate calculus with identity when this calculus is formulated in* $\mathcal{L}'$ *and the sign of identity is* F^2_1*, then* $\vdash_{SI} \mathbf{A}$.

Proof We can axiomatize classical first-order logic with identity by restricting our metalinguistic variables to $\mathcal{L}'$ and using QKA1, QKA6, $ST5^*$, $SIT1^*$, QIKA2, QKR1, and QKR3. In that axiomatization, all the axioms are theorems of SI, and the primitive rules of inference preserve theoremhood in SI (see Metatheorem 3.19). □

Metatheorem 3.28 *If* $\mathbf{A}$ *is a wff of* $\mathcal{L}'$ *and* $\vdash_{\mathrm{QIK}} \mathbf{A}$*, then* $\mathbf{A}$ *is one of the wffs that make up the classical predicate calculus with identity when this calculus is formulated in* $\mathcal{L}'$ *and the sign of identity is* F_1^2*.*

Proof By the soundness of QIK, since $\vdash_{\mathrm{QIK}} \mathbf{A}$, $\mathbf{A}$ is true in every identity model. Now, if $\mathbf{A}$ is not part of classical first-order logic with identity, there will be a classical interpretation $\langle \mathcal{D}, \mathcal{I} \rangle$ for $\mathcal{L}'$ in which $\mathcal{I}$ assigns the set $\{\langle d, d \rangle : d \in \mathcal{D}\}$ to F_1^2, and in which a sequence s of elements of $\mathcal{D}$ does not satisfy $\mathbf{A}$. Then we can construct an identity model $\langle W, R, D, Q, w^*, V \rangle$ such that $D_{w^*} = \mathcal{D}$, $V(\mathbf{a}) = \mathcal{I}(\mathbf{a})$ for every $\mathbf{a}$, and $V(F_j^i, w^*) = \mathcal{I}(F_j^i)$ for every F_j^i. By (3.21), $\mathbf{A}$ will not be true in that model. □

For each one of the systems QID, QIT, QIS4, QIB, QIS5, QIK1, QIK2, QIK3, QIK4, and QIK1/2, we can similarly prove an analogue of (3.28).

Now, in order to establish compactness and strong completeness, we can extend the language $\mathcal{L}$ to a language $\mathcal{L}^+$ by adding denumerably many individual constants and denumerably many predicate letters of degree 1. The new individual constants will be $a_1^+, a_2^+, \ldots$. Just as the concept of a model was defined in the context of $\mathcal{L}$, so the concept of a *model*$^+$ can be defined in the context of $\mathcal{L}^+$. The axiomatic system QK$^+$ will be just like QK except that it concerns the language $\mathcal{L}^+$. Other model-theoretic and proof-theoretic concepts are also extended to the new language. I shall assume that we have proved the counterparts in $\mathcal{L}^+$ of several metatheorems we have proved above for $\mathcal{L}$. We also need the following two statements.

Metatheorem 3.29 *For any model* $M = \langle W, R, D, Q, w^*, V \rangle$ *and any model*$^+$ $M^+ = \langle W, R, D, Q, w^*, V^+ \rangle$ *where* $V \subset V^+$*, any wff* $\mathbf{A}$ *of* $\mathcal{L}$*, any set* I *of positive integers, any* $w \in W$*, and any sequence* s *of elements of* D *such that, for every* $i \in I$*,* $s_i = V^+(a_i^+)$*:* s *satisfies* $\mathbf{A}$ *at* w *in* M *if and only if* s *satisfies* $\mathbf{E}$ *at* w *in* M^+*, where* $\mathbf{E}$ *is the wff of* $\mathcal{L}^+$ *that results from* $\mathbf{A}$ *when, for every* $i \in I$*, we replace every free occurrence of* x_i *with* a_i^+*.*

Metatheorem 3.30 *For any model* $M = \langle W, R, D, Q, w^*, V \rangle$ *and any model*$^+$ $M^+ = \langle W, R, D, Q, w^*, V^+ \rangle$ *where* $V \subset V^+$*, any wff* $\mathbf{A}$ *of* $\mathcal{L}$*, any* $w \in W$*, and any sequence* s *of elements of* D*:* s *satisfies* $\mathbf{A}$ *at* w *in* M *if and only if* s *satisfies* $\mathbf{A}$ *at* w *in* M^+*.*

(3.29) can be proved by induction on the number of occurrences of $\rightarrow$, $\neg$, $\Box$, and $\forall$ in $\mathbf{A}$, and (3.30) is a corollary of (3.29).

Metatheorem 3.31 *Let* Λ *be any set of wffs of* $\mathcal{L}$*. If, for each finite subset of* Λ*, there is a structure in which that subset is satisfiable, then* Λ *is satisfiable in a structure.*

Proof We assume that Λ is not finite, for the case with finite Λ is trivial. Let Λ^+ be the set of wffs of $\mathcal{L}^+$ which results from Λ when, in each member of Λ and for every i, we replace every free occurrence of x_i with a_i^+.

Take any finite subset Θ^+ of Λ^+, and let Θ be the subset of Λ from whose members we can get all members of Θ^+ by means of the replacement just described.

We know that there is a model $M = \langle W, R, D, Q, w^*, V\rangle$ in which, for some sequence s of elements of D_{w^*}, s satisfies all members of Θ at w^*. It is easy to construct a model$^+$ $M^+ = \langle W, R, D, Q, w^*, V^+\rangle$ where $V^+ \supset V$ and, for every i, $V^+(a_i^+) = s_i$. Then, by (3.29), we infer that in M^+ the sequence s satisfies all members of Θ^+ at w^*. In other words, Θ^+ is satisfiable in a structure.

Suppose that Λ^+ is not QK$^+$-consistent. Then there will be wffs $\mathbf{E}_1, \ldots, \mathbf{E}_j$ of $\mathcal{L}^+$ such that $\mathbf{E}_1, \ldots, \mathbf{E}_j \in \Lambda^+$ and $\vdash_{\mathrm{QK}^+} \neg[\mathbf{E}_1 \wedge \cdots \wedge \mathbf{E}_j]$. So, by the soundness of QK$^+$, $\neg[\mathbf{E}_1 \wedge \cdots \wedge \mathbf{E}_j]$ will be valid in every structure, contrary to the satisfiability of $\{\mathbf{E}_1, \ldots, \mathbf{E}_j\}$ in some structure. Hence Λ^+ is QK$^+$-consistent.

Λ^+ is also a nonempty set, and it is finite* in $\mathcal{L}^+$. Therefore, by the counterparts of (3.11) and (3.12) for $\mathcal{L}^+$, there is a QK$^+$-canonical model$^+$ $M^+ = \langle W, R, D, Q, w^*, V^+\rangle$ in which $(x_1, x_2, \ldots)$ satisfies every member of Λ^+ at w^*. So, by the analogue of (3.1) for $\mathcal{L}^+$, we have that in M^+ the sequence $(a_1^+, a_2^+, \ldots)$, which is a sequence of elements of D_{w^*}, satisfies every member of Λ^+ at w^*. Hence, by repeated applications of the counterpart of (3.2) for $\mathcal{L}^+$, we conclude that, in M^+, $(a_1^+, a_2^+, \ldots)$ satisfies every member of Λ at w^*.

Now consider the model M that results from M^+ when we delete the valuation of the individual constants and predicate letters that extend $\mathcal{L}$ to $\mathcal{L}^+$. Then, by (3.30), in M $(a_1^+, a_2^+, \ldots)$ satisfies every member of Λ at w^*. □

Metatheorem 3.32 *Let* $\mathbf{A}$ *be a wff of* $\mathcal{L}$, *and* Λ *be a set of wffs of* $\mathcal{L}$. *If* $\mathbf{A}$ *is a consequence of* Λ *in every structure, then for some wffs* $\mathbf{B}_1, \ldots, \mathbf{B}_n \in \Lambda$ $(n \geq 0)$ $\vdash_{\mathrm{QK}} \mathbf{B}_1 \wedge \cdots \wedge \mathbf{B}_n \to \mathbf{A}$.

Proof By hypothesis there is no structure in which $\Lambda \cup \{\neg\mathbf{A}\}$ is satisfiable. Hence, by (3.31), for some wffs $\mathbf{B}_1, \ldots, \mathbf{B}_n \in \Lambda$ there is no structure in which $\{\mathbf{B}_1, \ldots, \mathbf{B}_n, \neg\mathbf{A}\}$ is satisfiable. Thus $\mathbf{B}_1 \wedge \cdots \wedge \mathbf{B}_n \to \mathbf{A}$ is valid in every structure. Then invoke the weak completeness of QK. □

(3.31) says that satisfiability in a structure is compact. We can likewise demonstrate that satisfiability in a serial structure is compact, as well as satisfiability in a reflexive structure, in a transitive reflexive one, in a symmetrical reflexive one, in an equivalence structure, in a universal one, in a structure that meets condition (1), in one that meets (2), in a structure meeting (3), in one meeting (4), and in one that meets (6). But in order to demonstrate that satisfiability in a structure meeting condition (5) is compact, we should rely on (3.31) and argue as in the proof of (3.15). Then, proceeding as in the proof of (3.32), we can prove the strong completeness of QD, QT, QS4, QB, QS5, QK1, QK2, QK3, QK4, and QK1/2, as well as the strong completeness of QK regarding consequence in the structures that meet condition (5).

The proof of compactness and strong completeness for the axiomatizations with identity is similar.

Notes

1. One might define first-order modal logics with two kinds of quantifiers: one that involved a constant domain and gave rise to a classical nonmodal fragment and one that involved world-variable domains and gave rise to a nonclassical fragment. But it seems more interesting to define first-order modal logics in which the same quantifiers both involve world-variable domains and give rise to a classical fragment.

2. As usual, a statement such as '$\mathbf{A} \vee \neg\mathbf{A}$ is a theorem' will not mean 'The formula $\mathbf{A} \vee \neg\mathbf{A}$ is a theorem' but 'The wff abbreviated as $\mathbf{A} \vee \neg\mathbf{A}$ is a theorem'.

3. Outermost brackets are omitted. The rule of the association to the left applies; for example, $\mathbf{A} \to \mathbf{B} \to \mathbf{C}$ abbreviates $[[\mathbf{A} \to \mathbf{B}] \to \mathbf{C}]$. The brackets that enclose a disjunction or conjunction are omitted when the disjunction or conjunction is an argument of an occurrence of $\to$ or $\leftrightarrow$; so $\mathbf{A} \to \mathbf{B} \vee \mathbf{C}$ abbreviates $[\mathbf{A} \to [\mathbf{B} \vee \mathbf{C}]]$. No other brackets are omitted.

4. In the sense of 'proof' used in the text, a proof consists of wffs each of which is annotated with a reference to either an axiom schema or a primitive rule of inference. From a more formal viewpoint, a proof can be seen as just a k-tuple of wffs. In order to count as a proof in QK, a k-tuple $\langle \mathbf{A}_1, \ldots, \mathbf{A}_k \rangle$ must satisfy the following condition: if some wff $\mathbf{A}_i$ $(1 \leq i \leq k)$ is an instance of QKA6, but not also an instance of QKA1, and there are among $\mathbf{A}_1, \ldots, \mathbf{A}_{i-1}$ no wffs from which it can be inferred in accordance with one of QKR1–5, then every $\mathbf{A}_j$ such that $i < j \leq k$ either is also an instance of QKA6 or can be inferred in accordance with QKR1 from two wffs from among $\mathbf{A}_1, \ldots, \mathbf{A}_{j-1}$. Opting for this, more formal, sense of 'proof' does not change the class of theorems of QK.

5. A world w' is 0 R-steps away from w if and only if it is w itself, and it is $m + 1$ R-steps away from w if and only if there is a world w'' that is m R-steps away from w while $w'' R w'$.

6. The requirement that each set in W should contain a wff of the form $(\forall \mathbf{x})\mathbf{Fx}$ is needed for proving the completeness of QK1, QK2, QK3, QK1/2, and the corresponding systems with identity.

7. To get such an enumeration, we can first define a one-to-one correspondence between wffs and positive integers. Each one of the k-tuples we want to enumerate is assigned the sum of the integers corresponding to the wffs that occupy its positions. We then arrange the k-tuples in increasing order of the sums assigned to them; k-tuples having the same sum can be arranged in a lexicographic manner.

8. Usually, a recursive definition defines a sequence. The definition in the text does not have that form, since it constructs a matrix rather than a sequence. We can, however, give it the usual form by viewing it as defining a sequence of finite matrices, each of which has fewer columns, and possibly fewer rows, than its successor. Then the matrix we want to construct will be the union of those finite matrices.

9. A statement such as '$\mathbf{b} = \mathbf{c}$ belongs to w' will not mean 'The formula $\mathbf{b} = \mathbf{c}$ belongs to w' but 'The wff abbreviated as $\mathbf{b} = \mathbf{c}$ belongs to w'.

10. In order to deduce (3.24), we should realize that if a member Γ of an SI-full set contains an atomic wff in which a constant $\mathbf{a}$ occurs, then Γ contains $\mathbf{a} = f(\mathbf{a})$ and so contains an atomic wff in which $f(\mathbf{a})$ occurs.

References

[1] Fine, K., "Plantinga on the reduction of possibilist discourse," pp. 145–86 in *Alvin Plantinga*, edited by J. E. Tomberlin and P. van Inwagen, D. Reidel Publishing Company, Dordrecht, 1985. 195

[2] Fine, K., "Model theory for modal logic. I. The de re/de dicto distinction," *Journal of Philosophical Logic*, vol. 7 (1978), pp. 125–56. Zbl 0375.02008. MR 81f:03024. 196

[3] Forbes, G., *Languages of Possibility: An Essay in Philosophical Logic*, vol. 9 of *Aristotelian Society Series*, Basil Blackwell, Oxford, 1989. 195

[4] Garson, J. W., "Applications of free logic to quantified intensional logic," pp. 111–42 in *Philosophical Applications of Free Logic*, edited by K. Lambert, Oxford University Press, New York, 1991. 196

[5] Hughes, G. E., and M. J. Cresswell, *A New Introduction to Modal Logic*, Routledge, London, 1996. Zbl 0855.03002. MR 99j:03011. 196, 205

[6] Kripke, S. A., "Semantical considerations on modal logic," *Acta Philosophica Fennica*, vol. 16 (1963), pp. 83–94. Zbl 0131.00602. MR 30:1035. 194

[7] Menzel, C., "The true modal logic," *Journal of Philosophical Logic*, vol. 20 (1991), pp. 331–74. Zbl 0748.03015. MR 93b:03026. 194, 195

[8] Plantinga, A., "On existentialism," *Philosophical Studies*, vol. 44 (1983), pp. 1–20. 195

[9] Plantinga, A., "Replies to my colleagues," pp. 313–96 in *Alvin Plantinga*, edited by J. E. Tomberlin and P. van Inwagen, D. Reidel Publishing Company, Dordrecht, 1985. 195

[10] Pollock, J., "Plantinga on possible worlds," pp. 121–44 in *Alvin Plantinga*, edited by J. E. Tomberlin and P. van Inwagen, D. Reidel Publishing Company, Dordrecht, 1985. 195

[11] Stephanou, Y., "Necessary beings," *Analysis*, vol. 60 (2000), pp. 188–93. Zbl 01494340. MR 1760075. 199

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