

Nonclassical Mereology and Its Application to Sets

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Abstract Part One of this paper is a case against classical mereology and for Heyting mereology. This case proceeds by first undermining the appeal of classical mereology and then showing how it fails to cohere with our intuitions about a measure of quantity. Part Two shows how Heyting mereology provides an account of sets and classes without resort to any nonmereological primitive.

1 Introduction

Leśniewsky developed his mereology, which I call “classical”, in the hope of providing an alternative to set theory.¹ More recently Lewis [5] used classical mereology to give an account of sets and classes, taking as a primitive the singleton operator. In this paper I shall continue this tradition, although in a heterodox fashion. I argue that classical mereology is mistaken and I provide a replacement, which I call Heyting mereology.² By this I mean that when a fictitious empty or null item, \emptyset , is adjoined, the mereology becomes a *frame*. Without adding any nonmereological primitives, Heyting mereology can be used either to provide a theory of pure sets or a theory of sets with urelements, or, with a bit more artifice, both together. Precisionists should note at the outset that the theory provided is one of *pseudosets* not sets strictly speaking. That is because it will turn out that there are many things that play the role of a given set. Nonetheless these pseudosets turn out to be intuitively setlike in a way that the pseudosets discussed by Lewis are not.³ This paper is in two self-contained parts. The first is the case against classical mereology, together with motivation for the alternative Heyting mereology.⁴ The second part shows how Heyting mereology can provide us with all the sets we need.

2 Heyting Mereology

In this part I motivate Heyting mereology. First I argue against the presumption in favor of classical mereology. Next I focus on the contrast between classical and

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Heyting mereology that is of interest to metaphysicians, namely, the explication of the intuitive idea of the *sum* of things. After that I consider the way in which a Heyting mereology has a classical subsystem consisting of the *regular* items. I note that this does not provide any motivation for restricting attention to classical mereology. I then turn to an argument against classical mereology based upon its failure to cohere with a satisfactory measure theory.

2.1 Against the presumption in favor of classical mereology Mereology—the theory of parts and wholes—has several distinct paradigms. One is something we can take apart and put back together again, say a bicycle made up of a hundred parts including the nuts and bolts. Another is something that is divisible in many different ways such as a cake. A third is political geography, where shires, counties, cities, and so on, make up states, which make up nations, and so on.⁵ Not surprisingly these paradigms concern systems with only finitely many parts even when, as in the cake paradigm, there is no precise point at which parts of cake cease to be cake. So we should not carelessly assume that a system with infinitely many parts satisfies axioms extrapolated from the finite case.

Perhaps, however, we are entitled to extrapolate from finite paradigms to arrive at axioms that concern only finitely many things. So we might assume that mereology, with the fictitious null object \emptyset adjoined, is a *distributive lattice*. To say it is a *lattice* is to say that given any two things x and y there is a join (least upper bound) $x \vee y$ and a meet (greatest lower bound) $x \wedge y$, which is \emptyset if and only if x and y are disjoint. To say that it is a *distributive* lattice is to say that for any three things, x , y , and z , $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ —from which it follows that for any x , y , and z , $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$. I should stress that the null object here is merely a fiction. To say that $x \wedge y = \emptyset$ is merely shorthand for saying that x and y have no common part, and the distributive law is interpreted accordingly. So if x and y are disjoint but x and z are not, the distributive law states that $x \wedge (y \vee z) = x \wedge z$, and if y and z are disjoint it states that for any $x = (x \vee y) \wedge (x \vee z)$.

The importance of mereology for philosophy derives largely from our having an intuitive notion of the *sum* of things, and there is considerable controversy over the principle that any things have a sum. To avoid confusion I restrict the term ‘sum’ to this intuitive idea, and I stipulate that it is not a synonym for ‘fusion’ which I take to be a technical term. Because the paradigms of mereology all concern the finite case, we should be especially cautious about the principles governing the sum of infinitely many things. In fact it is not initially obvious how we should even *explicate* the intuitive idea of summation in the infinite case. Moreover, the intuitive idea of the *complement* of x is the sum of everything disjoint from x , so our caution concerning infinite summation should lead to corresponding caution concerning complements. We should, for instance, resist the “intuitive obviousness” of the principle that the complement of the complement of x is always x .

Classical mereology, I submit, derives its appeal as a careless extrapolation from the finite to the infinite case. My own diagnosis of just where that extrapolation fails is in the explication of the intuitive idea of the sum of things, to be discussed in Subsection 2.3. That it derives a spurious appeal as a careless extrapolation does not prevent it being put forward as a hypothesis, but it does undermine any presumption in its favor. My aim in this part of the paper is to criticize that hypothesis and to propose in its place a weaker hypothesis, which I call Heyting mereology.

2.2 Heyting mereology Formally, a Heyting mereology is an extension of the calculus of predicates with identity, with the proper part relation $<$ taken as primitive and with a name ' \emptyset ', designating the fictitious null thing. We use \sim , $\&$, \vee , \supset , \equiv for negation, conjunction, disjunction, material implication, and material equivalence; ' \forall ' and ' \exists ' for universal and existential quantification; and ' ι ' for the definite description operator. The symbols ' $\&$ ' and ' \vee ' are used because ' \wedge ' and ' \vee ' are reserved for the lattice-theoretic operations of join and meet. The definitions, axioms, and rules of inference are taken as schemata where required.

Definitions

HMD1 (parthood)	$x \leq y \equiv (x < y) \vee (x = y)$.
HMD2 (overlap)	$x \circ y \equiv \exists z((z \neq \emptyset) \& (z \leq x) \& (z \leq y))$.
HMD3	x and y are disjoint $\equiv \sim x \circ y$.
HMD4 (join)	$\forall y Fy = \iota x (\forall z (Fz \supset (z \leq x)) \& \forall w (\forall z (Fz \supset (z \leq w)) \supset x \leq w))$.
HMD5	$x \vee y = \vee z ((z = x) \vee (z = y))$.
HMD6 (meet)	$\wedge y Fy = \iota x (\forall z (Fz \supset (x \leq z)) \& \forall w (\forall z (Fz \supset (w \leq z)) \supset (w \leq x)))$.
HMD7	$x \wedge y = \wedge z ((z = x) \vee (z = y))$.
HMD8 (complement)	$\neg x = \vee y (\sim y \circ x)$.
HMD9	x is an <i>atom</i> $\equiv (x \neq \emptyset) \& (\forall y (y < x) \supset x)$.
HMD10	x is <i>simple</i> $\equiv (x \neq \emptyset) \& (\forall y (y < x) \supset (y = \emptyset))$.
HMD11	x is a <i>fusion</i> of the y such that $Fy \equiv (\forall z (Fz \supset (z \leq x)) \& \forall w (\forall z (Fz \& w \circ z) \supset w \circ x))$.

Axioms

HMA0	Any axioms sufficient for the first-order predicate calculus with identity.
HMA1 (antireflexivity)	$(x < y) \supset \sim (y < x)$.
HMA2 (transitivity)	$(x < y) \& (y < z) \supset (x < z)$.
HMA3	$\emptyset \leq x$.
HMA4 (complete lattice schema)	$\exists x (x = \vee y Fy)$.
HMA5 (complete distributivity schema)	$x \wedge (\vee y Fy) = \vee y (\exists z (y = x \wedge z) \& Fz)$.

Rules of Inference Any rules of inference which, given HMA0, are sufficient for the first-order predicate calculus with identity.

Associated with Heyting mereology is the category of *frames*, that is, completely distributive complete lattices with minimum elements. Here by complete distributivity we mean only that finite meet distributes over arbitrary join. The morphisms are those mappings that preserve the partial ordering. If we wanted a model theoretic semantics for Heyting mereology we would need another category, that of quasi frames, where a quasi frame is any frame from which the bottom element has been excised. The morphisms are, again, the mappings that preserve the partial ordering. There is a restriction functor, which excises the bottom element, and its two-sided adjoint the null-addition functor, which adjoins a bottom element. Then we may use the frames to provide the models that Heyting mereology would have if \emptyset were not a fiction, and then take the restrictions to quasi frames as the correct models for Heyting mereology.

The further theory of Heyting mereology may be left to readers, for the theorems of set-based frame theory may be adapted to provide theorems of Heyting mereology. I shall, therefore, continue in a less formal fashion, providing the motivation for replacing classical by Heyting mereology.

The case against taking classical mereology for granted was based upon a caution about extrapolation from the finite to the infinite case. We might well, then, assume on intuitive grounds that mereology (with \emptyset adjoined) is a distributive lattice. Beyond that we are hypothesizing, and I have yet to make out a case against hypothesizing classical mereology. In its place I shall hypothesize Heyting mereology, as described above. In addition to the intuitively obvious principles that parthood is transitive and proper parthood antireflexive, its characteristic principles are that even infinitely many things have a join and that the *infinite distribution law* holds. This states that for any thing x and for any F the meet of x with the join of the F is the join of the meets of x with each of the F . In symbols, $x \wedge (\vee y Fy) = \vee y (\exists z (y = x \wedge z) \& Fz)$. Here I have replaced the more familiar ' $\vee \{y : Fy\}$ ' by ' $\vee y Fy$ ' to indicate that it is the F whose join we are considering and there is no need to mention sets or classes, which in the present context would be circular. A familiar example of a frame is provided by the set of open sets in a topology, ordered by inclusion. Then the join is the union, and finite meets are finite intersections.

Assuming that we identify sums with joins, $\neg x = \vee y (y \wedge (x = \emptyset))$ and we find that $x \leq \neg\neg x$. If the equality $x = \neg\neg x$ held in all cases then we would have a complete Boolean algebra which would, jettisoning \emptyset , restore classical mereology. The significance of this is that there is an analogy between logic and mereology, and that just as we have come to take seriously logics without the equivalence of a proposition and its double negation we should also take seriously mereology without the identity of x and $\neg\neg x$.

In Heyting mereology the whole can, as the saying goes, be more than the sum of its parts. By that we often merely mean that the value of the whole cannot be inferred from knowing the value of the parts. That is the *organic unity thesis*, which need not concern us here. But taking 'part' to mean 'proper part', as it ordinarily does, then we might be making a quite literal claim that the whole is more than the sum of its proper parts. In that case the sum of its proper parts must be a unique maximal proper part. I call this the *holistic mereology thesis*; it is the key to a thoroughly mereological account of set theory. For the sum of the proper parts of the (pseudo)singleton $\{x\}$ turns out to be x itself.

2.3 Summation in classical and Heyting mereology Formally the difference between classical and Heyting mereology is that the latter is weaker than the former, but for metaphysicians the significant difference concerns the explication of the intuitive concept of a mereological sum.

Leśniewski first discussed mereology in a systematic way and what I call classical mereology is the system proposed by him. It is deceptively simple. There are only three axioms required. And I accept the first two, namely, that the part/whole relation is transitive, that is, if $x \leq y$ and $y \leq z$ then $x \leq z$, and that x and y are parts of each other if and only if they are identical, that is, if $x \leq y$ and $y \leq x$ then $x = y$. The third, which I reject, is the *principle of unique fusion*. It states that any things have a unique fusion, where a *fusion* of the F is some z that contains every F and such that anything overlapping z overlaps some F . There is nothing in this definition that

implies uniqueness. Nor is there anything in the definition that requires there to be more than one F . So we may, for instance, refer to the fusions of x .

Notice that x is the fusion of itself and that if x is a part of y either y is also a fusion of x or y has a part that is disjoint from x .⁶ Hence unique fusion implies weak supplementation, namely, that if x is a proper part of y then y must have a part disjoint from x . Conversely the existence of fusions together with weak supplementation implies unique fusion [7].

I shall not be arguing against the existence of fusions but merely against their uniqueness. Hence I reject weak supplementation. And once weak supplementation is rejected there is no longer any obstacle in the way of holistic mereology.

As I have said there is nothing in the *concept* of a fusion that requires uniqueness. By contrast we have a simpler concept, the least upper bound or *join*, which does imply uniqueness. We also have the intuitive idea of the sum, namely, something made up of all the R s. The basic assumption of mereology in the spirit of Leśniewsky is that any things, however numerous, have a unique sum. Although I shall be arguing against classical mereology I here accept, without argument, the existence and the uniqueness of sums.

If fusion is unique then the unique fusion is the join, but we should not use the uniqueness of the sum to argue for the uniqueness of fusion. For, instead, the sum could be the join and the fusion might fail to be unique. So which is the sum of the F , a fusion or the join $\vee xFx$? It cannot be anything greater than the join, for the join contains every F and the intuitive idea of the sum of the F is of something that has nothing more than the F . And if there is a fusion of the F then any fusion contains the join. So if there is a sum it must be the join. But the join is not a sum unless it is a fusion as well as the join. For a join of the F that is not a fusion of them contains something disjoint from every F , contrary to the idea of the sum as something made up of the F . We may illustrate this need for the sum to be a fusion as well as a join by looking at one of the reasons why someone might deny that sums always exist. Suppose b and c are two segments of a line each of one unit length but separated by one unit of length. These two segments have a sum, I say, namely, a part of the line that has a one unit gap in it. But someone might insist that there is no such sum, but merely two distinct segments. In that case the join of the two segments would be a three-unit-long segment d that included both b and c and also filled the gap between them. For that would be the smallest segment containing them both. It is not, however, the fusion of b and c , because if e is a smaller segment inside the gap then e is contained in d but overlaps neither b nor c . This sort of example motivates the analysis of the sum as a join that is also a fusion.

The principle that any things have a unique sum may be explicated then as asserting that

1. mereology is a complete lattice, that is, any things have a join (and hence it follows that any things F have a, possibly null, meet, namely, the join of everything contained in all the F), and
2. the join is always a fusion, that is, if x is disjoint from every F then x is disjoint from the join of the F .

Along with the first two mereological axioms this is all we *have to* assume for non-classical mereology. But, as I have already indicated, I shall assume finite distributivity as an intuitively plausible and, in the interests of providing a mathematically well-behaved mereology, I tentatively suggest infinite distributivity so that all the axioms of a frame hold. (These entail that any join is also a fusion.)

In hindsight we may see how classical mereology has an illusory appeal. In none of the paradigms can we distinguish between joins and other fusions, so it is not surprising that when we extrapolate to the case of an infinity of parts we extrapolate the existence of unique fusions.

2.4 The spuriously classical subsystem of regular elements In this section I shall be assuming only the rather weak mereology in which, once \emptyset is adjoined, the part-hood relation results in a complete lattice in which every join is also a fusion. In such a mereology we may characterize *regular* things and it turns out that the subsystem consisting of the regular things forms a complete Boolean lattice. Hence, it might seem, we could justify classical mereology as the result of ignoring any nonregular items and stipulating that we are only concerned with the regular ones. Now as far as sets are concerned this justification would be irrelevant. For to develop my account of (pseudo)sets I need the nonclassical part of the mereology. Because, however, I have another aim, namely, undermining classical mereology, it is worth pointing out that this justification fails.

The join of the F is a fusion, so it is the smallest fusion of the F . Is there also a greatest fusion? Yes, for any fusion of some of the fusions of the F is itself a fusion of the F . So the fusion of all fusions of the F is the greatest fusion. Moreover, any thing that contains the least fusion and is contained in the greatest fusion is also a fusion.

Now consider something x . It might have many fusions. Clearly the least fusion of x is x again. But the greatest fusion of x could be something larger, x^* . It is easy to see that $x^{**} = x^*$. Things for which $x = x^*$ are called *regular*. So for any x , x^* is regular.⁷

Given any things there will only be one regular fusion of them, namely, the greatest fusion. So if we restrict attention to the regular things we find that there is indeed unique fusion, and the regular things satisfy the axioms of classical mereology. Does it follow that we may for most purposes simply ignore the nonclassical character of a Heyting mereology? No, for what counts as a join depends on the context, but the intuitive idea of the sum is independent of context. The sum of regular things is to explicate then their join in the system of all things, not their join in the system of regular things.

2.5 The case against classical mereology In this section I sketch the argument to be developed in the remainder of this part of the paper. In classical mereology we may consider the sum of all atoms, on the one hand, and its complement, consisting of ‘atomless gunk’ on the other. Classical mereology might be defended by someone who insisted that there could be no gunk and hence mereology was just isomorphic to the theory of the nonempty sets, or maybe classes, of atoms. To that I reply that necessities are not to be multiplied more than is necessary and so we should at least countenance the possibility of gunk, even if the actual universe should turn out to be gunk-free, perhaps because a region of finite diameter must contain only finitely

many parts.⁸ Now, the fundamental principles of mereology should hold even for merely possible things, including gunk, and I argue against classical mereology for gunk by introducing the idea of a measure of the quantity of gunk and showing that intuitions about the measure fail to cohere with classical mereology.

I use as a standard example the quasi-Euclidean space in which the regions are represented by some of the sets of triples of reals. I call it *quasi-Euclidean* because there are no points, and I concentrate on quasi-*Euclidean* space for simplicity. Presumably it is metaphysically possible that there be a universe whose space-time structure is represented using a differentiable manifold. And I could have considered an open region in that manifold diffeomorphic to a Euclidean space of n dimensions where $n \geq 3$. Readers will note that my argument holds in any number of dimensions at least two.

By a *simple* I mean some (nonnull) thing with no proper (nonnull) part. By an *atom* I mean some (nonnull) thing that is indivisible in the sense of not being the sum of all its proper parts. Obviously every simple is an atom. In classical mereology every atom is a simple, but that is not the case in the mereology I am describing.⁹ I reserve the word ‘gunk’ for that which has no atomic parts. I am interested in the extreme cases of a piece of gunk on the one hand and a sum of atoms on the other. My strategy will be first to argue that gunk is nonclassical and then to use that as a precedent for a nonclassical theory of atoms.

2.6 Representing gunk In order to explore further the mereology of gunk, I shall consider the case of gunk that fills space. It could *be* space, or it could be thought of as energy or something else which has no atomic parts—ectoplasm maybe. It could vary in density but for simplicity I shall consider it to be uniform. Now ‘gunk’ is a generic term, so we need a name for the species of gunk being considered. Let us call it *spaffle* (for “space-filler”). Postmodernists will no doubt offer a gastronomic deconstruction of the widespread rejection of gunk in general and spaffle in particular. And its bland texture-free character might seem to prevent us saying much about it. But in fact there is much to be said. To develop gunk-theory, we represent chunks of spaffle as sets of coordinate triples, with parts being represented by subsets. It is tempting to think of a coordinate triple as specifying a point-piece of spaffle. We should resist that temptation, for theories with point pieces of spaffle are far more exotic than the gunk theories I am considering. For by the definition of gunk, such theories require that the point pieces are divisible into proper parts. Instead I assume that spaffle is nice gunk, more precisely that every chunk of spaffle, however small, contains a spherical chunk of spaffle—one represented by a set of coordinate triples of the form $\{(r, s, t) : (r - a)^2 + (s - b)^2 + (t - c)^2 < d^2\}$. Something else we might as well assume for simplicity is that the representation is faithful. That is, no two chunks of spaffle are represented by the very same set of coordinate triples. In the context of the argument against classical mereology these assumptions need not be actually correct. It suffices that they are jointly possible.

Given these constraints we are representing the chunks of spaffle by some but not all the sets of coordinate triples, and in particular we insist that every representing set contains some sphere. There seems to be considerable variety in the possible choices for the representing sets: far from spaffle being bland, it is bewildering in its possibilities. My chief aim is to argue that spaffle does not satisfy classical mereology. Before doing so, however, I note the hypothesis to beat, namely, that the chunks

of spaffle are represented by all and only the *regular* open sets of coordinate triples, where an open set is said to be regular (or *perfectly open*) if it is the interior of its closure. Any convex open set is regular open and any finite union of regular open sets is regular open. For simple examples of open sets that are not regular we excise from some regular open set finitely many points, curves, or surfaces, which we may think of as cracks. When we take the closure we fill in the cracks as well as adding an outer boundary to the set. When we then take the interior we strip off this outer boundary but we do not open up the cracks again. If the chunks of spaffle were represented by all and only the regular open sets, they would form a classical mereology.¹⁰ By denying that the mereology of spaffle is classical I am in effect proposing additional chunks of spaffle. (All the regular chunks are, as one might expect, represented by regular open sets of triples). So the burden of proof is on me.

2.7 Measuring gunk We may assign a measure of quantity to gunk. Because spaffle is of uniform density, the quantity of a chunk of spaffle should equal the volume (i.e., Lebesgue measure) of the representing set of triples. Intuitively, the quantity of a sum of countably many things is no greater than the sum of their quantities. Call this the *countable sum principle*. Elsewhere I have used it to argue against weak supplementation [2]. But although I still accept this countable sum principle, I shall here rely upon an intuitively weaker principle, which I call *countable rearrangement*:

Suppose we have two countable sequences u_n and v_n of chunks of spaffle where the u_n are pairwise disjoint, and so are the v_n . Suppose also that each u_n is the same shape as each of the v_n and has the same finite quantity (volume). Then the quantity of the sum of the u_n equals the quantity of the sum of the v_n .

Some readers might wonder why we restrict this to countable sums. A stronger principle would do no harm, but the weaker is all we need. And, again, it should be noted that all I need for the argument against classical mereology is the *possibility* of some nice gunk in a quasi-Euclidean space with a measure satisfying the countable rearrangement principle.

2.8 The Devil's Comb Given a *skerrick* (for precision taken to be a cubic centimeter) of spaffle, we can divide half the skerrick up into infinitely many teeth and use half to form the spine of a comb, which, again for precision, can be taken to be 100mm long and 5mm wide by 1mm thick. (Each tooth is a cuboid of 15mm by 1mm by an amount to be specified.) We need the spine because otherwise some curmudgeon would insist that a collection of teeth is not one thing but infinitely many things. Of the half skerrick used to make up the comb half of what is left is used for the first tooth, a quarter for the second and so on. And we choose the teeth to contain all the points that are a rational number (between 0 and 10) of cm from one end of the comb. We may do this because there are only countably many rationals. The result is the Devil's Comb. I call it that because, however fine a hair is, there is no space between teeth wide enough for it to go through. The Devil's Comb fits neatly into a case whose shape is a thin hollow cuboid and whose volume is many skerricks. (Yet again for precision we may take it to be 5mm by 20mm by 100mm, and so of volume 10cc.)

The first point to notice is that the Devil's Comb is not represented by a regular set of coordinate triples. For the smallest regular open set containing the representative of the comb is the interior of the comb's case. Suppose we now fill the rest of the

interior of the case with spaffle, obtaining a thin but solid cuboid of spaffle that I refer to as the *case-filler*. The quantity of the case-filler is many (10 for precision) skerricks. The Devil's Comb is part of the case-filler but it is not a regular part. In fact the case-filler is the largest fusion of the Devil's Comb, or if you prefer the largest fusion of the teeth and spine of the Devil's Comb.

This shows, I claim, that the mereology of spaffle, which, recall, is the nicest gunk, is nonclassical and hence that weak supplementation fails. The latter may be seen directly by noting that there can be no chunk of spaffle contained in the case-filler but disjoint from the Devil's Comb. For we are assuming that any piece of gunk contains a small sphere of gunk and any sphere inside the case-filler will overlap either the spine or one of the teeth.

In response to this, a defender of classical mereology might insist that there could be no Devil's Comb and hence the sum of the spine and the teeth is the whole case-filler. (It will not do to take the sum to be any other proper part of the case-filler for then there would still not be a unique fusion of the teeth and spine.) But that violates the countable rearrangement principle, which tells us that rearranging disjoint parts cannot increase the quantity of the sum. Suppose we arranged the teeth in a much more boring way, taking the thickest first then putting the next thickest next to it and so on. In that case we would obtain a regular L-shaped region of spaffle with total quantity one skerrick (1cc). It follows that, however they are arranged, the sum of the teeth and spine cannot contain more than a skerrick of spaffle, and so the sum is not the case-filler, which is many skerricks (10cc).

2.9 The mereological structure of spaffle I have already supposed that every chunk of spaffle, however small, contains a spherical chunk. Economy suggests that all chunks of spaffle are sums of spheres. One hypothesis that is economical in this fashion is that the chunks of spaffle are represented faithfully by the regular open sets of coordinate triples. But this hypothesis has been excluded by the Devil's Comb example using countable rearrangement. So we should seek an alternative. The chunks of spaffle are represented by sets of coordinate triples. We may take the interiors of these representing sets to form, if necessary, a new representation by open sets. And we may also insist that the representation is faithful. In both cases we will have, if anything, arrived at a more economical hypothesis. But that still leaves open a considerable variety of representations by open sets. We might, for instance, suppose that there are enough chunks of spaffle so that every open set represents one. This has the advantage, pointed out to me by Mormann, that the mereology specifies the topology without any additional mereotopological primitive. For we obtain the mereotopological relation in which x is a *bounded interior part* of y ($x \ll y$) simply by requiring that x be represented by a set whose closure is compact and is included in the set representing y . Using the standard definition of compactness this amounts to

$$x \ll y \text{ iff for any set } Z \text{ such that } y \leq \vee z(z \in Z) \\ \text{there is finite subset } W \text{ of } Z \text{ such that } y \leq \vee w(w \in W).$$

According to this hypothesis the chunks of spaffle would indeed form a frame.

Readers should decide whether these advantages outweigh the marked lack of economy of this hypothesis, a lack of economy shown by its failure to satisfy the intuitively appealing *Roeper's principle*¹¹:

If x and y are two chunks of spaffle of finite quantity and $x < y$, then y has greater quantity than x .

This principle excludes the hypothesis that all open sets represent chunks of spaffle. For, assuming that the quantity is equal to the Lebesgue measure of the chunks represented, there would be two chunks, c and d , with d represented by $\{(r, s, t) : r^2 + s^2 + t^2 < 1\}$ and c represented by $\{(r, s, t) : 0 < r^2 + s^2 + t^2 < 1\}$. Then $c < d$ but c and d both contain the same finite quantity of spaffle.

There is, however, a hypothesis according to which

1. every chunk is the sum of spherical chunks;
2. every chunk is faithfully represented by an open set of coordinate triples;
3. the countable rearrangement principle holds;
4. Roeper's principle holds.

This is the hypothesis that the chunks are represented by precisely the *maximal open sets*, where an open set U is said to be maximal open if given any open V such that $U \subset V$ then the difference $V - U$ has positive Lebesgue measure.

Moreover, this too is a Heyting mereology. (See [Appendix A](#).) I commend it as the most likely hypothesis for the structure of spaffle.

2.10 Summary of Part One I have made a case against classical mereology. One part of this case was the undermining of the apparent intuitive appeal of classical mereology by means of an examination of how we should explicate the intuitive idea of a sum. The other part was an argument based upon the difficulty of combining an adequate account of the measure of gunk with classical mereology.

If we suppose that any things have a sum and we explicate the sum as the join, as I have advocated, then the mereology will, adjoining the null object, be a complete lattice. But I have not provided any very strong justification for assuming the infinite distributivity principle that makes it a Heyting lattice. What I have provided is a plausible account of which open sets of triples of reals represent regions of space, namely, the measure maximal ones, which account has the implication that regions of space, if they are gunk, form a Heyting mereology. This is sufficient motivation for a further exploration of the hypothesis that the axioms governing mereology are those of a Heyting lattice, provided, as usual, that we adjoin the fictitious null object. In the second part of this paper I show how the hypothesis that there are nonsimple atoms in a Heyting mereology may be used to provide an account of sets.

3 A Theory of Sets

3.1 Toward a mereological theory of sets Once we abandon weak supplementation in the case of gunk we have no good reason to retain it for the atom-packed part of reality. So atoms need not be simples and the proper parts of atoms might themselves be gunk. Things can get complicated.

Given any nonsimple atom we may consider the sum of all its proper parts. By the definition of an atom this is the unique maximal proper part (mpp). Conversely anything with a nonnull mpp is a nonsimple atom. An atom is a simple if and only if its mpp is \emptyset .

Heyting mereology provides us with a way of identifying various things with sets or, more accurately, with pseudosets. I define a relation A by xAy if x is the maximal proper part of a part of y . The pseudomembership relation E is defined by xEy if

xAy and there is no u such that xAu and uAy . Then any y such that there are x, z for which xEy and yEz is a pseudoset. So there are no empty pseudosets. Any y such that for some x xEy is a pseudoclass; any pseudoclass that is not a pseudoset is a proper pseudoclass. Anything that is not a pseudoclass is a pseudourelement. Given this characterization, and recalling that the null thing is merely a fiction, the pseudourelements are precisely the mereological simples.

I have used the symbol ' E ' for the relation of pseudomembership in place of ' \in ' which I reserve for the fundamental dyadic set-membership predicate of a mathematical theory of sets and classes. I take it as true by definition that extensionality holds for ' \in '. That is, if, for all u , $u \in x$ if and only if $u \in y$, then $x = y$. But extensionality will not hold for the relation E . For suppose that y is a nonsimple atom with mpp x , then y could be interpreted as the set $\{x\}$ but so could the sum of y and some gunk. This is not, however, a serious problem. Quite generally if we have a denoting expression of the form "The unique u such that . . ." and there are in fact many such u , then we may reinterpret the expression as referring indefinitely to any of these u . Thus if x is a simple, the set-denotation ' $\{x\}$ ' which is synonymous with 'the unique thing y such that for precisely one x $x \in y$ ' refers, in a non-unique fashion, to any u such that for precisely one x xEu .¹² When we think of pseudosets as sets we deliberately ignore the set-theoretically irrelevant differences between things with the same members. If we assume the axiom of choice holds then we could take set-denotations to be equivocal rather than indefinite. That is, there would be many candidates for set theory in each of which extensionality holds, instead of one candidate for a pseudoset theory in which extensionality fails.¹³ I see, however, no reason to prefer equivocation to indefiniteness, and I note that my pseudosets have a great advantage over the ones considered by Lewis. I do not have to resort to an arbitrary coding of set theory. Instead I have a straightforward case of an oversupply of referents.

If we take the set-denotations to refer to pseudosets in the fashion just described, then not every set-denotation refers. For instance ' $\{b, \{b, c\}\}$ ' fails to refer. For if it referred to u , u would have some parts v, w that have mpps b and x , respectively, where x has parts y and z with mpps b and c ; but in that case bAx and xAu , so even though bAu it is not the case that bEu . Nonetheless, I shall argue, there are enough pseudosets to do mathematics. At least there are enough if there are big enough mereologies. Because I take pure mathematics to concern what is necessary, all I require is that there are big enough possible worlds to provide the requisite pseudosets.¹⁴

3.2 Simply founded sets I assume that we have a consistent set theory with fundamental dyadic predicate ' \in '. We restrict attention, if necessary, to *well-founded* sets. Let α be the ancestral of \in . Then to say that a set u is well-founded is to say that given any nonempty set x for which $x\alpha u$ there is some y such that $y\alpha x$ but there is no z such that $z \in y$. In particular this excludes \in -cycles, that is, for no x do we have $x\alpha u$ and $u \in x$.

I am assuming, for the moment, that the empty set \emptyset is a fiction so that well-foundedness implies that every descending chain (that is, every sequence $x_n, n = 1, 2, \dots$ such that for all $n, x_{n+1} \in x_n$, terminates in an urelement x_k . I say that a set z is *simply founded* if

1. given any w, x, y such that $w\alpha x\alpha y\alpha\{z\}$ then $w \notin y$; and

2. given w, x, y such that $w\alpha x\alpha z$ and $w\alpha y\alpha z$ then either $x = y$ or $x\alpha y$ or $y\alpha x$.

This amounts to saying that the relation \in restricted to $\{x : x\alpha\{z\}\}$ is a *tree*.

Given any simply founded set s we may define a frame $H(s)$ and a one to one mapping φ of $H(s) - \{\emptyset\}$ onto $\{s\} \cup \{x : xAs\}$. Moreover, if we define E as in the previous section $\varphi(x) \in \varphi(y)$ if and only if xEy . (See [Appendix B.1.](#)) This shows that the theory of mereology has the resources to provide pseudosets corresponding to arbitrarily large, simply founded sets. All that is required is to posit that the actual mereology has, when \emptyset is adjoined, the structure of a frame with a subsystem isomorphic to $H(s)$.

3.3 Pure sets A theory of pure sets is one in which there are no urelements. Pseudosets may be reinterpreted as performing the role of pure sets by taking the set-denotation ' \emptyset ' to refer to any of the mereological simples, which previously played the role of the urelements. A simply founded set with urelements is then interpreted as a well-founded but not necessarily simply founded pure set. This gives us all the pure sets we want provided there are enough urelements. More formally, given any pure set p then there is a simply founded set s with urelements and a mapping θ of $\{s\} \cup \{x : xAs\}$ onto $\{p\} \cup \{x : xAp\}$ such that $\theta(x) \in \theta(y)$ if and only if $x \in y$. (See [Appendix B.2.](#)) Then the composite $\theta\varphi$ of mappings φ and θ maps $H(s)^*$ onto $\{p\} \cup \{x : xAp\}$ in such a way that $\theta\varphi(x) \in \theta\varphi(y)$ if and only if xEy , showing that the mereology can provide at least one pseudoset corresponding to every pure (and well-founded) set in a consistent set theory.

3.4 Modulo junk Heyting mereology has the resources then to perform all the work done by simply founded sets, or by pure sets. I now show how, with a little added complexity, it can do the work performed by any sets with urelements. Given any regular thing j we may consider things *mod* j , that is, we may consider the system in which x and y are equivalent, written $x = y \pmod{j}$, just in case $x \wedge \neg j = y \wedge \neg j$. Hence if $x \leq j$ then $x = \emptyset \pmod{j}$. What this amounts to is a way of referring indefinitely to equivalent things by ignoring j and all its parts. (Hence ' j ' for junk.) Since we already have an account in which set-denotations refer to indefinitely many things, this is just more of the same, and no (further) drawback to the account is being proposed.

The effect of considering things *mod* j is to treat any atom whose mpp is x as a referent of the set-denotation ' $\{x \wedge \neg j\}$ '. If $x \leq j$ then this is one of many things to which ' \emptyset ' refers. If $x \wedge \neg j = y$ then this is one of many things to which ' $\{y\}$ ' refers. Hence by considering things *mod* j we can do the work both of pure sets that are not simply founded and sets with urelements that are not simply founded. Assuming j has enough parts the result is an adequate account of sets provided the urelements are restricted to those disjoint from j . For details see [Appendix B.2.](#)

4 Conclusion

The study of gunk motivates a nonclassical Heyting mereology. This in turn enables us to give a thoroughly mereological treatment of pseudoclasses so that members of pseudoclasses are indeed parts of pseudoclasses. The drawback is that we have to divide the whole of reality into a large (regular) part j and its complement $\neg j$ and restrict the pseudoclasses to those of which the urelements, if any, are parts of $\neg j$.

More modestly, Heyting mereology provides a theory of pseudosets that will do all the work of pure sets.

Appendix A Maximal Open Sets Form a Frame

The set of n -tuples (for some finite n) equipped with the usual topology has a countable basis for open sets. Therefore it suffices to prove the following.

Theorem A.1 *Consider a topological space with a countable basis of open sets and a countably additive measure that is positive on all nonempty open sets. Then the maximal open sets form a frame. (Notice that the Devil's Comb example shows that the maximal open sets in \mathbb{R}^3 do not form a complete Boolean algebra. Similar examples can be provided in \mathbb{R}^n for any positive integer n .) First define an equivalence relation on the open sets by $U \sim V$ if and only if both $U - V$ and $V - U$ are of zero measure.*

Lemma A.2 *If U is any open set there is a unique maximal open set equivalent to U .*

Proof of lemma It suffices to consider W the union of all the open sets equivalent to U . Provided W is itself equivalent to U then W is the unique maximal open set equivalent to U . Let B be a countable basis of open sets. Let C be the set of those B contained in some open set equivalent to U . Then $W = \cup C$. Now for all $X \in C$, $X - U$ is null, hence $W - U$ is the countable union of null sets and so is itself null. Since U is contained in W this proves that W is equivalent to U . \square

Proof of theorem The join of any set X of maximal open sets is the unique maximal open set equivalent to $\cup X$. Likewise the meet of maximal open sets U and V is the unique maximal open set equivalent to $U \cap V$. So the maximal open sets form a complete lattice. Now, for any complete lattice, $\forall z(\exists y(By \ \& \ (z = x \wedge y))) \leq x \wedge (\forall y By)$. So we have only to show that for any maximal open X and maximal open Ds , $X \wedge (\forall Y DY) \subseteq \forall Z(\exists Y(DY \ \& \ Z = X \wedge Y))$. Because there is a countable basis for the open sets there are countably many of the Ds , call them the Cs such that $\cup\{Y : DY\} = \cup\{Y : CY\}$. Hence $\forall Y DY$ is the unique maximal open set equivalent to $\cup\{Y : DY\}$, which is the same as the unique maximal open set equivalent to $\cup\{Y : CY\}$, that is, $\forall CY$. Now the measure is countably additive so the countable union of null sets is null. Therefore, for the countably many Cs , $X \wedge (\forall Y CY) = \forall Z(\exists Y(CY \ \& \ (Z = X \vee Y)))$. Therefore $X \wedge (\forall Y DY) = \cup\{X \wedge Y : CY\} = X \wedge (\forall Y CY) \subseteq \forall Z(\exists Y(DY \ \& \ (Z = X \wedge Y)))$, proving that the maximal open sets form a frame. \square

Appendix B

B.1 The frame $H(s)$ associated with a simply founded set s I assume some formulation of well-founded set theory without an empty set. Its set-membership relation is \in , the ancestral of this relation is α , and the singleton with member x is $\{x\}$. ' \emptyset ' denotes the empty set. If z is any set let $z^\circ = \{x : x\alpha z \ \& \ x \neq \emptyset\}$. Let $[z] = \{z\}^\circ$.

Definition B.1 A set z is *simply founded* if

1. given any w, x, y such that $w\alpha x\alpha y\alpha\{z\}$ then $w \notin y$; and
2. given w, x, y such that $w\alpha x\alpha z$ and $w\alpha y\alpha z$ then either $x = y$ or $x\alpha y$ or $y\alpha x$.

The following then are obvious.

Proposition B.2 $y^\circ = \cup\{[x] : x \in y\}$.

Proposition B.3 *If z is simply founded and $x\alpha z$ then x is also simply founded.*

Proposition B.4 *If z is simply founded then $\{z\}$ is also simply founded.*

We may give $[s]$ a topology in which the open subsets of $[s]$ are $[s]$, \emptyset , and all y° such that $y \subseteq [s]$. (That it is a topology follows immediately from Proposition B.2.) The open sets ordered by inclusion form the items in a frame $H(s)$ in which \emptyset is the null item.

Theorem B.5 *There is a 1 to 1 mapping φ of $H(s)$ onto $[s]$ such that xEy if and only if $\varphi(x) \in \varphi(y)$.*

Lemma B.6 *The atoms of $H(s)$ are precisely the $[x]$ such that $x \in [s]$.*

Proof of Lemma B.6 $[x]$ has x° as an mpp and so is an atom. Conversely suppose w is an atom. Then $w = \cup\{[x] : x \in y\} \subseteq y \subseteq [s]$. Now, since w is an atom, not all of the $[x]$ such that $x \in y$ are proper parts of w . That is, for some $x \in y$, $w = [x]$, completing the proof of the lemma. \square

Lemma B.7 *If x and y are both subsets of $[s]$ and if $x^\circ = y^\circ$ then $x = y$.*

Proof of Lemma B.7 Suppose $x \subseteq [s]$, $y \subseteq [s]$, $x^\circ = y^\circ$, and $x \in y$. We have to show that $z \in y$. So to obtain a proof by reductio assume $z \notin y$. Because $z \in x$, $z\alpha x$, so $z\alpha y$ and hence, since $z \notin y$, for some u , $u \in y$ and $z\alpha u$. Because $z\alpha x$ and $z\alpha u$ it follows from the definition of a simply founded set that either $x = u$ or $x\alpha u$ or $u\alpha x$. But if either $x = u$ or $x\alpha u$ then $x\alpha y$. Since $x^\circ = y^\circ$ it follows that $x\alpha x$ contradicting the acyclic character of \in . Therefore $u\alpha x$. But $z\alpha u$ so by the definition of a simply founded set $z \notin x$. This contradiction shows that $z \in y$. \square

Proof of Theorem B.5 By Lemma B.7 a mapping $\varphi : H(s) \rightarrow [s]$ may be defined by $\varphi(y^\circ) = y$. Clearly this is a 1 to 1 onto mapping. By Lemma B.6 the relation A on $H(s)$ may be defined by uAv if $u = y^\circ$ and $[y] \subseteq v$. Now $[y] = \{y\}^\circ$. So uAv if and only if $\{\varphi(u)\}^\circ \subseteq \varphi(v)^\circ$, which holds if and only if either $\{\varphi(u)\} \subseteq \varphi(v)$ or $\{\varphi(u)\}\alpha\varphi(v)$. In either case $\{\varphi(u)\}\alpha\varphi(v)$. Hence uAv if and only if $\varphi(u)\alpha\varphi(v)$. Since φ is 1 to 1 onto and s is simply founded it follows that uEv if and only if $\varphi(u) \in \varphi(v)$ as required. \square

B.2 The mapping θ and sets mod j Let k be the set of all $z \in H(s)$ for which there is a unique onto mapping $f_z : z \rightarrow p_z$ for some pure set p_z such that $f_z(\{u\}) = \emptyset$ for all urelements u and such that $f_z(x) \in f_z(y)$ if and only if $x \in y$. Let $m = \cup k$. It is easy to check that $m \in k$, and so m is k 's unique maximum member. If $m \neq [s]$ then either (1) there is an urelement u such that $\{u\} \cap m = \emptyset$, or (2) there is some $v \in k$ such that $\{v\} \in H(s)$ but $\{v\} \notin k$. In either case we can extend f_m to $f_{m'}$ contradicting the maximality of m . Hence $m = [s]$. The required mapping θ is then $f_{[s]}$.

The theory of sets mod j works in an analogous fashion. Let n be the set of all $z \in H(s)$ for which there is a unique onto mapping $g_z : z \rightarrow p_z$ for some set p_z such that $g_z(\{u\}) = u \wedge \neg j$ for all urelements u and such that $g_z(x) \in g_z(y)$ if and only if $x \in y$. Then replacing k by n and f by g in the above, we can prove that there is

a $g_{[s]}$. This shows that the sets $[s] \bmod j$ may be interpreted as $\{g_{[s]}(x) : x \in [s]\}$, which is a set of both some pure sets and some sets with urelements disjoint from j .

Notes

1. See Simons [6]. The standard work on classical mereology is Simons [7].
2. Classical mereology is a special case of Heyting mereology, so the contrast is between a weaker and a stronger hypothesis about mereology.
3. [5], Section 1.9.
4. That is, an infinitely distributive complete lattice. The category of frames differs from the category of complete Heyting algebras not in the objects but in the morphisms. See Johnstone [4], Chapter 3. If we were interested in morphisms of mereologies, we would, I suppose, require a different category again. For the morphisms would not merely preserve parthood but map the nonnull elements to nonnull elements and hence preserve the overlap relation.
5. I owe the importance of political geography as an application of mereology to the influence of Barry Smith.
6. If y is not a fusion of x then there is some z that overlaps y but not x . But, by the definition of overlap, y and z have a common part w , which must therefore be disjoint from x .
7. We may define the complement $\neg x$ of x as the sum of everything disjoint from x . Then in a Heyting mereology $x^* = \neg\neg x$. So the definition I have given of regularity coincides with the usual one for a Heyting algebra.
8. In Forrest [3] I argue that either every region of finite diameter contains only finitely many parts or space is atomless gunk.
9. Between the concept of an atom and the concept of a simple we have such intermediaries as not being the sum of two proper parts; not being the sum of nonoverlapping proper parts; and not being the sum of two nonoverlapping proper parts.
10. Tarski proposed that these were precisely the regions of space in [8]. See Postulate Two.
11. After Peter Roeper who suggested it to me.
12. We may formalize this by using van Fraassen's method of supervaluation. See [9].
13. Hence, following Lewis, we could "Ramsify out" \in . See [5], Section 2.6.
14. It is a matter of controversy whether there is an upper bound to the number of things there might be. (See [1], Section 6.6.) But even if there is such a bound it would be rather large. Lewis has suggested beth omega. Hence the restriction on sets that results would not devastate pure mathematics.

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