

## Logics of Relative Identity

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**Abstract** This paper is the first part of an exploration into the logical properties of relative identity. After providing the semantic grounds for various monadic logics of relative identity, I define the minimal system and its nine extensions. It is suggested that despite their purely formal origin at least some of them may contain nontrivial philosophical insights. All logics are axiomatized by means of sound and complete sequent calculi. I show their affinities with existing formalizations.

### 1 The Minimal Theory of Relative Identity

One of the most notorious logical inventions of Geach is relative identity (from Geach [4] to [5]). Due to the extravagance of his theory, relative identity (RI for short) has not been warmly welcomed by his contemporaries. In particular Geach's rejection of absolute identity (AI) turned out to be fatal for any future theory of RI.

However, fairly recently Deutsch [2] has tried to revive the notion in his logic of general similarity. One of the most striking features of his account is its philosophical modesty. Deutsch confines himself to stating linguistic facts without fortifying them by means of ontological or epistemological explanations. Needless to say, Deutsch does not reject AI. He merely proposes examining the logical properties of the relation that is expressed in our everyday language by formulas of the form “. . . is the same . . . as . . .”.

In what follows I will sketch a formal landscape for monadic logics of relative identity (henceforth MLRIs).<sup>1</sup> My aim is to expose the possible logical properties of RI and explore the logical consequences of these properties. The nonformal framework of my formalization consists of four assumptions.

**Assumption 1.1** RI is denoted by ‘. . . is the same . . . as . . .’. Paradigmatic occurrences of this expression include the following:

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1. *Goldberg Variations* by Glenn Gould is the same piece of music as Murray Perahia's interpretation, but they are not the same interpretation of Bach's masterpiece;
2. The ship of Theseus is the same collection of planks as the reassembled ship, but they are not the same artefacts;
3. The inscription 'identity' is the same word (i.e., so-called type-word) as the inscription 'identity', but not the same inscription (token-word);
4.  $\frac{1}{2}$  is the same rational number as  $\frac{2}{4}$ , but they are not the same fractions;
5. Cockney English is the same language as BBC English, but is not the same dialect;
6. Karol Wojtyła is the same man as John Paul II, but not the same member of the Catholic Church.

**Assumption 1.2** Such sentences will be rendered by schemata ' $x$  is the same  $A$  as  $y$ ', where ' $x$ ' and ' $y$ ' are individual variables of first-order logic, and ' $A$ ' is a predicate variable. Some more restrictions may be imposed on nonformal grounds.<sup>2</sup>

**Assumption 1.3** 'is the same  $A$  as' denotes an equivalence relation on the set of all objects falling under a predicate  $A$ . Hence this set is exhaustively divided by it into disjoint nonempty subsets.

**Assumption 1.4** The semantics of MLRIs will be formulated in the standard set theory (ZFC) and share the alphabet with first-order nonmodal classical predicate logic.<sup>3</sup>

If  $x = y$ , then  $x$  and  $y$  will be called *AI-objects*. Because of (1.4) the definition of an RI-object is bound to refer to AI-objects; namely, an *RI( $\delta$ )-object* is the set of AI-objects that are the same  $\delta$ . For example, RI (piece of music) contains AI-objects that are the same pieces of music.

My *minimal theory of RI* consists of (1.1) and (1.3). The former merely reports unusual identity sentences and the latter imposes the minimal condition for RI being identity. Observe that my theory

1. does not exclude that RI may collapse into AI, that is, it does not proscribe reading ' $x$  is the same  $A$  as  $y$ ' as ' $x = y$  and  $A(x)$ ',
2. does not require the standard set theory.

The prospects for developing logics of RI in a nonstandard set theory are rather poor, but I think that Blizard's theory of multisets or the Krause conception of quasi sets may be useful in this respect (cf. Blizard [1] and Krause [7]).

A full theory of RI should furthermore plausibly explain why we are entitled to treat non-AI objects as identical in some derivative sense and on the ground of this justification "predict" the behavior of RI sentences. It is hoped that MLRIs will provide formal tools useable in this enterprise and give some clues for possible justifications.

However small my minimal theory might seem to be, it is not small enough to encompass all accounts of RI. In particular, it excludes Zemach's version of RI, according to which relativity of identity derives from the incompleteness of objects to which we refer (Zemach [16]). Zemach makes the controversial claim that if  $x$  is the same  $A$  as  $y$ , then neither  $x$  nor  $y$  need be  $A$  (cf. (1.3) above). On the other hand, (1.3) follows from theories in [2], [5], Griffin [6], Noonan [8], and van Inwagen [14].

My minimal theory of relative identity deliberately avoids some controversial theses on RI:

1. AI is a syntactically and/or semantically incomplete expression ([5]).
2. RI is indispensable when it comes to individuating things ([6], pp. 156–61).
3. Every AI statement is a disguised RI statement (Strawson [13], pp. 47–74, and a consequence of [16]).
4. Only RI may provide a criterion of identity ([8], pp. 1–81; renounced in Noonan [9]).
5. RI holds only among vague objects ([16] and Zemach [18]).

For those reasons, I find (1.1)–(1.4) rather innocuous and acceptable even for a moderate opponent to RI.

## 2 Language

Let a nonempty set of signs be called an *alphabet*. If  $X$  is an alphabet, then  $X^*$  will denote the set of all finite sequences of signs from  $X$ .  $A$  will denote the alphabet of the language of MLRIs (the monadic language, for short).

**Definition 2.1**  $A$  is the union of the following sets:

1.  $x_1, x_2, \dots$  (individual variables);
2.  $\neg, \wedge, \vee, \rightarrow, \equiv$ ;
3.  $\forall, \exists$ ;
4.  $=$ .<sup>4</sup>

The set (1) will be referred to as **Var**.

A set  $S$  of *symbols* of the monadic language is the union of a (possibly empty) set of monadic predicates,  $A_1, A_2, \dots$ , and a (possibly empty) set of constants,  $c_1, c_2, \dots$ . A set  $T^S$  of *terms* of the monadic language consists of **Var** and the set of constants. The *monadic language*  $\mathcal{L}^S$  is the smallest subset of  $(A \cup S)^*$  satisfying the standard conditions on formula-construction and the condition concerning RI.

**Definition 2.2** If  $\alpha_1, \alpha_2 \in T^S$  and  $\delta \in S$ , then  $\alpha_1 =_\delta \alpha_2 \in \mathcal{L}^S$ .

The elements of  $\mathcal{L}^S$  will be called *S-formulas*. Sets of *S-formulas* will be called *S-sets*.

I assume the usual definition of substitution. If  $\varphi \in \mathcal{L}^S$ , then  $\varphi[\beta_1, \dots, \beta_n / \alpha_1, \dots, \alpha_n]$  will denote the result of uniform substitution of terms  $\alpha_1, \dots, \alpha_n$  for variables  $\beta_1, \dots, \beta_n$ .

## 3 Semantics

I depart from the standard semantics at Definition 3.1.

**Definition 3.1** An *S-structure* is a pair  $\mathfrak{S} := (U, \mathcal{F})$ .

1.  $U$  is a nonempty set called the *universe* of  $\mathfrak{S}$ .
2.  $\mathcal{F}$  is a map on  $S$  such that
  - (a)  $\mathcal{F} : \{A_n : A_n \text{ is a monadic predicate}\} \rightarrow \wp(\wp(X) \setminus \{\emptyset\})$ ,
  - (b)  $\mathcal{F} : \{c_n : c_n \text{ is a constant}\} \rightarrow U$ .

Therefore, in my semantics a monadic predicate is interpreted by a *family* of sets. An *assignment* in an *S-structure*  $\mathfrak{S}$  is a map  $\mathcal{G} : \mathbf{Var} \rightarrow U$ . An *S-interpretation* is a pair  $\mathfrak{I} := (\mathfrak{S}, \mathcal{G})$ . If  $\mathcal{G}$  is an assignment in an *S-structure*  $\mathfrak{S}$  and  $u \in U$ , then we have the following.

**Definition 3.2**  $\mathcal{G}_u^{\beta_1}(\beta_2) := \begin{cases} \mathcal{G}(\beta_2) & \text{if } \beta_1 \neq \beta_2. \\ u & \text{if } \beta_1 = \beta_2. \end{cases}$

Respectively,  $\mathfrak{F}_u^{\beta} := (\mathfrak{J}, \mathcal{G}_u^{\beta})$ .

**Definition 3.3** An  $S$ -interpretation  $\mathfrak{F}$  is (also) a map  $\mathfrak{F} : S \rightarrow U$  such that

1.  $\mathfrak{F}(\beta) := \mathcal{G}(\beta)$ .
2.  $\mathfrak{F}(\chi) := \mathcal{F}(\chi)$ .

Let ' $\mathfrak{F} \models \varphi$ ' abbreviate 'An  $S$ -interpretation  $\mathfrak{F}$  is a *model* for an  $S$ -formula  $\varphi$ '.

**Definition 3.4** If  $\mathfrak{F} = (\mathfrak{J}, \mathcal{G})$  is an  $S$ -interpretation, then

$$\mathfrak{F} \models \alpha_1 = \alpha_2 \quad \text{iff} \quad \mathfrak{F}(\alpha_1) = \mathfrak{F}(\alpha_2), \quad (1)$$

$$\mathfrak{F} \models \alpha_1 =_{\delta} \alpha_2 \quad \text{iff} \quad \exists X \in \mathcal{F}(\delta) \mathfrak{F}(\alpha_1), \mathfrak{F}(\alpha_2) \in X, \quad (2)$$

$$\mathfrak{F} \models \delta(\alpha) \quad \text{iff} \quad \exists X \in \mathcal{F}(\delta) \mathfrak{F}(\alpha) \in X, \quad (3)$$

$$\mathfrak{F} \models \neg\varphi \quad \text{iff} \quad \neg\mathfrak{F} \models \varphi, \quad (4)$$

$$\mathfrak{F} \models \varphi_1 \vee \varphi_2 \quad \text{iff} \quad \mathfrak{F} \models \varphi_1 \text{ or } \mathfrak{F} \models \varphi_2, \quad (5)$$

$$\mathfrak{F} \models \exists\beta\varphi \quad \text{iff} \quad \text{for some } u \in U, \mathfrak{F}_u^{\beta} \models \varphi. \quad (6)$$

If an  $S$ -interpretation  $\mathfrak{F}$  is a model for all formulas from an  $S$ -set  $\Phi$ , then we say that it is a model for this set:  $\mathfrak{F} \models \Phi$ . The sign ' $\models$ ' is commonly used also in another context: ' $\Phi \models \varphi$ ' means 'An  $S$ -formula  $\varphi$  is a *semantic consequence* of an  $S$ -set  $\Phi$ '. As MLRIs will be identified by means of sets of structures, we need a relativization thereof to a set  $\Sigma$  of  $S$ -interpretations.

**Definition 3.5**  $\Phi \models_{\Sigma} \varphi$  if and only if for every  $S$ -interpretation  $\mathfrak{F}$  from  $\Sigma$ , if  $\mathfrak{F}$  is a model for  $\Phi$ , then it is a model for  $\varphi$ .

We may treat a consequence operation  $\models_{\Sigma}$  as a logic characterized by  $\Sigma$ . For short, such logic will be referred to as a  $\Sigma$  logic.

**Definition 3.6** A set  $\Phi$  of formulas is  $\Sigma$ -*satisfiable* if and only if  $\exists \mathfrak{F} \in \Sigma \forall \varphi \in \Phi \mathfrak{F} \models_{\Sigma} \varphi$ .

#### 4 Minimal MLRI

Assumption 1.3 from Section 1 establishes the minimal condition for  $\mathcal{F}$ .

(C1) If  $X, Y \in \mathcal{F}(\delta)$  and  $X \neq Y$ , then  $X \cap Y = \emptyset$ .<sup>5</sup>

$S$ -structures satisfying (C1) will be called *minimal  $S$ -structures*. The minimal MLRI is characterized by the class of all minimal  $S$ -structures. It will be referred to as ' $\models_{C1}$ ' or (C1). It is easy to verify the following facts.

**Fact 4.1**

1.  $\models_{C1} x_1 = x_2 \wedge A_1(x_1) \rightarrow x_1 =_{A_1} x_2$ .
2.  $\models_{C1} x_1 =_{A_1} x_2 \rightarrow A_1(x_1)$ .
3.  $\models_{C1} A_1(x_1) \rightarrow x_1 =_{A_1} x_1$ .
4.  $\models_{C1} x_1 =_{A_1} x_2 \rightarrow x_2 =_{A_1} x_1$ .
5.  $\models_{C1} x_1 =_{A_1} x_2 \wedge x_2 =_{A_1} x_3 \rightarrow x_1 =_{A_1} x_3$ .

All these facts but the last obtain without (C1).

To be an MLRI is to contain (C1). All structures below (both nondeviant and deviant) are assumed to satisfy (C1).

### 5 Some Nondeviant MLRIs

Since RI sentences come from miscellaneous domains (witness Assumption 1.1 (1)–(6) above) I am inclined to recognize various logics of RI, at least as a formal possibility. From the “point of view of the monadic language” their variety may be described in terms of relations among RI-objects falling under predicates that contain or intersect one another.

Let ‘ $\cup$ ’ denote the union of a family of sets. More interesting cases of containment conditions include the following.

- (C2) If  $\cup \mathcal{F}(\delta_1) = \cup \mathcal{F}(\delta_2)$ , then  $\mathcal{F}(\delta_1) = \mathcal{F}(\delta_2)$ .
- (C3) If  $\cup \mathcal{F}(\delta_1) \subseteq \cup \mathcal{F}(\delta_2)$ , then  $\forall X \in \mathcal{F}(\delta_1) \exists Y \in \mathcal{F}(\delta_2) X \subseteq Y$ .
- (C4) If  $\cup \mathcal{F}(\delta_1) \subseteq \cup \mathcal{F}(\delta_2)$ , then  $\forall X \in \mathcal{F}(\delta_2) \exists Y \in \mathcal{F}(\delta_1) (X \subseteq \mathcal{F}(\delta_1) \rightarrow X \subseteq Y)$ .
- (C5) If  $\cup \mathcal{F}(\delta_1) \subseteq \cup \mathcal{F}(\delta_2)$ , then  $\mathcal{F}(\delta_1) \subseteq \mathcal{F}(\delta_2)$ .

(C2) may be called the *extensionalist* MLRI since it guarantees that (extensionally) identical predicates possess (extensionally) identical RI-objects. The nontrivial intersection restrictions involve the following conditions.

- (C6) If  $X \in \mathcal{F}(\delta_1)$ ,  $Y \in \mathcal{F}(\delta_2)$ , and  $X \cap Y \neq \emptyset$ , then either  $X \subseteq Y$  or  $Y \subseteq X$ .
- (C7) If  $X \in \mathcal{F}(\delta_1)$ ,  $Y \in \mathcal{F}(\delta_2)$ , and  $X \cap Y \neq \emptyset$ , then  $X \subseteq \cup \mathcal{F}(\delta_2)$ .
- (C8) If  $X \in \mathcal{F}(\delta_1)$ ,  $Y \in \mathcal{F}(\delta_2)$ ,  $Z \in \mathcal{F}(\delta_2)$ ,  $X \cap Y \neq \emptyset$ , and  $X \cap Z \neq \emptyset$ , then  $Y = Z$ .
- (C9) If  $X \in \mathcal{F}(\delta_1)$ ,  $Y \in \mathcal{F}(\delta_2)$ , and  $X \cap Y \neq \emptyset$ , then  $X \in \mathcal{F}(\delta_2)$ .

**Fact 5.1** One may prove by inspection that

1. (C3) entails (C2).
2. (C4) entails (C2).
3. (C5) entails (C3) and (C4).
4. (C8) entails (C3).
5. (C9) entails (C5), (C6), (C7), and (C8).
6.  $(C7 \wedge C8)$  entails (C9).

A  $C_n$  logic is a logic characterized by the class of all  $S$ -structures satisfying  $C_n$  (and (C1) as well). In Sections 8 and 9 it will be proved that characteristic theorems for  $C_n$  logics are as follows.

**Fact 5.2**

1.  $\models_{C2} \forall x_1 (A_1(x_1) \equiv A_2(x_1)) \rightarrow (x_1 =_{A_1} x_2 \rightarrow x_1 =_{A_2} x_2)$ .
2.  $\models_{C3} \forall x_1 (A_1(x_1) \rightarrow A_2(x_1)) \rightarrow (x_1 =_{A_1} x_2 \wedge x_2 =_{A_2} x_3 \rightarrow x_1 =_{A_2} x_3)$ .
3.  $\models_{C4} \forall x_1 (A_1(x_1) \rightarrow A_2(x_1)) \rightarrow (x_1 =_{A_1} x_2 \wedge x_2 =_{A_2} x_3 \rightarrow x_1 =_{A_1} x_3)$ .
4.  $\models_{C5} \forall x_1 (A_1(x_1) \rightarrow A_2(x_1)) \rightarrow (x_1 =_{A_1} x_2 \wedge x_2 =_{A_2} x_3 \rightarrow x_1 =_{A_1} x_3 \wedge x_1 =_{A_2} x_3)$ .
5.  $\models_{C6} x_1 =_{A_1} x_2 \wedge x_2 =_{A_2} x_3 \rightarrow x_1 =_{A_1} x_3 \vee x_1 =_{A_2} x_3$ .
6.  $\models_{C7} x_1 =_{A_1} x_2 \wedge A_2(x_1) \rightarrow A_2(x_2)$ .
7.  $\models_{C8} x_1 =_{A_1} x_2 \wedge A_2(x_1) \wedge A_2(x_2) \rightarrow x_1 =_{A_2} x_2$ .
8.  $\models_{C9} x_1 =_{A_1} x_2 \wedge A_2(x_1) \rightarrow x_1 =_{A_2} x_2$ , however,
9.  $\not\models_{C9} x_1 =_{A_1} x_2 \rightarrow x_1 = x_2$ .

Incidentally, compare (2)–(4) with

**Fact 5.3**  $\models_{C9} x_1 =_{A_1} x_2 \wedge x_2 =_{A_2} x_3 \rightarrow x_1 =_{A_1} x_3 \wedge x_1 =_{A_2} x_3$ .

Observe also the following.

**Fact 5.4**  $\models_{C3,5} \forall x_1 (A_1(x_1) \rightarrow A_2(x_1)) \rightarrow (x_1 =_{A_1} x_2 \rightarrow x_1 =_{A_2} x_2)$ .

## 6 Some Deviant MLRIs

The formula in Fact 5.2(9), which is invalid in (C9), holds if we assume the following.

(C10) If  $X \in \mathcal{F}(\delta)$ , then  $\exists u \in \cup \mathcal{F}(\delta) X = \{u\}$ .

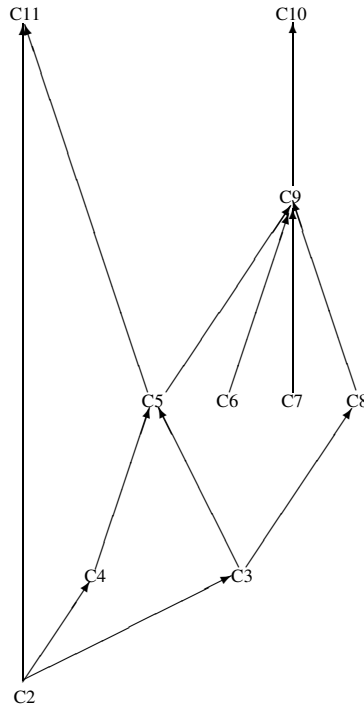
And here is another deviant condition on  $\mathcal{F}$ .

(C11)  $\cup \mathcal{F}(\delta) \in \mathcal{F}(\delta)$ .

Then the following is obvious.

**Fact 6.1**

1. (C10) entails (C9).
2. (C11) entails (C3).



**Figure 1**

The deviancy of (C10) and (C11) is revealed by the following facts.

**Fact 6.2**

1.  $\models_{\mathbf{C10}} x_1 = x_2 \wedge A_1(x_1) \equiv x_1 =_{A_1} x_2$ .
2.  $\models_{\mathbf{C11}} A_1(x_1) \wedge A_1(x_2) \rightarrow x_1 =_{A_1} x_2$ .

Notice that (C11) validates inferences like ‘if  $x$  is an artefact and  $y$  is an artefact, then  $x$  is the same artefact as  $y$ ’.

Relations among nonminimal MLRIs are depicted in Figure 1.

## 7 Sequent Calculi

Let  $\varphi_1, \varphi_2, \dots, \varphi_n$  be a nonempty sequence of  $S$ -formulas. Such a sequence will be called an  $S$ -sequent, the subsequence  $\varphi_1, \varphi_2, \dots, \varphi_{n-1}$  its *antecedent* and the  $S$ -formula  $\varphi_n$  its *succedent*. Sequents are to correspond to “stages” in formal proofs. A sequent calculus is a set of sequence derivation rules. A *rule of derivation* is a set of pairs  $(X, s)$  in which  $X$  is a finite (possibly empty) set of sequents and  $s$  is a single sequent. Elements of  $X$  may be called *premises* of the rule and  $s$  its *conclusion*. Usually rules of derivation are introduced as schemes of the following form:

$$\frac{\begin{array}{c} \Phi_1\varphi_1 \\ \Phi_2\varphi_2 \\ \dots \\ \Phi_n\varphi_n \end{array}}{\Phi\varphi}$$

**Definition 7.1** A sequent  $\Phi\varphi$  is derivable in a sequent calculus  $\mathbf{SQ} = \{\mathbf{R}_1, \mathbf{R}_2, \dots, \mathbf{R}_n\}$  if and only if there exists a finite sequence of sequents  $\Phi_1\varphi_1, \Phi_2\varphi_2, \dots, \Phi_n\varphi_n$  such that

1.  $\Phi_n = \Phi$  and  $\varphi_n = \varphi$ ,
2. every element  $\Phi_i\varphi_i$  in that sequence either is the conclusion of some non-premise rule from  $\mathbf{SQ}$  or there is a rule in  $\mathbf{SQ}$  such that  $\Phi_i\varphi_i$  is its conclusion and its premises are among precedent sequents in the sequence.

If there is a derivation of the sequent  $\Phi\varphi$  in  $\mathbf{SQ}$ , then we write  $\vdash_{\mathbf{SQ}} \Phi\varphi$ .

**Definition 7.2** A formula  $\varphi$  is derivable in  $\mathbf{SQ}$  from a set  $\Phi$  of formulas,  $\Phi \vdash_{\mathbf{SQ}} \varphi$ , if and only if there is a finite number of formulas  $\varphi_1, \varphi_2, \dots, \varphi_n$  in  $\Phi$  such that  $\vdash_{\mathbf{SQ}} \varphi_1, \varphi_2, \dots, \varphi_n\varphi$ .

The calculi below are to yield  $S$ -sequents in such a way that for every  $\mathbf{Cn}$  logic, there is a sequent calculus  $\mathbf{SQn}$  such that  $\varphi$  is a consequence of  $\Phi$  in  $\mathbf{Cn}$  if and only if  $\varphi$  is derivable in  $\mathbf{SQn}$  from  $\Phi$ . My monadic calculi of relative identity (MCRI) will be based on the standard sequent calculus for classical first-order logic with identity defined by the following rules.

$$(R1) \quad \frac{\Phi_1\varphi}{\Phi_2\varphi} \quad \text{if } \Phi_1 \subseteq \Phi_2$$

$$(R2) \quad \frac{}{\Phi\varphi} \quad \text{if } \varphi \in \Phi$$

$$(R3) \quad \frac{\Phi\varphi_1\varphi_2 \quad \Phi\neg\varphi_1\varphi_2}{\Phi\varphi_2}$$

$$(R4) \quad \frac{\Phi \neg \varphi_1 \varphi_2 \quad \Phi \neg \varphi_1 \neg \varphi_2}{\Phi \varphi_1}$$

$$(R5) \quad \frac{\Phi \varphi_1 \varphi_3 \quad \Phi \varphi_2 \varphi_3}{\Phi (\varphi_1 \vee \varphi_2) \varphi_3}$$

$$(R6) \quad \begin{array}{l} \text{(i)} \quad \frac{\Phi \varphi_1}{\Phi (\varphi_1 \vee \varphi_2)} \\ \text{(ii)} \quad \frac{\Phi \varphi_1}{\Phi (\varphi_2 \vee \varphi_1)} \end{array}$$

$$(R7) \quad \frac{\Phi \varphi[\beta/\alpha]}{\Phi \exists \beta \varphi}$$

$$(R8) \quad \frac{\Phi \varphi_1[\beta_1/\beta_2] \varphi_2}{\Phi \exists \beta_1 \varphi_1 \varphi_2} \quad \text{if } \beta_2 \text{ is not free in the sequent } \Phi \exists \beta_1 \varphi_1 \varphi_2$$

$$(R9) \quad \overline{\alpha \equiv \alpha}$$

$$(R10) \quad \frac{\Phi \varphi[\beta/\alpha_2]}{\Phi \alpha_1 = \alpha_2 \varphi[\beta/\alpha_2]}$$

**Definition 7.3**  $SQ0 = \{R1, R2, \dots, R10\}$ .

For example, I will prove in SQ0 the sequent  $(\varphi \vee \neg \varphi)$ :

1.  $\varphi \varphi$             R2
2.  $\varphi(\varphi \vee \neg \varphi)$    R6(i):1
3.  $\neg \varphi \neg \varphi$         R2
4.  $\neg \varphi(\varphi \vee \neg \varphi)$  R6(ii):3
5.  $(\varphi \vee \neg \varphi)$      R3:2,4

MCRIs will be characterized by the following additional rules.

$$(R11A) \quad \frac{\Phi \alpha_1 = \alpha_2 \quad \Phi \delta(\alpha_1)}{\Phi \alpha_1 =_{\delta} \alpha_2}$$

$$(R11B) \quad \frac{\Phi \alpha_1 =_{\delta} \alpha_2}{\Phi \delta(\alpha_1)}$$

$$(R11C) \quad \frac{\Phi \alpha_1 =_{\delta} \alpha_2}{\Phi \alpha_2 =_{\delta} \alpha_1}$$

$$(R11D) \quad \frac{\Phi \alpha_1 =_{\delta} \alpha_2 \quad \Phi \alpha_2 =_{\delta} \alpha_3}{\Phi \alpha_1 =_{\delta} \alpha_3}$$

$$(R12) \quad \frac{\Phi \forall \beta (\delta_1(\beta) \equiv \delta_2(\beta)) \quad \Phi \alpha_1 =_{\delta_1} \alpha_2}{\Phi \alpha_1 =_{\delta_2} \alpha_2}$$

$$(R13) \quad \frac{\Phi \forall \beta (\delta_1(\beta) \rightarrow \delta_2(\beta)) \quad \Phi (\alpha_1 =_{\delta_1} \alpha_2) \wedge (\alpha_2 =_{\delta_2} \alpha_3)}{\Phi \alpha_1 =_{\delta_2} \alpha_3}$$



$$(R14) \quad \frac{\Phi \forall \beta (\delta_1(\beta) \rightarrow \delta_2(\beta)) \quad \Phi(\alpha_1 =_{\delta_1} \alpha_2) \wedge (\alpha_2 =_{\delta_2} \alpha_3)}{\Phi(\alpha_1 =_{\delta_1} \alpha_3)}$$

$$(R15) \quad \frac{\Phi \forall \beta (\delta_1(\beta) \rightarrow \delta_2(\beta)) \quad \Phi(\alpha_1 =_{\delta_1} \alpha_2) \wedge (\alpha_2 =_{\delta_2} \alpha_3)}{\Phi(\alpha_1 =_{\delta_1} \alpha_3 \wedge \alpha_1 =_{\delta_2} \alpha_3)}$$

$$(R16) \quad \frac{\Phi(\alpha_1 =_{\delta_1} \alpha_2) \wedge (\alpha_2 =_{\delta_2} \alpha_3)}{\Phi(\alpha_1 =_{\delta_1} \alpha_3) \vee (\alpha_1 =_{\delta_2} \alpha_3)}$$

$$(R17) \quad \frac{\Phi(\alpha_1 =_{\delta_1} \alpha_2) \quad \Phi \delta_2(\alpha_1)}{\Phi \delta_2(\alpha_2)}$$

$$(R18) \quad \frac{\Phi(\alpha_1 =_{\delta_1} \alpha_2) \quad \Phi(\delta_2(\alpha_1) \wedge \delta_2(\alpha_2))}{\Phi \alpha_1 =_{\delta_2} \alpha_2}$$

$$(R19) \quad \frac{\Phi(\alpha_1 =_{\delta_1} \alpha_2) \quad \Phi \delta_2(\alpha_1)}{\Phi(\alpha_1 =_{\delta_2} \alpha_2)}$$

$$(R20) \quad \frac{\Phi(\alpha_1 =_{\delta} \alpha_2)}{\Phi(\alpha_1 = \alpha_2)}$$

$$(R21) \quad \frac{\Phi(\delta(\alpha_1) \wedge \delta(\alpha_2))}{\Phi \alpha_1 =_{\delta} \alpha_2}$$

**Definition 7.4**  $SQ1 := SQ0 \cup \{R11A, R11B, R11C, R11D\}$ .

**Definition 7.5**  $SQn := SQ1 \cup \{R(10 + n)\}$ .<sup>6</sup>

The following facts will be useful later.

**Fact 7.6**

1.  $\Phi \vdash_{SQn} \delta(\alpha)$  iff  $\Phi \vdash_{SQn} \alpha =_{\delta} \alpha$ .
2. If  $\forall \beta (\Phi \vdash_{SQn} \beta =_{\delta_1} \beta$  only if  $\Phi \vdash_{SQn} \beta =_{\delta_2} \beta)$ ,  
then  $\Phi \vdash_{SQn} \forall \beta (\delta_1(\beta) \rightarrow \delta_2(\beta))$ .

**Fact 7.7** If  $\Phi \vdash_{SQ3,5} \forall \beta (\delta_1(\beta) \rightarrow \delta_2(\beta))$  and  $\Phi \vdash_{SQ3,5} \alpha_1 =_{\delta_1} \alpha_2$ , then  $\Phi \vdash_{SQ3,5} \alpha_1 =_{\delta_2} \alpha_2$ .

MCRIs' counterparts of Facts 5.1 and 10.1 are collected below.

**Fact 7.8**

1. If  $\Phi \vdash_{SQ1} \varphi$ , then  $\Phi \vdash_{SQ2,6,7} \varphi$ .
2. If  $\Phi \vdash_{SQ2} \varphi$ , then  $\Phi \vdash_{SQ3} \varphi$ .
3. If  $\Phi \vdash_{SQ3} \varphi$ , then  $\Phi \vdash_{SQ5,8,11} \varphi$ .
4. If  $\Phi \vdash_{SQ4} \varphi$ , then  $\Phi \vdash_{SQ5} \varphi$ .
5. If  $\Phi \vdash_{SQ5,6,7,8} \varphi$ , then  $\Phi \vdash_{SQ9} \varphi$ .
6. If  $\Phi \vdash_{SQ9} \varphi$ , then  $\Phi \vdash_{SQ10} \varphi$ .

**Proof** Let  $1 \leq n \leq 8$  and  $2 \leq m \leq 10$ . Due to Definitions 7.1 – 7.5, in order to prove that if  $\Phi \vdash_{SQn} \varphi$ , then  $\Phi \vdash_{SQm} \varphi$ , it is sufficient to prove that the conclusion of the rule  $R(n + 10)$  is derivable from its premises in  $SQm$ . The cases (1), (2), (4), and (6) are obvious.

(3) Assume first that  $m = 5$ . The conclusion of (R13) follows from the conclusion of (R15). Now let  $m = 8$ . In SQ1 the sequents  $\Phi\delta_2(\alpha_1)$  and  $\Phi\delta_2(\alpha_2)$  follow from the premises of (R13). From these sequents and the left conjunct of the second premise we obtain (using (R18)) the sequent  $\Phi\alpha_1 =_{\delta_2} \alpha_2$  and then the desired result by (R11D).

Now let  $m = 11$ . The second premise of (R13) entails the sequents  $\Phi\delta_1(\alpha_1)$  and  $\Phi(\delta_2)(\alpha_3)$ . The former with the first premise leads to the sequent  $\Phi\delta_2(\alpha_1)$ , a sequent that together with the latter results in the conclusion of (R13).

(5) Let  $n = 5$ . In SQ1 the sequents  $\Phi\alpha_2 =_{\delta_1} \alpha_2$  and  $\Phi\alpha_2 =_{\delta_2} \alpha_3$  are derivable from the second premise of (R15). (R11B) and the former entail the sequent  $\Phi\delta_1(\alpha_2)$ , a sequent that together with the latter in SQ9 gives the sequent  $\Phi\alpha_2 =_{\delta_1} \alpha_3$ . The sequent  $\Phi\alpha_1 =_{\delta_1} \alpha_3$  is now derivable by means of (R11C)–(R11D).

If  $n = 6$ , then as in the case of  $n = 5$  we get the sequent (\*):

$$(*) \quad \Phi\alpha_1 =_{\delta_1} \alpha_3.$$

The second premise of (R16) entails in SQ9 that  $\Phi\delta_2(\alpha_3)$ , the sequent that yields the sequent  $\Phi\alpha_1 =_{\delta_2} \alpha_3$  by means of (\*) and (R19). The proofs for  $n = 7, 8$  are similar.  $\square$

In the proof of the completeness theorem I will use the standard notion of consistency of a set of formulas with respect to a calculus (cf. Ebbinghaus et al. [3], p. 72).

## 8 Soundness

**Theorem 8.1** *If  $\Phi \vdash_{\text{SQ}n} \varphi$ , then  $\Phi \vdash_{\text{C}n} \varphi$ .*

**Proof** If  $\Phi \vdash_{\text{SQ}n} \varphi$ , then there is a finite set  $\Phi'$  such that  $\Phi' \subseteq \Phi$  and  $\vdash_{\text{SQ}n} \Phi' \varphi$  (cf. Definition 7.2). If we show that for every rule  $R_m$  of SQ $n$ ,  $R_m = (\{\Phi_{i_1}\varphi_{i_1}, \Phi_{i_2}\varphi_{i_2}, \dots, \Phi_{i_m}\varphi_{i_m}\}, \Phi_{j_m}\varphi_{j_m})$ , it holds that

$$(*) \quad \text{if } \Phi_{i_1} \models_{\text{C}n} \varphi_{i_1}, \Phi_{i_2} \models_{\text{C}n} \varphi_{i_2}, \dots, \Phi_{i_m} \models_{\text{C}n} \varphi_{i_m}, \text{ then } \Phi_{j_m} \models_{\text{C}n} \varphi_{j_m},$$

then we prove that  $\Phi' \models_{\text{C}n} \varphi$ , and for that reason that  $\Phi \models_{\text{C}n} \varphi$ .

The proof of (\*) consists of the subproof of the classical rules (R1)–(R10) and the subproof for the rules for RI. The former is a trivial extension of the proof of the counterpart of Theorem 8.1 in the standard first-order sequent calculus. The cases of the rules (R11A)–(R11C) are obvious.

**Case R11D** Assume that  $\Phi \models_{\text{C}1} \alpha_1 =_{\delta} \alpha_2$  and  $\Phi \models_{\text{C}1} \alpha_2 =_{\delta} \alpha_3$ . If  $\mathfrak{F} \models_{\text{C}1} \Phi$ , then  $\exists X \in \mathcal{F}(\delta)\mathfrak{F}(\alpha_1), \mathfrak{F}(\alpha_2) \in X$  and  $\exists X \in \mathcal{F}(\delta)\mathfrak{F}(\alpha_1), \mathfrak{F}(\alpha_2) \in X$ . The disjointness condition (i.e., (C1)) entails that  $\exists X \in \mathcal{F}(\delta)\mathfrak{F}(\alpha_1), \mathfrak{F}(\alpha_3) \in X$ . Hence,  $\Phi \models_{\text{C}1} \alpha_1 =_{\delta} \alpha_3$ .

**Case R12** Let  $\Phi \models_{\text{C}2} \forall\beta(\delta_1(\beta) \equiv \delta_2(\beta))$ ,  $\Phi \models_{\text{C}2} \alpha_1 =_{\delta_1} \alpha_2$ , and  $\mathfrak{F} \models_{\text{C}2} \Phi$ . Then for all  $u \in U$ ,  $\mathfrak{F}_u^{\beta} \models_{\text{C}2} \delta_1(\beta) \equiv \delta_2(\beta)$  and  $\exists X \in \mathcal{F}(\delta_1)\mathfrak{F}(\alpha_1), \mathfrak{F}(\alpha_2) \in X$ . The former entails  $\cup\mathcal{F}\delta_1 = \cup\mathcal{F}\delta_2$ , so the latter by (C2) results in  $\mathfrak{F} \models_{\text{C}2} \alpha_1 =_{\delta_2} \alpha_2$ .

**Case R13** If  $\mathfrak{F} \models_{\text{C}3} \forall\beta(\delta_1(\beta) \rightarrow \delta_2(\beta))$ , then by (C3)

$$(*) \quad \forall X \in \mathcal{F}(\delta_1)\exists Y \in \mathcal{F}(\delta_2)X \subseteq Y.$$

If  $\mathfrak{F} \models_{C3} (\alpha_1 =_{\delta_1} \alpha_2) \wedge (\alpha_2 =_{\delta_2} \alpha_3)$ , then  $\exists X \in \mathcal{F}(\delta_1) \mathfrak{F}(\alpha_1), \mathfrak{F}(\alpha_2) \in X$  and  $\exists X \in \mathcal{F}(\delta_2) \mathfrak{F}(\alpha_2), \mathfrak{F}(\alpha_3) \in X$ . The former with (\*) and (C1) entail that  $\mathfrak{F}(\alpha_1)$  belongs to the set referred to in the latter. This means that  $\mathfrak{F} \models_{C3} \alpha_1 =_{\delta_2} \alpha_3$ .

**Case R14** Let  $\mathfrak{F} \models_{C4} \forall \beta (\delta_1(\beta) \rightarrow \delta_2(\beta))$ . Then by (C4)

$$(*) \quad \forall X \in \mathcal{F}(\delta_2) \exists Y \in \mathcal{F}(\delta_1) (X \subseteq \mathcal{F}(\delta_1) \rightarrow X \subseteq Y).$$

If  $\mathfrak{F} \models_{C4} (\alpha_1 =_{\delta_1} \alpha_2) \wedge (\alpha_2 =_{\delta_2} \alpha_3)$ , then  $\exists X \in \mathcal{F}(\delta_1) \mathfrak{F}(\alpha_1), \mathfrak{F}(\alpha_2) \in X$  and  $\exists X \in \mathcal{F}(\delta_2) \mathfrak{F}(\alpha_2), \mathfrak{F}(\alpha_3) \in X$ . The latter with (\*) and (C1) entail that  $\mathfrak{F}(\alpha_2)$  belongs to the set referred to in the former. This means that  $\mathfrak{F} \models_{C4} \alpha_1 =_{\delta_1} \alpha_3$ .

**Case R15** Assume that  $\Phi \models_{C5} \forall \beta (\delta_1(\beta) \rightarrow \delta_2(\beta))$  and  $\Phi \models_{C5} (\alpha_1 =_{\delta_1} \alpha_2) \wedge (\alpha_2 =_{\delta_2} \alpha_3)$ . It follows that if  $\mathfrak{F} \models_{C5} \Phi$ , then for every  $u \in U$ ,

- (\*) if  $\exists X \in \mathcal{F}(\delta_1) u \in X$ , then  $\exists X \in \mathcal{F}(\delta_2) u \in X$ , and
- (\*\*)  $\exists X \in \mathcal{F}(\delta_1) \mathfrak{F}(\alpha_1), \mathfrak{F}(\alpha_2) \in X$ ,
- (\*\*\*)  $\exists X \in \mathcal{F}(\delta_2) \mathfrak{F}(\alpha_2), \mathfrak{F}(\alpha_3) \in X$ .

(\*) is the antecedent of (C5). Thus, (\*\*) and (\*\*\*) together with (C1) guarantee that there is some set in  $\mathcal{F}(\delta_1)$  that contains  $\mathfrak{F}(\alpha_1)$ ,  $\mathfrak{F}(\alpha_2)$ , and  $\mathfrak{F}(\alpha_3)$  and that there is some set in  $\mathcal{F}(\delta_2)$  that also contains  $\mathfrak{F}(\alpha_1)$ ,  $\mathfrak{F}(\alpha_2)$ , and  $\mathfrak{F}(\alpha_3)$ . This entails  $\mathfrak{F} \models_{C5} \alpha_1 =_{\delta_1} \alpha_3 \wedge \alpha_1 =_{\delta_2} \alpha_3$ .

**Case R16** Let  $\mathfrak{F} \models_{C6} (\alpha_1 =_{\delta_1} \alpha_2) \wedge (\alpha_2 =_{\delta_2} \alpha_3)$ . Definition 3.4(2) guarantees that there are two sets  $X$  and  $Y$  such that

- (\*)  $X \in \mathcal{F}(\delta_1)$  and  $\mathfrak{F}(\alpha_1), \mathfrak{F}(\alpha_2) \in X$ .
- (\*\*)  $Y \in \mathcal{F}(\delta_2)$  and  $\mathfrak{F}(\alpha_2), \mathfrak{F}(\alpha_3) \in Y$ .

Thus,  $X \cap Y \neq \emptyset$ , and because of (C6), either  $X \subseteq Y$  or  $Y \subseteq X$ , that is, either  $\mathfrak{F} \models_{C6} \alpha_1 =_{\delta_2} \alpha_3$  or  $\mathfrak{F} \models_{C6} \alpha_1 =_{\delta_1} \alpha_3$ .

**Case R17** If  $\mathfrak{F} \models_{C7} \delta_2(\alpha_1)$  and  $\mathfrak{F} \models_{C7} \alpha_1 =_{\delta_1} \alpha_2$ , then there are sets  $X, Y$  such that  $\mathfrak{F}(\alpha_1), \mathfrak{F}(\alpha_2) \in X \in \mathcal{F}(\delta_1)$ , and  $\mathfrak{F}(\alpha_1) \in Y \in \mathcal{F}(\delta_2)$ . Therefore,  $X \cap Y \neq \emptyset$ , and due to (C7),  $X \subseteq \cup \mathcal{F}(\delta_2)$ . Consequently,  $\exists X \in \mathcal{F}(\delta_2) \mathfrak{F}(\alpha_2) \in X$  and  $\mathfrak{F} \models_{C7} \delta_2(\alpha_2)$ , as desired.

**Case R18** If  $\mathfrak{F} \models_{C8} \alpha_1 =_{\delta_1} \alpha_2$  and  $\mathfrak{F} \models_{C8} \delta_2(\alpha_1) \wedge (\delta_2(\alpha_1))$ , then there are three sets  $X, Y, Z$  such that  $X \in \mathcal{F}(\delta_1) \mathfrak{F}(\alpha_1), \mathfrak{F}(\alpha_2) \in X, Y \in \mathcal{F}(\delta_2) \mathfrak{F}(\alpha_1) \in Y$ , and  $Z \in \mathcal{F}(\delta_2) \mathfrak{F}(\alpha_2) \in Z$ . (C8) entails that  $Y = Z$ , thus  $\exists X \in \mathcal{F}(\delta_2) \mathfrak{F}(\alpha_1), \mathfrak{F}(\alpha_2) \in X$ , as desired.

**Case R19** Assume that  $\Phi \models_{C9} \delta_2(\alpha_2)$  and  $\Phi \models_{C9} \alpha_1 =_{\delta_1} \alpha_2$ . If  $\mathfrak{F} \models_{C9} \Phi$ , then

- (\*)  $\exists X \in \mathcal{F}(\delta_2) \mathfrak{F}(\alpha_2) \in X$ ,
- (\*\*)  $\exists X \in \mathcal{F}(\delta_1) \mathfrak{F}(\alpha_1), \mathfrak{F}(\alpha_2) \in X$ .

By (C9) these two consequences entail that there is a set contained both in  $\mathcal{F}(\delta_1)$  and  $\mathcal{F}(\delta_2)$  that contains  $\mathfrak{F}(\alpha_2)$ . (\*\*) implies that  $\mathfrak{F}(\alpha_1)$  belongs to this set as well. That suffices for  $\mathfrak{F} \models_{C9} \alpha_1 =_{\delta_2} \alpha_2$ .

**Case R20** Suppose that  $\mathfrak{F} \models_{C10} \alpha_1 =_{\delta} \alpha_2$ . It follows that  $\exists X \in \mathcal{F}(\delta) \mathfrak{F}(\alpha_1), \mathfrak{F}(\alpha_2) \in X$ . By (C10), we have  $\mathfrak{F}(\alpha_1) = \mathfrak{F}(\alpha_2)$ , that is,  $\mathfrak{F} \models_{C10} \alpha_1 = \alpha_2$ .

**Case R21** Assume that  $\mathfrak{F} \models_{\text{C11}} \delta(\alpha_1) \wedge \delta(\alpha_2)$ . Thus,  $\exists X \in \mathcal{F}(\delta) \mathfrak{F}(\alpha_1) \in X$  and  $\exists X \in \mathcal{F}(\delta) \mathfrak{F}(\alpha_2) \in X$ , that is,  $\mathfrak{F}(\alpha_1), \mathfrak{F}(\alpha_2) \in \cup \mathcal{F} \delta$ . From (C11) it follows that  $\exists X \in \mathcal{F}(\delta) \mathfrak{F}(\alpha_1), \mathfrak{F}(\alpha_2) \in X$ , so  $\Phi \models_{\text{C11}} \alpha_1 =_{\delta} \alpha_2$ .  $\square$

## 9 Completeness

**Theorem 9.1** *If  $\Phi \models_{\text{Cn}} \varphi$ , then  $\Phi \vdash_{\text{SQn}} \varphi$ .*

In order to justify Theorem 9.1, I will modify Henkin's proof for the standard first-order system. First, I show the following.

**Lemma 9.2** *Every SQn-consistent S-set is Cn-satisfiable.*

**Proof** Let  $\Phi$  be a set of S-formulas. In order to verify Lemma 9.2, I will construct  $n$  term-structures  $\mathfrak{S}_n^{\Phi} = (U_n^{\Phi}, \mathcal{F}_n^{\Phi})$  in which every SQn-consistent set is Cn-satisfiable. Their universes consist of elements identical "according to  $\Phi$ ."

**Definition 9.3**  $\underline{\alpha}^n := \{\alpha' \in T^S : \Phi \vdash_{\text{SQn}} \alpha = \alpha'\}$ .

**Definition 9.4**  $U_n^{\Phi} := \{\underline{\alpha}^n : \alpha \in T^S\}$ .

Their functions of structure may be defined as follows.

**Definition 9.5**  $X \in \mathcal{F}_n^{\Phi}(\delta)$  iff  $\forall \underline{\alpha}^n, \underline{\alpha}'^n, (\underline{\alpha}^n \in X \rightarrow (\underline{\alpha}'^n \in X \equiv \Phi \vdash_{\text{SQn}} \alpha =_{\delta} \alpha'))$ .

**Definition 9.6**  $\mathcal{F}_n^{\Phi}(\chi) := \underline{\chi}^n$ .

Fixing an assignment  $\mathfrak{g}_n^{\Phi}(\beta) := \underline{\beta}^n$  we get  $n$  term-interpretations  $\mathfrak{F}_n^{\Phi} := (\mathfrak{S}_n^{\Phi}, \mathfrak{g}_n^{\Phi})$ . In order to show that they characterize Cn logics we must prove the following.

**Lemma 9.7** *Every  $\mathcal{F}_n^{\Phi}$  function satisfies the Cn condition.*

**Proof**

**Definition 9.8**  $\text{RI}_{\alpha, \delta}^n := \{\underline{\alpha}'^n : \Phi \vdash_{\text{SQn}} \alpha =_{\delta} \alpha'\}$ .

In what follows it is helpful to notice that by Definition 9.8 and (R11D) we have the following.

**Fact 9.9**  $\text{RI}_{\alpha, \delta}^n \in \mathcal{F}_n^{\Phi}(\delta)$ .

For the sake of simplicity,  $\underline{\alpha}^n, \underline{\alpha}'^n, \mathcal{F}_n^{\Phi}$  will be in the  $n$ th case abbreviated by  $\underline{\alpha}, \underline{\alpha}', \mathcal{F}$ .

( $n = 1$ ) Assume otherwise. Then there exist two distinct sets  $X, Y \in \mathcal{F}(\delta)$  such that  $X \cap Y \neq \emptyset$ . Let  $\underline{\alpha} \in X, Y$ . Now by Definition 9.5, for all  $\underline{\alpha}'$ , it is the case that  $\underline{\alpha}' \in X \equiv \Phi \vdash_{\text{SQ1}} \alpha =_{\delta} \alpha'$  and  $\underline{\alpha}' \in Y \equiv \Phi \vdash_{\text{SQ1}} \alpha =_{\delta} \alpha'$ , that is,  $X = Y$ . This contradicts our assumption to the effect that  $X$  and  $Y$  are distinct.

( $n = 2, 3$ ) Proofs are similar to the proof for the case when  $n = 5$ .

( $n = 4$ ) Assume that  $\cup \mathcal{F}(\delta_1) \subseteq \cup \mathcal{F}(\delta_2)$ . Because of Definition 9.5 this gives

(\*) for all  $\underline{\alpha} \in U$ , if  $\exists X[X \in \cup \mathcal{F}(\delta_1)$  and  $\underline{\alpha} \in X$ ],  
then  $\exists X[\forall \underline{\alpha}', \underline{\alpha}''(\underline{\alpha}' \in X \rightarrow (\underline{\alpha}'' \in X \equiv \Phi \vdash_{\text{SQ4}} \alpha' =_{\delta_2} \alpha''))$  and  $\underline{\alpha} \in X]$ .

It is clear that  $\underline{\alpha} \in \text{RI}_{\alpha, \delta}^4$  if  $\Phi \vdash_{\text{SQ6}} \alpha =_{\delta} \alpha$ . Therefore from (\*) it follows that

(\*\*)  $\forall \beta(\text{if } \Phi \vdash_{\text{SQ4}} \beta =_{\delta_1} \beta, \text{ then } \Phi \vdash_{\text{SQ4}} \beta =_{\delta_2} \beta)$ .

Facts 7.6 and 7.8 and (\*\*) entail that

$$(***) \quad \Phi \vdash_{\text{SQ4}} \forall \beta (\delta_1(\beta) \rightarrow \delta_2(\beta)).$$

Suppose that

$$(****) \quad X \in \mathcal{F}(\delta_2),$$

and

$$(***) \quad X \subseteq \mathcal{F}(\delta_1).$$

Let  $\underline{\alpha} \in X$  and  $\underline{\alpha}' \in X$ . Let  $Y_0 \in \mathcal{F}(\delta_1)$  denote the set to which  $\underline{\alpha}'$  belongs (cf. Definition 3.1 and (\*\*\*)). I will show that  $\underline{\alpha}$  belongs to  $Y_0$  as well. By (\*\*\*) and Definition 9.5 we get that  $\Phi \vdash_{\text{SQ4}} \alpha =_{\delta_2} \alpha'$ . By (\*\*\*), Definition 9.5, and Fact 7.8 we get that  $\Phi \vdash_{\text{SQ4}} \alpha' =_{\delta_1} \alpha'$ . Thus,  $\Phi \vdash_{\text{SQ4}} \alpha =_{\delta_1} \alpha'$  due to (R14). (9.5) entails now that  $\underline{\alpha}$  belongs to the same element of  $\mathcal{F}(\delta_1)$  as  $\underline{\alpha}'$ , namely,  $Y_0$ .

( $n = 5$ ) Assume that  $\cup \mathcal{F}(\delta_1) \subseteq \cup \mathcal{F}(\delta_2)$ . As in the previous case this assumption entails that

$$(***) \quad \Phi \vdash_{\text{SQ5}} \forall \beta (\delta_1(\beta) \rightarrow \delta_2(\beta)).$$

But now due to Fact 7.7, (\*\*\*) entails that

$$(***) \quad \text{If } \Phi \vdash_{\text{SQ5}} \alpha_1 =_{\delta_1} \alpha_2, \text{ then } \Phi \vdash_{\text{SQ5}} \alpha_1 =_{\delta_2} \alpha_2.$$

Suppose that  $X \in \mathcal{F}(\delta_1)$ . Let  $\underline{\alpha} \in X$  and  $\underline{\alpha}' \in X$ . Then  $\Phi \vdash_{\text{SQ5}} \alpha =_{\delta_1} \alpha'$ , and by (\*\*\*)  $\Phi \vdash_{\text{SQ5}} \alpha =_{\delta_2} \alpha'$ . Let now  $\Phi \vdash_{\text{SQ5}} \alpha =_{\delta_2} \alpha'$  and  $\underline{\alpha} \in X$ . The latter entails that  $\Phi \vdash_{\text{SQ5}} \alpha =_{\delta_1} \alpha$ , consequently the former and (R15) gives the sequent  $\Phi \vdash_{\text{SQ5}} \alpha =_{\delta_1} \alpha'$ . Since  $\underline{\alpha} \in X$ , therefore  $\underline{\alpha}' \in X$ . In this way we have shown that  $X \in \mathcal{F}(\delta_2)$ .

( $n = 6$ ) Assume otherwise. Then there are two sets  $X, Y$  such that  $X \in \mathcal{F}(\delta_1)$ ,  $Y \in \mathcal{F}(\delta_2)$ ,  $X \cap Y \neq \emptyset$ ,  $X \not\subseteq Y$ , and  $Y \not\subseteq X$ . Subsequently, there are three  $U$ -elements  $\underline{\alpha}, \underline{\alpha}', \underline{\alpha}''$  such that  $\underline{\alpha} \in X \cap Y$ ,  $\underline{\alpha}' \in X \setminus Y$ , and  $\underline{\alpha}'' \in Y \setminus X$ . Therefore,

$$(*) \quad \Phi \vdash_{\text{SQ6}} \alpha =_{\delta_1} \alpha' \text{ and } \Phi \vdash_{\text{SQ6}} \alpha =_{\delta_2} \alpha''.$$

$$(**) \quad \Phi \not\vdash_{\text{SQ6}} \alpha =_{\delta_2} \alpha' \text{ and } \Phi \not\vdash_{\text{SQ6}} \alpha =_{\delta_1} \alpha''.$$

(R16) applied to (\*) gives

$$(***) \quad \Phi \vdash_{\text{SQ6}} (\alpha' =_{\delta_1} \alpha'') \vee (\alpha' =_{\delta_2} \alpha'').$$

Now, the left disjunct of (\*\*\*) with the left conjunct of (\*) contradicts, by (R11C)–(R11D), the right part of (\*\*), and the right disjunct with the right conjunct contradicts the left part.

( $n = 7$ ) Suppose that  $X \in \mathcal{F}(\delta_1)$ ,  $Y \in \mathcal{F}(\delta_2)$ , and  $X \cap Y \neq \emptyset$ . Then there exist  $\underline{\alpha}$  such that

$$(*) \quad \forall \underline{\alpha}' (\underline{\alpha}' \in X \equiv \Phi \vdash_{\text{SQ7}} \alpha =_{\delta_1} \alpha').$$

and because of (R11B)

$$(**) \quad \Phi \vdash_{\text{SQ7}} \delta_2(\alpha).$$

Let  $\underline{\alpha}'' \in X$ . From (\*) it follows that  $\Phi \vdash_{\text{SQ7}} \alpha =_{\delta_1} \alpha''$ , so in view of (\*\*) it is the case that  $\Phi \vdash_{\text{SQ7}} \alpha'' =_{\delta_2} \alpha''$  (cf. (R17) and Fact 7.6). Consequently,  $\underline{\alpha}'' \in \text{RI}_{\alpha'', \delta_2}^7$  and  $\underline{\alpha}'' \in \cup \mathcal{F}(\delta_2)$  (Fact 9.9). This completes the proof that  $X \subseteq \cup \mathcal{F}(\delta_2)$ .

( $n = 8$ ) Let  $X \in \mathcal{F}(\delta_1)$ ,  $Y \in \mathcal{F}(\delta_2)$ ,  $Z \in \mathcal{F}(\delta_2)$ ,  $X \cap Y \neq \emptyset$ ,  $X \cap Z \neq \emptyset$ . This means that there are  $\underline{\alpha}, \underline{\alpha}'$ , such that  $\underline{\alpha} \in X \cap Y$ ,  $\underline{\alpha}' \in X \cap Z$ . Thus,

- (\*)  $\forall \underline{\alpha}'' [(\underline{\alpha}'' \in X \equiv \Phi \vdash_{\text{SQ8}} \alpha =_{\delta_1} \alpha'') \wedge (\underline{\alpha}'' \in X \equiv \Phi \vdash_{\text{SQ8}} \alpha' =_{\delta_1} \alpha'')]$ .
- (\*\*)  $\forall \underline{\alpha}'' (\underline{\alpha}'' \in Y \equiv \Phi \vdash_{\text{SQ8}} \alpha =_{\delta_2} \alpha'')$ .
- (\*\*\*)  $\forall \underline{\alpha}'' (\underline{\alpha}'' \in Z \equiv \Phi \vdash_{\text{SQ8}} \alpha' =_{\delta_2} \alpha'')$ .

From (\*) it follows that  $\Phi \vdash_{\text{SQ8}} \alpha =_{\delta_1} \alpha'$  (R11D). Since  $\underline{\alpha} \in Y$  and  $\underline{\alpha}' \in Z$ ,  $\Phi \vdash_{\text{SQ8}} \delta_2(\alpha)$  and  $\Phi \vdash_{\text{SQ8}} \delta_2(\alpha')$  because of (\*\*), (\*\*\*), and 7.6(1). Due to (R18),  $\Phi \vdash_{\text{SQ8}} \alpha =_{\delta_2} \alpha'$  holds. Therefore, if  $\underline{\alpha}'' \in Y$ , then  $\Phi \vdash_{\text{SQ8}} \alpha' =_{\delta_2} \alpha''$  by (\*\*) and (R11D). Thus  $\underline{\alpha}'' \in Z$  owing to (\*\*\*). The proof that  $Z \subseteq Y$  is analogous.

( $n = 9$ ) Let  $X \in \mathcal{F}(\delta_1)$ ,  $Y \in \mathcal{F}(\delta_2)$ , and  $X \cap Y \neq \emptyset$ . Then there is some  $\underline{\alpha}$  such that

- (\*)  $\forall \underline{\alpha}' (\underline{\alpha}' \in X \equiv \Phi \vdash_{\text{SQ9}} \alpha =_{\delta_1} \alpha')$ .
- (\*\*)  $\Phi \vdash_{\text{SQ9}} \delta_2(\alpha)$ .

In order to prove that  $X \in \mathcal{F}(\delta_2)$  assume  $\underline{\alpha}', \underline{\alpha}'' \in X$ . Then  $\Phi \vdash_{\text{SQ9}} \alpha' =_{\delta_1} \alpha''$  and  $\Phi \vdash_{\text{SQ9}} \alpha =_{\delta_1} \alpha'$ . The latter and (\*\*) entail  $\Phi \vdash_{\text{SQ9}} \delta_2(\alpha')$ ; therefore the former gives  $\Phi \vdash_{\text{SQ9}} \alpha' =_{\delta_2} \alpha''$ . If  $\underline{\alpha}' \in X$  and  $\Phi \vdash_{\text{SQ9}} \alpha' =_{\delta_2} \alpha''$ , then we obtain  $\Phi \vdash_{\text{SQ9}} \delta_1(\alpha')$  from the former and (R11B). Consequently, the latter entail  $\Phi \vdash_{\text{SQ9}} \alpha' =_{\delta_1} \alpha''$ . The desired result,  $\underline{\alpha}'' \in X$ , follows from (\*).

( $n = 10$ ) If  $X \in \mathcal{F}(\delta)$ , then for all  $\underline{\alpha}, \underline{\alpha}' \in X$ ,  $\Phi \vdash_{\text{SQ10}} \alpha =_{\delta} \alpha'$ , and by (R20)  $\Phi \vdash_{\text{SQ10}} \alpha = \alpha'$ . According to the definition of the term-structure, this means that  $\underline{\alpha} = \underline{\alpha}'$ . Therefore,  $X$  has exactly one element (Definition 3.1(1)), that is to say,  $\exists u \in \cup \mathcal{F}(\delta) X = \{u\}$ .

( $n = 11$ ) If  $\underline{\alpha}, \underline{\alpha}' \in \cup \mathcal{F}(\delta)$ , then  $\Phi \vdash_{\text{SQ11}} \alpha =_{\delta} \alpha$  and  $\Phi \vdash_{\text{SQ11}} \alpha' =_{\delta} \alpha'$ . Now by (R11B)  $\Phi \vdash_{\text{SQ11}} \delta(\alpha)$  and  $\Phi \vdash_{\text{SQ11}} \delta(\alpha')$ . (R21) yields that  $\Phi \vdash_{\text{SQ11}} \alpha =_{\delta} \alpha'$ . Now let  $\underline{\alpha} \in \cup \mathcal{F}(\delta)$  and  $\Phi \vdash_{\text{SQ11}} \alpha =_{\delta} \alpha'$ . The former entails that there is some set  $X \in \mathcal{F}(\delta)$  such that if  $\Phi \vdash_{\text{SQ11}} \alpha =_{\delta} \alpha'$ , then  $\underline{\alpha}' \in X$ . Hence, due to the latter,  $\underline{\alpha}' \in X \subseteq \cup \mathcal{F}(\delta)$ . This fulfills Definition 9.5 condition for  $\cup \mathcal{F}(\delta) \in \mathcal{F}(\delta)$ .  $\square$

Now, following Henkin's proof, I will prove two facts from which Lemma 9.2 follows immediately.

**Fact 9.10** For every SQ $n$ -consistent and negation-complete (with respect to SQ $n$ ) set  $\Phi$  that contains witnesses (with respect to SQ $n$ ), and for every formula  $\varphi$ ,  $\mathfrak{S}_n^{\Phi} \models_{\text{Cn}} \varphi$  if and only if  $\Phi \vdash_{\text{SQn}} \varphi$ .

**Fact 9.11** Every SQ $n$ -consistent set  $\Phi$  is contained in some SQ $n$ -consistent and negation-complete (with respect to SQ $n$ ) set  $\Phi'$  that contains witnesses (with respect to SQ $n$ ).

The definition of a negation-complete set (with respect to a calculus) and the definition of a set containing witnesses (with respect to a calculus) are standard (cf. [3], p. 78).

**Proof of 9.10** The proof is by induction on the construction of  $\varphi$ .

(i) The basis of the induction

**Case of  $\varphi = \delta(\alpha)$**   $\delta_n^\Phi \models_{Cn} \delta(\alpha)$  if and only if  $\exists X \in \mathcal{F}_n^\Phi(\delta) \delta_n^\Phi(\alpha) \in X$  if and only if  $\exists X[\forall \underline{\alpha}^n, \underline{\alpha}'^n, (\underline{\alpha}^n \in X \rightarrow (\underline{\alpha}'^n \in X \equiv \Phi \vdash_{SQn} \alpha =_\delta \alpha'))$  and  $\underline{\alpha}^n \in X$  only if  $\Phi \vdash_{SQn} \alpha =_\delta \alpha$  if and only if (by Facts 7.6(1) and 7.8)  $\Phi \vdash_{SQn} \delta(\alpha)$ .

$\Phi \vdash_{SQn} \delta(\alpha)$  if and only if (by Facts 7.6(1) and 7.8)  $\Phi \vdash_{SQn} \alpha =_\delta \alpha$ . Therefore,  $\underline{\alpha}^n \in RI_{\alpha, \delta}^n$ . Consequently (by Fact 9.9),  $\exists X \in \mathcal{F}_n^\Phi(\delta) \delta_n^\Phi(\alpha) \in X$ , and  $\delta_n^\Phi \models_{Cn} \delta(\alpha)$ .

**Case of  $\varphi = \alpha =_\delta \alpha'$**   $\delta_n^\Phi \models_{Cn} \alpha =_\delta \alpha'$  if and only if  $\exists X \in \mathcal{F}_n^\Phi(\delta) \delta_n^\Phi(\alpha), \delta_n^\Phi(\alpha') \in X$  if and only if  $\exists X[\forall \underline{\alpha}^n, \underline{\alpha}'^n (\underline{\alpha}^n \in X \rightarrow (\underline{\alpha}'^n \in X \equiv \Phi \vdash_{SQn} \alpha =_\delta \alpha'))$  and  $\underline{\alpha}^n, \underline{\alpha}'^n \in X$  only if  $\Phi \vdash_{SQn} \alpha =_\delta \alpha'$ .

If  $\Phi \vdash_{SQn} \alpha =_\delta \alpha'$ , then (by (R11B) and Fact 7.8),  $\underline{\alpha}^n, \underline{\alpha}'^n \in RI_{\alpha, \delta}^n$ . Since this set belongs to  $\mathcal{F}_n^\Phi(\delta)$  by Fact 9.9, we have  $\delta_n^\Phi \models_{Cn} \alpha =_\delta \alpha'$ .

(ii) The inductive step involves only those satisfaction conditions and rules for deriving sequents that are shared with the standard system.  $\square$

**Proof of 9.11** In order to verify (9.11) we need two more lemmas.

**Lemma 9.12** *Every SQn-consistent S-set  $\Phi$  such that the set of free variables in formulas of  $\Phi$  is finite can be extended to an SQn-consistent S-set that contains witnesses (with respect to SQn).*

**Lemma 9.13** *Every SQn-consistent S-set  $\Phi$  can be extended to an SQn-consistent and negation-complete (with respect to SQn) S-set.*

Then, using Henkin's device of extending the set of constants, we get (9.11). The proof of this fact and proofs of Lemmas 9.12 and 9.13 are similar to the standard derivations.  $\square$

**Proof of Theorem 9.1** If  $\Phi \models_{Cn} \varphi$  and  $\Phi \not\vdash_{SQn} \varphi$ , then  $\Phi \cup \{\neg\varphi\}$  is SQn-consistent but not Cn-satisfiable, contrary to Lemma 9.2.  $\square$

## 10 Logical Objection

One of the most significant objections to RI comes from Wiggins. He argues at some length that RI is to be interpreted as AI confined to some set of objects, that is,

$$(*) \quad x =_A y \equiv A(x) \wedge x = y.^7$$

In terms of my formal landscape, his position has it that the only plausible logic of RI is deviant (C10). That strongly suggests that the RI phenomena stated in Assumption 1.1 are deceptive.

Wiggins admits that the force of his arguments comes mainly from Leibniz's Law:

(LL) If  $x$  is the same as  $y$ , then all properties of  $x$  are properties of  $y$ .

Assume that RI is governed by an MLRI different from (C10). This entails that RI need not satisfy (LL). Wiggins argues that then there is no reason to classify it as identity.

Leibniz's Law marks off what is peculiar to real identity and what differentiates it in a way in which transitivity, symmetry, and reflexivity . . . do not.

How if  $a$  is  $b$  could be something true of the object  $a$  which was untrue of the object  $b$ ? After all they are the same object.

If Leibniz's Law is dropped . . . , then we need some formal principle or other, and one of at least compared universality, to justify the instances of the intersubstitution of identicals that evidently are valid. The instability, indeterminacy or arbitrariness of all extant emendations or relativizations of Leibniz's Law constitutes an important part of the case for a pure congruence principle such as Leibniz's. (Wiggins [15], p. 27–28)

I propose calling Wiggins's position the *logical objection*. There are two possible answers to the logical objection. The first was suggested by Deutsch. He admits that there are no formal counterparts of (LL) for RI; nevertheless, there are material counterparts. Or better, there are no a priori justifiable counterparts and we are left with a posteriori principles. Deutsch claims that if  $x$  is the same  $A_1$  as  $y$  and  $x$  is  $A_2$ , then whether  $y$  is  $A_2$  depends on  $A_1$  and  $A_2$ . There are no logical warrants here. For example, if  $x$  is the same ship as  $y$  and  $x$  is a galley, then  $y$  is also a galley, but  $x$  might be painted green without  $y$  being green. I would add that Wiggins is right up to a point: RI is a relation sui generis. Whether it is a real identity seems to be a verbal problem of little importance. It is the logical properties of RI that are far more important.

If it holds that if  $\alpha =_{\delta_1} \beta$  and  $\delta_2(\alpha)$ , then  $\delta_2(\beta)$ , then, following Deutsch, I will say that a predicate  $\delta_2$  is  $\delta_1$ -subscripted. Deutsch notes that RI and indiscernibility principles for subscripted predicates are to be respected because they help us solve problems about change, allographic objects, and the constitution of material things (cf. [2]).

The second answer points out that in (C7) and (C9) there are “restricted” counterparts of (LL).

**Fact 10.1**

- (i)  $\models_{C7} \alpha =_{\delta_1} \beta \rightarrow (\varphi \rightarrow \varphi[\beta/\alpha])$  provided that  $\varphi$  does not contain either ‘=’ or ‘ $=_{\delta_2}$ ’.
- (ii)  $\models_{C9} \alpha =_{\delta_1} \beta \rightarrow (\varphi \rightarrow \varphi[\beta/\alpha])$  provided that  $\varphi$  does not contain ‘=’.

## 11 Ambiguity Objection

Wiggins also provides a detailed analysis of alleged counterexamples to (LL) (they are similar to my examples from Assumption 1.1). His analysis rests on (LL) and the postulate of referential univocity of the terms and concepts we use. The upshot of that analysis is that most delusive RI phenomena are due to the vicious referential ambiguity of terms and concepts. Consider for example the term ‘an inscription’. Wiggins seems to require that we should determine whether, in using that term, we refer to token-words or to type-words. In the former case it is false that an inscription ‘identity’ is the same type-word as an inscription ‘identity’ and it is false that an inscription ‘identity’ is the same token-word as an inscription ‘identity’. In the latter case it is true that an inscription ‘identity’ is the same type-word as an inscription ‘identity’ and it is false that an inscription ‘identity’ is the same token-word as an inscription ‘identity’. Still, neither inscription is a token-word, so Wiggins's absolutism is not threatened: in both cases the RI-phenomenon disappears.

First, let me observe that Wiggins's postulate has never been realized. In this respect it resembles other postulates of ideal languages that up to now have never been constructed. At present we are doomed to use imperfect concepts and terms. Secondly, the epistemological cost of the postulate seems to be high. A lot of terms



we use are referentially ambiguous, so Wiggins's reform would have to modify a vast part of our language. If we wished to remove all such ambiguity, we would have to at least double the number of terms in the language. For example, if we wished to remove the RI phenomenon generated by the term '*Goldberg Variations*', we would have to introduce at least two terms, for example, 'score of *Goldberg Variations*' and 'interpretation of *Goldberg Variations*'. Thirdly, we would guarantee that a term we introduced is immune to referential ambiguity only if some day we closed our conceptual vocabulary. Otherwise, it will always be possible that new concepts and terms will 'discover' the referential ambiguity of the old concepts and terms. Consider our new allegedly univocal term 'interpretation of *Goldberg Variations*'. Suppose that some day music lovers will pay more attention to the kind of instrument an interpretation is performed on. The same instruments will correspond to the *same executions* of an interpretation and different instruments correspond to *different executions*. Then it is possible that one interpretation of *Goldberg Variations* is the same interpretation as the other interpretation but a different execution.

In conclusion, I find Wiggins's postulate untenable if it is construed as a global requirement that eliminates RI altogether. In my opinion we have to learn to live with RI phenomena and it is to be hoped that MLRIs will help us to do so.

## 12 *Embarras de Richesse*

The variety of MLRIs is likely to raise the old perplexing problem, Which logic is the correct one? (for monists) or How to choose a logic? (for pluralists). Unfortunately, we get little help here from the few proponents of RI as they mainly ramble on about epistemology and/or philosophy of language, which has hardly any bearing on the logic of RI.

There are at least two strategies for approaching this problem. First, if you wish to preserve RI-minimalism, you ought to compare MLRIs with paradigms of RI-sentences from Assumption 1.1. Then, either you assume that there is only one type of RI and choose from MLRIs the strongest logic that is consistent with those sentences or you maintain that RI is ambiguous and distinguish various kinds thereof. Secondly, you may be tempted to abandon minimalism and risk some philosophy in order to explain RI-phenomena. Then your explanations should provide a criterion for "valid" MLRIs.

I begin with the first strategy. Assume that RI is univocal. If you compare the examples of RI sentences to MLRIs, then you will have to content yourself with (C1). (C7) (and subsequently (C9)) is falsified by all examples. Assumption 1.1(4) undermines (C2) (and subsequently other MLRIs). For example, the fact that Gould's *Goldberg Variations* is shorter than Perahia's contradicts Fact 5.2(5). Similarly, since 'rational number' and 'fraction' are extensionally equivalent, Assumption 1.1(4) is a counterexample to Fact 5.2(1). On the other hand, if you consider RI ambiguous, then your choice depends on your views on the meanings of RI, but you must not choose (C9) or (C7) as they are falsified by all examples.

However, one may object that within RI-minimalism all nonminimal MLRIs are only formal structures without any philosophical merit. It is the second of the aforementioned strategies that can address such objections. In what follows I will present a philosophical conception that explains the main features of some MLRIs, but I do not claim that my proposal is the only reason for introducing MLRIs. I hope that MLRIs may be fortified on different grounds.

The philosophical conception behind my semantics begins with the assumption to the effect that the ontologically ultimate reality is determinately “carved up” into AI-objects. Let  $X$  denote a set of AI-objects and a family  $A$  of predicates represent monadic properties possessed by objects from  $X$ . (For the sake of brevity I will treat  $A$  also as a set of extensions of predicates.) Contrary to the classical case it is assumed that if we describe  $X$  by means of predicates from  $A$ , then, for some reason or other, only sets of AI-objects are accessible for us; that is to say, “from the point of view of  $A$ ” we are in a position to recognize RI-objects but not AI-objects. Single elements of  $X$  escape from our conceptual net. For example, if someone says “The ship of Theseus was magnificent,” she does not and cannot specify which of at least two, as Assumption 1.1(2) proves, ships of Theseus was magnificent.

This assumption strongly suggests defining relative identity in terms of indiscernibility. It seems to me that such a strategy could have been approved by Geach. Since there are some problems with interpretations of his views (see for example [6], pp. 142–54 and Noonan [10]), and since he himself complains about frequent misinterpretations, I will quote in extenso the latest formulation of his argument.

No criterion has been given, or, I think, could be given for a predicable’s being used in a language  $L$  to express absolute identity. The familiar axiom schemata for identity could at most guarantee, if satisfied, that the relative term under investigation will be true in  $L$  only of pairs that are indiscernible by description framed in terms of the other predicables of  $L$ . This cannot guarantee that there is no proper extension of  $L$ , with extra predicables, that makes possible the discrimination of things that were indiscernible by the resources of  $L$ . If there is such a proper extension of  $L$ , obviously the friend of absolute identity will not be able to say that a predicable of  $L$  which in  $L$  satisfied the axiom schemata for identity was an absolute-identity predicable. ([5], p. 297).

Presumably he argues here that the definition of AI is always relative to some stock of predicates and that absolutely identical objects may cease to be identical if we enlarge this stock. To wit, we are able to define only sets of objects that are indiscernible relative to some stock of predicates.

Geach would probably agree that it is possible that  $A \neq \emptyset(X)$ . Let ‘ $x \approx_A y$ ’ abbreviate ‘ $x$  is indiscernible from  $y$  in respect of properties from  $A$ ’.

**Definition 12.1**  $x \approx_A y \equiv \forall A_i \in A (A_i(x) \equiv A_i(y))$ .

Indiscernibility may be interpreted either as an epistemological or ontological phenomenon. *Epistemic indiscernibility* (for an epistemic subject  $P$ ) interprets  $A$  as a set of predicates with which  $P$  is familiar at a given time. Obviously, epistemic indiscernibility varies through times and subjects. On the other hand, *ontologically interpreted indiscernibility* assumes that  $A$  represents all properties that objects from  $X$  may possess.

**Definition 12.2**  $x =_A y \equiv x \approx_A y \wedge (A(x) \wedge A(y))$ .

If we now define RI in terms of indiscernibility as in Definition 12.2, then it can be proved that RI satisfies the (C9) characteristic theorem 5.2(7), but, provided that  $A \neq \emptyset(X)$ , (12.2) does not in general lead to (C10), that is, ‘ $=_A$ ’ need not be equivalent to ‘ $=$ ’. In order to see this, assume that  $x =_{A_1} y$  and  $A_2(x)$ . The former and (12.2) entail that

(\*)  $A_1(x)$

(\*\*)  $\forall A_i \in A (A_i(x) \equiv A_i(y))$ .

From (\*), (\*\*), and Definition 12.1 it follows that  $A_2(y)$ . Then  $x =_{A_2} y$  due to (\*\*) and (12.1).

Nevertheless, if one wishes to define RI by means of indiscernibility with respect only to some available properties, the ensuing logic might be different. For example, suppose that you think that there is an ontological reason for discriminating between different kinds of properties, that is, that you divide  $A$  into disjoint subsets. Moreover, you require that each such subset be a tree with respect to the set-theoretical inclusion. Your position then bears a close resemblance to that of Aristotle, because his *Categories* may be interpreted as imposing a tree-structure within one category. If  $A_i \subseteq A$  is a tree with respect to the set-theoretical inclusion and  $\text{Sup}_{A_i}$  is its greatest element, let  $A_i$  be called *an aristotelian family* and  $\text{Sup}_{A_i}$  be the *category* with respect to  $A_i$ . Aristotelian families should disjointly cover  $A$ , that is,

1.  $\Sigma\{A_i : A_i \text{ is an aristotelian family}\} = A$ .
2.  $\Pi\{A_i : A_i \text{ is an aristotelian family}\} = \emptyset$ .

If you claim further that only properties from one aristotelian family contribute to the ontological indiscernibility of objects falling under such a family, you may define RI as ontological indiscernibility with respect to that family.

**Definition 12.3**  $x \approx_A y \equiv A \in A_i \wedge \forall A_j \in A_i (A_j(x) \equiv A_j(y))$ .<sup>8</sup>

The RI defined by (12.2) now satisfies the characteristic theorem of (C3). I will briefly sketch the proof. Let  $\forall x (A_1(x) \rightarrow A_2(x))$  and  $x =_{A_1} y \wedge y =_{A_2} z$ . The former yields that there exists a unique aristotelian family  $A_0$  to which  $A_1$  and  $A_2$  belong. Then due to the latter it is the case that  $\forall A_j \in A_0 (A_j(x) \equiv A_j(y))$  and  $\forall A_j \in A_0 (A_j(y) \equiv A_j(z))$ . As a result, it holds that  $x \approx_{A_2} z$ , and subsequently that  $A_2(x) \wedge A_2(z)$ .

## 13 Comparisons

**13.1 MLRIs and a formal theory of sortal quantification (Stevenson)** Stevenson's [12] theory is a full first-order logic. Predicates (or more precisely, sortals) in ' $=_\delta$ ' behave here in a special way.

1. If predicates  $\delta_1$  and  $\delta_2$  intersect, then there is a predicate  $\delta$  such that  $\vdash_{\text{STEVENSON}} \forall \beta (\delta_1(\beta) \vee \delta_2(\beta) \rightarrow \delta(\beta))$ .
2. For every predicate  $\delta$ , there is a predicate  $\delta'$  such that  $\vdash_{\text{STEVENSON}} \forall \beta (\delta(\beta) \rightarrow \delta'(\beta))$ , and for no predicate  $\delta''$  other than  $\delta'$ ,  $\vdash_{\text{STEVENSON}} \forall \beta (\delta'(\beta) \rightarrow \delta''(\beta))$ .

Since in Stevenson's account it holds that  $\alpha_1 =_\delta \alpha_2 \equiv \alpha_1 = \alpha_2 \wedge \delta(\alpha_1)$ , my counterpart of it is (C10).

**13.2 MLRIs and Griffin-Routley's logics** Griffin's monograph *Relative Identity* [6] is devoted mainly to repudiating various arguments against RI. In its constructive part one may find, among other things, an argument for RI from the open-texture of language (it is a refinement of Zemach [17]). The notion of open-texture is not clearly defined here, but it seems that we may proceed with the following definition. A term  $t$  is *open-textured* with respect to a concept  $C$  ( $C$ -open for short) if and only if there is  $x$  that falls under  $t$  such that it is indeterminate whether  $x$  falls under  $C$ . Further, a term  $t_1$  *closes* a  $C$ -open term  $t_2$  if and only if every  $x$  that falls under  $t_1$

also falls under  $t_2$  and for every  $x$  that falls under  $t_1$  it is determinate whether  $x$  falls under  $C$  or not. For example, Griffin claims that the term ‘word’ is open-textured with respect to the concept ‘is the same as  $y$ ’, but is closed by the terms ‘token-word’ and ‘type-word’. In general, terms closing RI-open terms may be such that sentences of the form ‘ $x$  is the same  $A_1$  as  $y$ ’ and ‘ $x$  is not the same  $A_2$  as  $y$ ’ are both true.

In a later paper with Routley [11], Griffin presents us with a sequence of RI logics. The most advanced one is a second-ordered nonclassical system based on a three-valued logic of significance, and for that reason their proposal is essentially incommensurable to mine. However, one of Routley-Griffin’s preliminary logics, namely, the classical second-order system, may be taken to express their views on the logical properties of RI. Routley and Griffin strive to define RI as a kind of restricted congruence relation.

- (i)  $\alpha_1 =_{\delta} \alpha_2$  iff  $\delta(\alpha_1), \delta(\alpha_2)$  and (if  $\delta' \in \Delta_{\delta}$ , then  $\delta'(\alpha_1)$  iff  $\delta'(\alpha_2)$ ).

In [11] the set  $\Delta_{\delta}$  is said to contain ‘ $\delta$ ’ and be closed under negation, conjunction, and implication; that is, cf. [6], pp. 140–41:

- (ii) if  $\delta \in \Delta_{\delta}$ , then  $(\neg\delta) \in \Delta_{\delta}$ .  
 (iii) if  $\neg\forall\beta(\delta_1(\beta) \equiv \neg\delta_2(\beta))$  and  $\delta_1, \delta_2 \in \Delta_{\delta}$ , then  $(\delta_1 \wedge \delta_2) \in \Delta_{\delta}$ .  
 (iv) if  $\delta_1 \in \Delta_{\delta}$  and  $\forall\beta(\delta_1(\beta) \rightarrow \delta_2(\beta))$ , then  $\delta_2 \in \Delta_{\delta}$ .

If we add that

- (v)  $\Delta_{\delta}$  is the least set meeting (ii)–(iv),

then we will be in a position to show that (C3) is my counterpart of Routley-Griffin logic. First, observe that

- (vi) if  $\forall\beta(\delta_1(\beta) \rightarrow \delta_2(\beta))$ , then  $\Delta_{\delta_2} \subseteq \Delta_{\delta_1}$ .

I prove (vi) by induction on the construction of predicates. Assume that  $\forall\beta(\delta_1(\beta) \rightarrow \delta_2(\beta))$ .

#### The basis of induction

- (\*)  $\delta_2 \in \Delta_{\delta_1}$ .  
 (\*\*) If  $\forall\beta(\delta_2(\beta) \rightarrow \delta_3(\beta))$ , then  $\delta_3 \in \Delta_{\delta_1}$  because  $\forall\beta(\delta_1(\beta) \rightarrow \delta_3(\beta))$  holds as well.

#### The inductive step

- (\*\*) If  $\neg\delta \in \Delta_{\delta_2}$ , then by the hypothesis of induction  $\delta \in \Delta_{\delta_1}$ . Therefore,  $\neg\delta \in \Delta_{\delta_1}$ .  
 (\*\*\*) If  $\delta \wedge \delta' \in \Delta_{\delta_2}$ , then by the hypothesis of induction  $\delta \in \Delta_{\delta_1}$  and  $\delta' \in \Delta_{\delta_1}$ . Therefore,  $\delta \wedge \delta' \in \Delta_{\delta_1}$ .

Assume now that

- (\*)  $x =_{A_1} y$  and  $y =_{A_2} z$ ,  
 (\*\*)  $\forall x(A_1(x) \rightarrow A_2(x))$ .

I will prove that from (\*) and (\*\*) it follows that  $x =_{A_2} z$  (cf. 5.2(2)) provided that we define RI by means of (i). Obviously,  $A_2(x)$  and  $A_2(z)$  hold. The fact (vi) and (\*\*) entail that (\*\*\*)  $\Delta_{A_2} \subseteq \Delta_{A_1}$ .

Suppose now that  $A \in \Delta_{A_2}$ . Then it follows from the right conjunct of (\*) that  $A(y) \equiv A(z)$  and from (\*\*\*) and the right conjunct of (\*) that  $A(x) \equiv A(y)$ . Thus  $A(x) \equiv A(z)$ , as desired (cf. (i)).

**13.3 MLRIs and Zemach's logic of incompleteness** Zemach's [16] logic for RI is based on his theory of vagueness which postulates two kinds of objects: *complete* and *incomplete*. The former are apt, as it were, to submit a determinate answer to every question of the form 'Is  $x$  an  $A$ ?'. Consequently, they fall under AI. The latter fail to be determinate in some respects, and therefore involve RI. For every set of complete objects that fall under some sortal, there exists an incomplete object that exhibits all of and only those properties that the objects have in common. Zemach says that incomplete objects are schemas of complete ones. An object that schematizes all objects falling under a sortal is called the *minimal schematic object* constituted by these objects with respect to this sortal.

Now, an object  $O_1$  is the same  $A$  as an object  $O_2$  if and only if both  $O_1$  and  $O_2$  constitute with respect to a sortal ' $A$ ' the same minimal schematic object. Since  $O_1$  and  $O_2$  may fall under two sortals ' $A_1$ ' and ' $A_2$ ' such that all determinate properties of  $O_1$  and  $O_2$  belong to the  $A_1$ -minimal schematic object but some, say,  $O_1$  properties do not belong to the  $A_2$ -minimal schematic object, therefore  $O_1$  may be the same  $A_1$  as  $O_2$  but a different  $A_2$ . Oddly enough Zemach insists that  $O_1$  and  $O_2$  that are the same  $A$  may not be  $A$ s themselves (Zemach [18], p. 259–60). Seemingly, schematic objects are considered genuine entities on a par with complete ones. His position, thus, entails rejection of (C1). The ensuing logic for RI is a second-order theory based on Bochvar's three-valued logic.

**13.4 MLRIs and van Inwagen's logic** Van Inwagen ([13], pp. 231–38) constructs a first-order logic for RI in order to formalize Christian doctrines of Trinity and Incarnation. He rejects any version of (LL) as containing metaphysical content and characterizes RI as a weak-reflexive (i.e.,  $A(x) \rightarrow x =_A x$ ), symmetrical, and transitive relation. As a result, (C1) is its counterpart.

**13.5 MLRIs and a logic of general similarity** Deutsch [2] sketches a logic of general similarity which he treats as a logic of RI. It is a classical elementary monadic theory with ' $=_{d_A}$ ' (the sign corresponding to RI) and ' $\mathbf{d}$ ' as nonlogical constants. Deutsch's way of rendering RI in the language of general similarity is a little unusual. According to his explanation we must not read ' $\alpha_1 =_{d_{A_k}} \alpha_2$ ' as ' $\alpha_1$  is the same  $A_k$  as  $\alpha_2$ ', but rather as ' $\alpha_1$  is the same  $\mathbf{d}A_k$  as  $\alpha_2$ ', where  $\mathbf{d}A_k$  is a determinable of which ' $A_k$ ' is a determinate predicate. For example, 'has a shape' is a determinable of which 'is round' is a determinant, that is, ' $\mathbf{d}(\text{is round}) = \text{has a shape}$ ', and ' $x =_{d_{\text{round}}} y$ ' is to be read as ' $x$  is of the same shape as  $y$ '. Consequently, there are three kinds of predicates in his theory: determinate predicates, determinable predicates, and subscripted predicates (cf. Section 10). The first and the third group need not be disjoint, but Deutsch implicitly presupposes that a predicate may be subordinate only to a determinable predicate.

His convention is a little awkward to follow since we do not know whether to express 'The piano concerto I performed yesterday was the same Chopin concerto as the concerto I had listened to the day before yesterday' as ' $\alpha_1 =_{d_{\text{Chopin's concerto in F minor}}} \alpha_2$ ' or as ' $\alpha_1 =_{d_{\text{Chopin's concerto in E minor}}} \alpha_2$ '. Deutsch's position does not, nonetheless, result in rejection of my Assumption 1.3 from Section 1. Both Chopin concertos, the one I performed yesterday and the one I listened to the day before, whether they were in F minor or in E minor, are still Chopin concertos.

The semantics of Deutsch's logic of general identity treats each kind of predicate differently. Determinate predicates are interpreted by sets. A determinable predicate is interpreted by a family of pairwise disjoint subsets. These subsets are interpretations of the determinate predicates of the determinable predicate. A  $\delta$ -subscripted predicate is interpreted by a family such that any of its members either includes, or is disjoint from, any set from the family that interprets the predicate  $\delta$ . Deutsch's satisfaction conditions guarantee that

- (i)  $\vdash_{\text{DEUTSCH}} x_1 = d_A x_2 \rightarrow (A(x_1) \rightarrow A(x_2))$ ;
- (ii)  $\vdash_{\text{DEUTSCH}} x_1 = d_{A_1} x_2 \wedge A_1(x_1) \rightarrow (\forall x(A_1(x) \rightarrow A_2(x)) \rightarrow x_1 = d_{A_2} x_2)$ ;
- (iii)  $\vdash_{\text{DEUTSCH}} A(x_1) \wedge A(x_2) \rightarrow x_1 = d_A x_2$ .

The last theorem does not prove the deviancy of the logic of general similarity but is a consequence of the aforementioned notational convention.

In order to compare Deutsch's logic with MLRIs, first we have to neutralize that convention. This is fairly simple. We may render his ' $x = d_{A_1} y$ ' as my ' $x = d_{A_2} y$ '. It is the semantics that should worry us. My semantics does not discriminate among determinate, determinable, and subscripted predicates. For that reason, I am not in a position to compare them with their ancestor without modifying the latter. One may argue that since Definition 3.1 interprets predicates as families of sets, we might exclude determinate predicates. What about subscripted predicates? There seem to be two options available. We might stipulate that every (determinable) predicate is subscripted by any other. Deutsch's semantics would require that if two predicates overlap, then the respective families share the same sets in the overlapping region. In other words, Deutsch's semantics would impose the condition (C9). However, one is justified in objecting that our stipulation is inconsistent with the initial distinction between subscripted and nonsubscripted predicates. Further, we might dispense with the distinction without that inconsistency simply by stipulating that there are no subscripted predicates at all. In that case, since Deutsch's semantics does not impose any condition on nonsubscripted predicates, we have to content ourselves with (C1).

**13.6** I hope my brief survey shows that the RI-logics so far developed either trivialize RI or try to display the peculiarity of RI on nonlogical grounds. Consequently, both strategies resulted in neglecting some formal possibilities, for example, (C2).

## 14 Open Problems

**Problem 14.1** Are there any MLRIs besides those defined above? Well, in general the answer is positive. One might come up with the following,

$$(C^*) \quad \text{if } \cup \mathcal{F}(\delta_1) \cap \cup \mathcal{F}(\delta_2) = \cup \mathcal{F}(\delta_3), \text{ then } \mathcal{F}(\delta_1) \cap \mathcal{F}(\delta_2) = \mathcal{F}(\delta_3),$$

and obtain a logic stronger than (C2) but different from (C3), (C6), and (C7). In some sense, however, it would be a trivial extension of (C2). Assume that we would introduce the notion of noteworthiness and make it sufficiently precise. Are there any noteworthy MLRIs except those defined above? Again, the answer is positive. Witness (C6)+(C8), that is, the logic characterized by the class of structures satisfying (C6) and (C8). Since (C6) $\wedge$ (C8) does not entail (C9) and (C6) neither entails (C8) nor vice versa, (C6)+(C8) is stronger than (C6) and (C8), but weaker than (C9). But we might further invent the notion of relative newness, and ask whether there are any noteworthy MLRIs that are new relative to (C1)–(C11).

**Problem 14.2** Are MLRIs conservative, that is, is there any formula from the intersection of the monadic language and the language of classical logic such that it is a tautology of an MLRI and is not a tautology of classical monadic predicate calculus?

### Notes

1. The forthcoming second part of my paper will contain an extension of my semantics to a full system with  $n$ -ary functions and relations. The idea of a semantic approach to the logic of relative identity originates from Deutsch's seminal article on a logic of general similarity [2].
2. Usually it is held that only predicates that are (or express) sortals may occur in RI. Following Strawson [13] I assume that *sortals* correspond to countable (in the grammatical sense) nouns, that is, sortals yield definite answers to questions 'How many . . . are there?'. Therefore, 'gold' (used as a mass term) cannot qualify as a sortal.
3. In my exposition of MLRIs I follow a formulation of first-order logic from Ebbinghaus [3]; my Sections 2 and 3 borrow from their Chapters II and III; Sections 9 and 10 copy Chapter IV; and the completeness result uses Henkin's proof from Chapter V. In particular, vast parts of proofs of theorems on MLRIs are identical to their classical counterparts and, for that reason, will be omitted.
4. More rigorously, there are two signs of identity in the monadic language: one for AI and the other for RI; the former forms sentences from terms, the latter requires terms and predicates. My compressing them into one symbol is not intended to have any formal or philosophical consequence. An extreme RI proponent who denies the meaningfulness of '=' is free to throw it out of the monadic language.
5. Observe that it does not preclude that if  $x$  is an artefact and  $y$  is an artefact, then  $x$  is the same artefact as  $y$ .
6. From here on,  $1 \leq n \leq 11$  unless stated otherwise.
7. In some cases RI is claimed not to be identity at all. Wiggins suggests that then it is the relation of composition or the relation of representation in disguise.
8. The consistency of Definition 12.3 is guaranteed by (i) and (ii).

### References

- [1] Blizard, W. D., "Multiset theory," *Notre Dame Journal of Formal Logic*, vol. 30 (1989), pp. 36–66. [Zbl 0668.03027](#). [MR 90b:03074](#). 28
- [2] Deutsch, H., "Identity and general similarity," pp. 177–99 in *Philosophical Perspectives, XII, Language, Mind, and Ontology*, edited by J. E. Tomberline, Blackwell Publishers, Cambridge, 1998. [Zbl 0953.03521](#). 27, 28, 42, 47, 49
- [3] Ebbinghaus, H.-D., J. Flum, and W. Thomas, *Mathematical Logic*, 2d edition, Undergraduate Texts in Mathematics. Springer-Verlag, New York, 1994. Translated from the German by Margit Meßmer. [Zbl 0795.03001](#). [MR 95e:03002](#). 36, 40, 49



- [4] Geach, P., "Identity," *Review of Metaphysics*, vol. 21 (1967), pp. 3–13. [27](#)
- [5] Geach, P., "Replies," pp. 276–99 in *Peter Geach: Philosophical Encounter*, edited by H. Lewis, Kluwer, Dordrecht, 1991. [27](#), [28](#), [29](#), [44](#)
- [6] Griffin, N., *Relative Identity*, Clarendon Press, Oxford, 1977. [28](#), [29](#), [44](#), [45](#), [46](#)
- [7] Krause, D., "On a quasi-set theory," *Notre Dame Journal of Formal Logic*, vol. 33 (1992), pp. 402–11. [Zbl 0774.03032](#). [MR 93f:03033](#). [28](#)
- [8] Noonan, H. W., *Objects and Identity: An Examination of the Relative Identity Thesis and its Consequences*, Martinus Nijhoff Publishers, The Hague, 1980. [28](#), [29](#)
- [9] Noonan, H. W., "Relative identity: A reconsideration," *Analysis*, vol. 46 (1986), pp. 6–10. [Zbl 0943.03508](#). [MR 826595](#). [29](#)
- [10] Noonan, H. W., "Relative identity," pp. 634–52 in *A Companion to the Philosophy of Language*, edited by B. Hale and C. Wright, Blackwell Publishers, Cambridge, 1996. [44](#)
- [11] Routley, R., and N. Griffin, "Towards a logic of relative identity," *Logique et Analyse. Nouvelle Série*, vol. 22 (1979), pp. 65–83. [Zbl 0416.03006](#). [MR 80j:03018](#). [46](#)
- [12] Stevenson, L., "A formal theory of sortal quantification," *Notre Dame Journal of Formal Logic*, vol. 16 (1975), pp. 185–207. [Zbl 0225.02014](#). [MR 51:2858](#). [45](#)
- [13] Strawson, P., *Individuals: An Essay in Descriptive Metaphysics*, Methuen, London, 1959. [29](#), [47](#), [49](#)
- [14] van Inwagen, P., "And yet they are not three Gods but one God," pp. 222–59 in *God, Knowledge, and Mystery*, edited by P. van Inwagen, Cornell University Press, Ithaca, 1989. [28](#)
- [15] Wiggins, D., *Sameness and Substance Renewed*, 2d edition, Cambridge University Press, Cambridge, 2001. [42](#)
- [16] Zemach, E. M., "Schematic objects and relative identity," *Noûs*, vol. 16 (1982), pp. 295–305. [MR 83h:03031](#). [28](#), [29](#), [47](#)
- [17] Zemach, E. M., "In defence of relative identity," *Philosophical Studies*, vol. 26 (1974), pp. 207–18. [45](#)
- [18] Zemach, E. M., "Identity and open texture," *Philosophia*, vol. 13 (1983), pp. 255–62. [29](#), [47](#)

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