Notre Dame Journal of Formal Logic
Volume 39, Number 4, Fall 1998

# Pitts' Quantifiers Are Not Topological Quantification

TOMASZ POŁACIK

**Abstract** We show that Pitts' modeling of propositional quantification in intuitionistic logic (as the appropriate interpolants) does not coincide with the topological interpretation. This contrasts with the case of the monadic language and the interpretation over sufficiently regular topological spaces. We also point to the difference between the topological interpretation over sufficiently regular spaces and the interpretation of propositional quantifiers in Kripke models.

When we consider propositional quantification and think of classical logic we easily find out that the problem is trivial: the truth functional interpretation allows us to express quantification by means of propositional connectives and constants only. However, in case of nonclassical logics the situation is different—there propositional quantification usually gives rise to interesting extensions of the logic in question. The problem of propositional quantification was investigated in case of modal logics (see e.g., Fine [2], Bull [1], Ghilardi and Zawadowski [6], Kaplan [8], Kremer [11]), relevance logic (see Kremer [12]) and intuitionistic logic (see Gabbay [4], Scedrov [17], Kremer [10], Pitts [13], Połacik [16], Ghilardi and Zawadowski [5], Visser [22], Skvortsov [19]). The study of propositional quantification in intuitionistic logic is continued in this paper.

We consider the Heyting calculus which corresponds to (the fragment of) intuitionistic propositional logic in the language of the standard propositional connectives: $\neg, \vee, \wedge, \rightarrow$. In this language, the constants $\top, \bot$, and equivalence $\equiv$ can be defined in the usual way. In the sequel, $p, q, r, s, \ldots$ will range over the set of propositional variables and the letters $F, G, H \ldots$ will serve as the metavariables for formulas. The symbol $\vdash$ will be used to denote provability in Heyting calculus. We extend the language by adding propositional quantifiers $\exists p, \forall p, \ldots$; the notions of formula (in the extended language) and free variable are as usual.

One way of introducing propositional quantification into a propositional logic is to specify the characteristic properties of quantification in the form of axioms and

*Received December 13, 1995; revised January 5, 1998*

rules of inference and add them to the list of the axioms and rules of inference of the appropriate logical system. In the case of intuitionistic logic the merits of propositional quantification can be given, for example, by the following schemata of formulas,

$$\forall p\, F(p) \to F(q) \qquad\qquad F(q) \to \exists p\, F(p),$$

and rules of inference,

$$\frac{F(p) \to G}{\exists p\, F(p) \to G} \qquad\qquad \frac{G \to F(p)}{G \to \forall p\, F(p)}\ ,$$

where $p$ is not free in $G$. It is natural to accept also the comprehension schema: for every $G$ in which $p$ is not free,

$$\exists p\, (p \equiv G).$$

These basic axioms and rules governing the quantifiers, together with a usual axiomatization of Heyting calculus, give rise to the system $\mathsf{IPC}^2$ which can be regarded as the minimal system corresponding to intuitionistic logic (in the standard language) with propositional quantification.

In [4], undecidability as well as soundness and completeness (with respect to a variant of Kripke semantics) of $\mathsf{IPC}^2$ is proved. However, it is easy to see that the system $\mathsf{IPC}^2$ cannot be complete with respect to any natural semantics of intuitionistic propositional quantification, since for example, the formula $\neg\forall p\, (p \vee \neg p)$—which is intuitively true intuitionistically—is not provable in $\mathsf{IPC}^2$. This fact shows that $\mathsf{IPC}^2$ covers only a fragment of intuitionistic logic with propositional quantification and motivates the semantical approach to propositional quantifiers. However, any natural interpretation of propositional quantification should validate the axioms and rules of $\mathsf{IPC}^2$.

An interpretation satisfying this property is the so-called Pitts' interpretation in which propositional quantifiers are interpreted within Heyting calculus. Let us sketch how it can be done. Recall that Heyting calculus enjoys the interpolation property: for all formulas $F$, $G$ such that the formula $F \to G$ is provable, there is a formula $I$—called their interpolant—involving only the variables involved both in $F$ and in $G$, such that the formulas $F \to I$ and $I \to G$ are also provable. It is shown in [13], that the interpolation property for Heyting calculus can be strengthened. Namely, for every propositional variable $p$ and every formula $F$ of the language of Heyting calculus, the set of interpolants of $F$ not involving $p$ is not merely nonempty but contains the weakest and strongest elements with respect to the provability ordering in Heyting calculus. More precisely, the following can be proven (see [13]).

**Theorem 1** (Uniform interpolation theorem for Heyting calculus) *Given a propositional variable $p$ and a formula $F$, one can effectively find formulas $\mathsf{A}p\, F$ and $\mathsf{E}p\, F$ containing only variables not equal to $p$ which occur in $F$, such that for all formulas $G$ not involving $p$,*

$$\vdash G \to \mathsf{A}p\, F \quad \text{iff} \quad \vdash G \to F,$$

$$\vdash \mathsf{E}p\, F \to G \quad \text{iff} \quad \vdash F \to G.$$

The uniform interpolants $\mathsf{A}p\,F$ and $\mathsf{E}p\,F$ satisfy the axioms and rules of the system $\mathsf{IPC}^2$ if we substitute $\mathsf{A}p\,F$ for $\forall p\,F$ and $\mathsf{E}p\,F$ for $\exists p\,F$. Thus we can model propositional quantification in Heyting calculus: given a formula $F$ we can interpret $\exists p\,F$ as $\mathsf{E}p\,F$ and $\forall p\,F$ as $\mathsf{A}p\,F$. This modeling will be called *Pitts' interpretation*; consequently, $\mathsf{E}$ and $\mathsf{A}$ will be called *Pitts' quantifiers*.

Recall that, although the existential quantifier is definable in intuitionistic logic by means of the universal quantifier and the implication, Pitts' quantifiers are defined simultaneously via mutual recursion. Moreover, $\mathsf{E}$ and $\mathsf{A}$ are defined for multisets $\Delta$ of formulas rather than for single formulas and have the form $\mathsf{E}p\,(\Delta)$ and $\mathsf{A}p\,(\Delta;\,F)$ where $F$ is a formula of the language of Heyting calculus. $\mathsf{A}p\,F$ is then defined as $\mathsf{A}p\,(\varnothing;\,F)$ and $F$ in $\mathsf{E}p\,F$ is to be treated as the one-element multiset. From the rules of calculating $\mathsf{E}$ and $\mathsf{A}$ we recall those employed in this paper. Here we give only the variants of the rules in the case of one-element multisets.

$$\mathsf{E}p\,((F \to G) \to H) = (\mathsf{E}p\,(G \to H) \to \mathsf{A}p\,(G \to H;\, F \to G)) \to \mathsf{E}p\,H,$$

$$\mathsf{A}p\,(\varnothing;\, F \to G) = \mathsf{E}p\,F \to \mathsf{A}p\,(F;\, G),$$

$$\mathsf{A}p\,(\varnothing;\, F \vee G) = \mathsf{A}p\,(\varnothing;\, F) \vee \mathsf{A}p\,(\varnothing;\, G).$$

However, whenever it is possible we shall avoid the laborious computations of Pitts' quantifiers and exploit the fact that they are the appropriate interpolants of the formulas in question.

A semantical proof of Uniform Interpolation Theorem was discovered recently by Ghilardi and Zawadowski [5] who used ideas from Shavrukov's proof of the Uniform Interpolation Theorem for $\mathsf{GL}$ (see [18]). Ghilardi and Zawadowski's proof also conveys a definite meaning on the Pitts' quantifiers in terms of Kripke semantics. These results were also obtained independently by Visser [22].

It should be pointed out that, since Pitts' quantifiers are definable in Heyting calculus, we can compare them with the specific meaning of propositional quantifiers in any other interpretation. Moreover, the formulas $\mathsf{A}p\,F \to \forall p\,F$ and $\exists p\,F \to \mathsf{E}p\,F$ are provable in $\mathsf{IPC}^2$, hence they are generally valid. This fact suggests the question: in which of the known models for propositional quantification—if there are any— are the formulas $\mathsf{E}p\,F \to \exists p\,F$ and $\forall p\,F \to \mathsf{A}p\,F$ also valid, that is, to which of known interpretations of propositional quantification—if there are any—is Pitts' interpretation equivalent?

When looking for an appropriate interpretation of propositional quantification in intuitionistic logic, one can begin with an arbitrary semantics for Heyting calculus and extend it by an appropriate interpretation of the quantifiers. We follow this way to specify the topological interpretation of propositional quantification which will be considered in this paper.

First, we establish some terminology and notation. By $\mathbf{T}$ we will denote a topological space. If $X \subseteq \mathbf{T}$, we write $-X$ for the complement of $X$, $\mathrm{int}\,X$ and $\mathrm{cl}\,X$ for the interior and closure of $X$, respectively. As usual, we say that a point $x$ is an *accumulation point* of a subset $X$ of $\mathbf{T}$ if $x \in \mathrm{cl}\,(X \setminus \{x\})$. Recall that a space $\mathbf{T}$ is *dense-in-itself* if all the points of $\mathbf{T}$ are its accumulation points.

The standard topological semantics for Heyting calculus can be defined as follows (see Tarski [20]). Given a topological space $\mathbf{T}$, to each propositional variable we assign an open subset of $\mathbf{T}$: $p \mapsto P, q \mapsto Q, r \mapsto R, s \mapsto S, \ldots$. (For convenience we shall always assume the appropriate correspondence of lower and uppercase letters as above.) Relative to this assignment of the propositional variables, for each formula $F(\vec{q})$ of the standard language, we define, by induction on the complexity of $F$, an open set $F[\vec{Q}]$, where $\vec{q}$, and $\vec{Q}$ denote finite sequences of propositional variables and the open sets being assigned to $\vec{q}$, respectively:

$$\begin{array}{rcl}
(\neg F')[\vec{Q}] & = & \mathrm{int}\,(-F'[\vec{Q}]) \\
(F' \vee F'')[\vec{Q}] & = & F'[\vec{Q}] \cup F''[\vec{Q}] \\
(F' \wedge F'')[\vec{Q}] & = & F'[\vec{Q}] \cap F''[\vec{Q}] \\
(F' \to F'')[\vec{Q}] & = & \mathrm{int}\,(-F'[\vec{Q}] \cup F''[\vec{Q}]).
\end{array}$$

Now the idea of interpreting quantifiers as supremum and infimum leads us to the following extension of the interpretation:

$$\begin{array}{rcl}
(\exists p\, F)[\vec{Q}] & = & \bigcup\{F[P, \vec{Q}] \,:\, P - \text{open}\} \\
(\forall p\, F)[\vec{Q}] & = & \mathrm{int} \bigcap\{F[P, \vec{Q}] \,:\, P - \text{open}\}.
\end{array}$$

For brevity we shall sometimes write $X \longrightarrow Y$ instead of $\mathrm{int}\,(-X \cup Y)$ and $\neg X$ instead of $\mathrm{int}\,(-X)$. Let $F(p, q, \ldots, r)$ be an arbitrary formula of the language of $\mathsf{IPC}^2$. In the case that for all assignments for the propositional variables $F[P, Q, \ldots, R] = \mathbf{T}$, that is, when the formula $F$ is valid in $\mathbf{T}$, we write $\mathbf{T} \models F$.

The topological semantics for intuitionistic logic are closely related to Kripke semantics, in which to every propositional variable $p$ we assign an upward closed subset of the frame and interpret propositional quantifiers as ranging over such sets (see [10]). In fact, in every Kripke frame $\mathbf{K} = (K, \leq)$ for intuitionistic propositional logic we can define a topology by assuming that a subset of $K$ is open if and only if it is upward closed with respect to the ordering $\leq$. Thus, such an extension of Kripke semantics can be regarded as a kind of topological semantics.

Notice that the topological space associated with a given Kripke frame need not satisfy stronger separation properties, for example, $T_1$. In this paper, we generally leave aside such spaces and direct toward the semantics over topological spaces with strong separation properties. The reason is twofold. First, such spaces appear naturally in mathematical context. Second, more importantly, they appear naturally when intuitionistic propositional logic is concerned—recall that every dense-in-itself metric space is a universal space for Heyting calculus, that is, for every dense-in-itself metric space $\mathbf{T}$, a formula $F$ is provable in Heyting calculus if and only if it is valid in $\mathbf{T}$ (see [20]). On the other hand, the full binary tree which is the universal Kripke model for Heyting calculus is, from the topological point of view, isomorphic to Cantor space and thus possesses very strong topological properties.

The semantics for propositional quantification in intuitionistic logic associated with Kripke models was studied in [10]. There, it is shown that the set of all valid formulas is recursively isomorphic to second-order predicate logic and hence, in particular, undecidable (see also [19]). Of course, since Heyting calculus is decidable, so is the set of the valid formulas for Pitts' quantifiers. Thus, from undecidability of the set of valid principles for quantifiers in the semantics of upward closed subsets in Kripke models follows that such quantifiers are very different from Pitts' quantifiers. In this paper—giving appropriate examples—we present direct arguments for this fact. We show that this also holds for topological quantifiers in the semantics of sufficiently regular spaces. More precisely, we show that there are formulas $F$, $G$ of Heyting calculus involving two propositional variables such that in the class of dense-in-themselves metric spaces the formulas $\mathsf{E}p\, F \equiv \exists p\, F$ and $\mathsf{A}p\, G \equiv \forall p\, G$ are not valid. It contrasts with the case of the monadic language where Pitts' interpretation coincides with topological interpretation of propositional quantification in the class of sufficiently regular spaces (see [16]).

Since the solution of our problem depends not only on the topological properties of the considered space, but also on the number of variables, our considerations will depend essentially on this parameter. First, let us consider the language with only the variable $p$. The problem of monadic language was studied in detail in [16]. Here—as Theorem 3—we just present the relevant result. In the proof of Theorem 3, we shall rely on the following, more general, fact. Its proof, however, is beyond the scope of this short paper, so it will be omitted (see [16]).

**Theorem 2** *Let $\mathbf{T}$ be an arbitrary dense-in-itself metric space and let $F(p)$ be a quantifier-free monadic formula which is not provable in Heyting calculus. Then, for every $x \in \mathbf{T}$, there is an open set $P_x$ such that $x \notin F[P_x]$.*

We employ Theorem 2 as the basic tool in proving the following theorem.

**Theorem 3** *For every dense-in-itself metric space, the topological interpretation of quantifiers coincides with Pitts' interpretation when restricted to the language of one variable.*

*Proof:* We put aside the obvious case of formulas $F(p)$ for which $\vdash F$ or $\vdash \neg F$. The case of the existential quantifier is trivial: for any formula $F$ in question we have $\vdash \mathsf{E}p\, F \equiv \top$ and also in every topological space $\mathbf{T}$, $\mathbf{T} \models \exists p\, F \equiv \top$. Notice that $\vdash \mathsf{A}p\, F \equiv \bot$. Assume that $\mathbf{T}$ is a dense-in-itself metric space and let $F(p)$ be a monadic quantifier-free formula which is not provable in Heyting calculus. Then, by Theorem 2, for every $x \in \mathbf{T}$, there is an open set $P_x$, such that $x \notin F[P_x]$. Thus we have

$$\mathrm{int} \bigcap \{F[P] \;:\; P - \mathrm{open}\} \subseteq \mathrm{int} \bigcap \{F[P_x] \;:\; x \in \mathbf{T}\} \subseteq \mathrm{int} \bigcap_{x \in \mathbf{T}} -\{x\} = \varnothing.$$

Hence $\mathbf{T} \models \forall p\, F \equiv \bot$. $\square$

Let us note that the situation here differs drastically from the situation in the topology associated with Kripke models. There, for example, the formula $\forall p\,(p \vee \neg p)$ defines the set of top nodes, but $\vdash \mathsf{A}p\,(p \vee \neg p) \equiv \bot$. So, the semantics of upward closed sets in Kripke models differ from Pitts' interpretation of propositional quantification even in the case of the monadic language.

**Corollary 4** *The semantics of upward closed sets in Kripke models differs from the topological semantics of dense-in-themselves metric spaces when restricted to the monadic language and the universal quantifier. In the case of the monadic language and the existential quantifier, the semantics in question coincide.*

Now consider the language of at least two variables. We shall prove that, in this case, the standard topological meaning of quantifiers is not, in general, the same as the meaning of Pitts' quantifiers. However, instead of proving this negative result for the existential quantifier only (which would imply the negative result for the universal quantifier), we divide the problem into two separate cases of the two quantifiers. The reason is that in the definition of the existential quantifier,

$$\exists p\, F \equiv \forall q (\forall p\, (F \to q) \to q),$$

a new variable occurs and proceeding this way we would not be able to cover the case of the universal quantifier and the language of two variables. In fact, we give the solution of our problem in the latter case before we turn to the case of the existential quantifier.

**Theorem 5** *For every topological space, the topological interpretation of the universal quantifier does not coincide with Pitts' interpretation when restricted to the language of at least two variables.*

*Proof:* We put

$$H(p, q) = p \vee (p \to q).$$

Obviously, $\vdash \mathsf{A}p\, (\varnothing;\, p) \equiv \bot$ and $\vdash \mathsf{A}p\, (\varnothing;\, p \to q) \equiv q$. Hence

$$\vdash \mathsf{A}p\, (\varnothing;\, p \vee (p \to q)) \equiv \mathsf{A}p\, (\varnothing;\, p) \vee \mathsf{A}p\, (\varnothing;\, p \to q) \equiv q.^{1}$$

Thus, in every topological space $\mathbf{T}$ we have

$$\mathbf{T} \models \mathsf{A}p\, H \equiv q. \tag{1}$$

Assume that $\mathbf{T}$ is an arbitrary space and $x \in \mathbf{T}$. Let $Q = \mathrm{int}\,(\mathbf{T} \setminus \{x\})$. Obviously, $Q \subseteq H[P, Q]$ for every open $P \subseteq \mathbf{T}$. Take an arbitrary open set $P$. There are two possibilities: $x \in P$ or $x \notin P$. It is clear that in the former case $\mathbf{T} \models H[P, Q]$. Assume the latter. Observe that then, since $P$ is open, $-Q = \mathrm{cl}\,\{x\} \subseteq -P$. Consequently $P {\to} Q = \mathbf{T}$ and hence $\mathbf{T} \models H[P, Q]$. So, for all $P$, $H[P, Q] = \mathbf{T}$. Thus we have

$$\forall p\, H\, [Q] = \mathbf{T}. \tag{2}$$

Finally, by (1) and (2) we get $\mathbf{T} \not\models \forall p\, H \to \mathsf{A}p\, H$. □

Notice that in Theorem 5 we do not assume any topological properties. So, this result is fully general, in particular, it is also valid for the semantics of upward closed sets in Kripke models. Moreover, proving Theorem 5, we show in fact that under the standard topological interpretation of $\mathsf{IPC}^2$ over any topological space $\mathbf{T}$ (including the semantics of upward closed sets in Kripke models), the formula $\forall p\, (F \vee G) \to \forall p\, F \vee \forall p\, G$ is not valid although, since $\vdash \mathsf{A}p\, (F \vee G) \equiv \mathsf{A}p\, F \vee \mathsf{A}p\, G$, it is valid under Pitts' interpretation.

Validity of the formula in question is intuitively unacceptable which makes Pitts' interpretation rather peculiar. The following formulas—in which validity is also intuitively questionable—are valid under Pitts' interpretation:

$$\begin{aligned}
&\exists p\, \neg(F \to G) \equiv \neg\forall p\, (F \to \neg\neg G),\\
&\exists p\, \neg\neg F \equiv \neg\forall p\, \neg F,\\
&\exists p\, (\neg F \to G) \equiv \forall p\, \neg F \to \exists p\, G.
\end{aligned}$$

The next two theorems give the solution of the problem of the existential quantifier. We now proceed as follows: first, we prove the negative result for the language of (at least) three variables; then, strengthening the assumptions on the space, we give the final solution in the language of two variables. It is perhaps worth noting that the mentioned theorems are in fact incomparable in strength, that is, in particular, Theorem 6 does not follow from Theorem 7.

**Theorem 6** *For every Hausdorff space with a countable basis in at least one point, the topological interpretation of the existential quantifier does not coincide with Pitts' interpretation when restricted to the language of at least three variables.*

*Proof:* We put

$$F(p,q) = \neg p \to q, \qquad G(p,q) = p \to q,$$

$$H(p,q,r) = (r \to F(p,q) \vee G(p,q)) \to r.$$

Obviously, $q$ is the greatest lower interpolant of both the formulas $F(p,q)$ and $G(p,q)$, that is, $\vdash \mathsf{A}p\,(\varnothing; F) \equiv \mathsf{A}p\,(\varnothing; G) \equiv q$. Hence

$$\vdash \mathsf{A}p\,(\varnothing; F \vee G) \equiv \mathsf{A}p\,(\varnothing; F) \vee \mathsf{A}p\,(\varnothing; G) \equiv q.$$

Let us compute $\mathsf{E}p\,H(p,q,r)$:[2]

$$\begin{aligned}
\vdash \mathsf{E}p\,H \quad &\equiv (\mathsf{E}p\,(F \vee G \to r) \to \mathsf{A}p\,(F \vee G \to r; r \to F \vee G)) \to \mathsf{E}p\,r\\
&\equiv \mathsf{A}p\,(\varnothing; (F \vee G \to r) \to (r \to F \vee G)) \to r\\
&\equiv \mathsf{A}p\,(\varnothing; r \to F \vee G) \to r\\
&\equiv (r \to \mathsf{A}p\,(\varnothing; F \vee G)) \to r\\
&\equiv (r \to q) \to r.
\end{aligned}$$

Thus, in every topological space $\mathbf{T}$,

$$\mathbf{T} \models \mathsf{E}p\,H \equiv (r \to q) \to r.$$

Let $\mathbf{T}$ be a Hausdorff space and $x$ its point with a countable basis $U_0 \supsetneq U_1 \supsetneq \cdots$. For every $n \in \mathbb{N}$, we choose $z_n \in U_n \setminus U_{n+1}$ and put $Z = \{z_n : n \in \mathbb{N}\}$. Observe that $x \notin Z$ and $Z^d = \{x\}$, that is, $x$ is the only accumulation point of $Z$. Indeed, $x$ is an accumulation point of $Z$ since for every open neighborhood $X$ of $x$, $U_n \subseteq X$ for a sufficiently large $n$; moreover, $x$ is unique, since for every $y \neq x$ we can find neighborhoods $X \ni x$ and $Y \ni y$ such that $X \cap Y = \varnothing$, and hence $Y \cap U_m = \varnothing$ for a sufficiently large $n$ and all $m \geq n$.

Let $Q = \mathbf{T} \setminus \mathrm{cl}\, Z$ and $R = \mathbf{T} \setminus \{x\}$. We have $((r \to q) \to r)\,[Q, R] = \mathrm{int}\,(\mathrm{cl}\,(R \setminus Q) \cup R) = \mathrm{int}\,(\mathrm{cl}\,Z \cup R) = \mathbf{T}$. Hence

$$\mathbf{T} \models \mathsf{E}p\,H[Q, R]. \tag{1}$$

Of course, $R \subseteq H[P, Q, R]$ for all $P$. Suppose that $\mathbf{T} \models \exists p\,H[Q, R]$, that is, $x \in \exists p\,H[Q, R]$. Then, for some $P$,

$$x \in \mathrm{int}\,(\mathrm{cl}\,(R \setminus F[P, Q] \setminus G[P, Q]) \cup R),$$

whence

$$x \in \mathrm{cl}\,(R \cap \mathrm{cl}\,(\mathrm{int}\,(-P) \setminus Q) \cap \mathrm{cl}\,(P \setminus Q)) \\ = \mathrm{cl}\,(R \cap \mathrm{cl}\,(\mathrm{int}\,(-P) \cap \mathrm{cl}\,Z) \cap \mathrm{cl}\,(P \cap \mathrm{cl}\,Z)).$$

Let $Z_0 = \mathrm{int}\,(-P) \cap \mathrm{cl}\,Z$ and $Z_1 = P \cap \mathrm{cl}\,Z$. We have $x \in \mathrm{cl}\,(R \cap \mathrm{cl}\,Z_0 \cap \mathrm{cl}\,Z_1)$. But, since $Z_0, Z_1 \subseteq \mathrm{cl}\,Z$, we get $Z_0^d, Z_1^d \subseteq Z^d = \{x\}$ and, since $Z_0 \subseteq \mathrm{int}\,(-P)$ and $Z_1 \subseteq P$, we get $Z_0 \cap Z_1 = \varnothing$. Hence $x \in \mathrm{cl}\,(R \cap \{x\}) = \varnothing$, a contradiction. Thus,

$$\exists p\, H\,[Q, R] = R, \tag{2}$$

and consequently, by (1) and (2) , $\mathbf{T} \not\models \mathsf{E} p\, H \to \exists p\, H$. □

Now, to give the final solution of our problem, we turn to the case of the language of two variables. However, as in Theorem 3, in this case, we shall restrict to the class of dense-in-themselves metric spaces.

**Theorem 7** *For every dense-in-itself metric space, the topological interpretation of the existential quantifier does not coincide with Pitts' interpretation when restricted to the language of two variables.*

Before we prove Theorem 7 let us state a useful property of dense-in-themselves metric spaces (see [16]). Recall that a subset $X$ of a topological space $\mathbf{T}$ is called *dense* if $\mathrm{cl}\,X = \mathbf{T}$; $X$ is called *regularly open*, provided $\mathrm{int}\,\mathrm{cl}\,X = X$.

**Lemma 8** *Every regularly open subset of a dense-in-itself metric space contains a proper dense subset which is a union of two disjoint regularly open sets.*

*Proof (sketch):* Let $W$ be a regularly open subset of a dense-in-itself metric space $\mathbf{T}$. Consider two cases: (1) every point of the set $W$ has a countable basis of closed and open neighborhoods in the relative topology of $W$; (2) some point $x \in W$ does not have a countable basis of closed and open sets.

*Case 1:* Let $x \in W$ and let $\{Y_n : n \geq 1\}$ be a countable basis of closed and open neighborhoods of $x$. We may assume that $\cdots \subsetneq Y_{n+1} \subsetneq Y_n \subsetneq \cdots \subsetneq Y_1$. In this case we define $Y_0 = W \setminus Y_1$ and $U = \bigcup_{n=0}^{\infty} (Y_{2n} \setminus Y_{2n+1})$, $V = \bigcup_{n=0}^{\infty} (Y_{2n+1} \setminus Y_{2n+2})$.

*Case 2:* Let $U$ be an open neighborhood of $x$ such that $U$ is regularly open but not closed. Then we define $V = W \setminus \mathrm{cl}\,U$.

In both cases, one can show that $U$ and $V$ are regularly open and $U \cup V \subsetneq W \subseteq \mathrm{cl}\,(U \cup V)$, that is, $U$ and $V$ satisfy the required property. □

*Proof of Theorem 7:* The idea of the proof is to construct both the sets $Q$ and $R$, which occur in the proof of Theorem 6, using only one set. Recall that we want $Q$ to be $\mathbf{T} \setminus Z$ for some $x \in \mathbf{T}$ and $Z \subseteq \mathbf{T}$ such that $x \notin Z$, $Z^d = \{x\}$, and $R$ to be $\mathbf{T} \setminus \{x\}$. Let $\mathbf{T}$ be a dense-in-itself space with a metric $\rho$. We fix $x \in \mathbf{T}$ and put

$$R = \mathbf{T} \setminus \{x\}.$$

Let $\{L_n : n \in \mathbb{N}\}$ be a countable basis (of neighborhoods) in $x$ such that the diameters of $L_n$'s, forming a strictly decreasing series, converge to 0 (recall that the diameter of $L_n$ is defined as $\sup\{\rho(x, y) : y \in L_n\}$, where $\rho$ is the metric of the space $\mathbf{T}$).

We define inductively open sets $K_n$, such that, $\mathrm{cl}\, K_n \subseteq L_n \setminus L_{n+1}$ and $\mathrm{cl}\, K_n \cap \mathrm{cl}\, K_m = \varnothing$ for all $n, m \in \mathbb{N}$, $n \neq m$. It is clear that a construction of $K_n$'s can be carried out in every metric space (using the fact that the diameters of the sets $L_n$ are decreasing).

Now, for every $n \in \mathbb{N}$ we choose an open set $W_n$ that $W_n \subseteq \mathrm{cl}\, W_n \subseteq K_n$. We can assume that $W_n$'s are regularly open, that is, $W_n = \mathrm{int}\,\mathrm{cl}\, W_n$ for all $n \in \mathbb{N}$. By Lemma 8, for every $W_n$ there is an open and dense proper subset which is the union of two disjoint regularly open sets. Let, for every $n \in \mathbb{N}$, $U_n$ and $V_n$ be such that

$$U_n \neq \varnothing \neq V_n, \ \mathrm{int}\,\mathrm{cl}\, U_n = U_n, \quad \mathrm{int}\,\mathrm{cl}\, V_n = V_n,$$

$$U_n \cap V_n = \varnothing, \quad U_n \cup V_n \subsetneq W_n \subseteq \mathrm{cl}\,(U_n \cup V_n).$$

For every $n \in \mathbb{N}$ we take $z_n \in U_n$ and put $Z = \{z_n : n \in \mathbb{N}\}$. Obviously, $Z^d = \{x\}$, that is, $x$ is the only point of accumulation of the set $Z$.

We put

$$Q = \mathbf{T} \setminus \mathrm{cl}\, Z.$$

Now we can define

$$S = \bigcup_{n\in\mathbb{N}} U_n \setminus \{z_n\}.$$

We show how the sets $Q$ and $R$ can be constructed from the set $S$. Let

$$F(s) = \neg\neg s \vee (\neg\neg s \to s), \qquad G(s) = \neg\neg s \to s.$$

Note that $\vdash G \to F$ and $\vdash (F \to G) \equiv G$ hence

$$\vdash (F \to G) \to F. \tag{1}$$

We shall show that

$$F[S] = \neg\neg S \cup (\neg\neg S {\longrightarrow} S) = R \qquad \text{and} \qquad G[S] = \neg\neg S {\longrightarrow} S = Q.$$

First, we show that

$$\mathrm{cl}\, S = \bigcup_{n\in\mathbb{N}} \mathrm{cl}\, U_n \cup \{x\}. \tag{2}$$

Observe that

$$\bigcup_{n\in\mathbb{N}} \mathrm{cl}\, U_n \subseteq \bigcup_{n\in\mathbb{N}} \mathrm{cl}\,(U_n \setminus \{z_n\}) \subseteq \mathrm{cl} \bigcup_{n\in\mathbb{N}} (U_n \setminus \{z_n\}) = \mathrm{cl}\, S.$$

Moreover, $x \in \mathrm{cl}\, S$, since an arbitrary open neighborhood of $x$ contains all but finitely many sets $L_n \supseteq U_{z_n} \setminus \{z_n\}$; hence $x \in \mathrm{cl} \bigcup_{n\in\mathbb{N}} (U_n \setminus \{z_n\})$. To prove the converse inclusion, assume that $y \notin \bigcup_{n\in\mathbb{N}} \mathrm{cl}\, U_n$ and $y \neq x$ and $Y$ is an arbitrary neighborhood of $y$. Then, for a sufficiently large $n$, $L_m \cap Y = \varnothing$ for all $m \geq n$, and consequently $U_m \cap Y = \varnothing$, hence $y \notin \mathrm{cl} \bigcup_{n\in\mathbb{N}} U_n$.

By (2) we get $\neg\neg S = \mathrm{int}\,\mathrm{cl}\, S = \mathrm{int}\,(\bigcup_{n\in\mathbb{N}} \mathrm{cl}\, U_n \cup \{x\})$. Note that every open neighborhood of $x$ contains all but finitely many sets $L_n$; hence, since $V_n \subseteq L_n$, we get

$x \in \mathrm{cl} \bigcup_{n\in\mathbb{N}} V_n$. But, since $\bigcup_{n\in\mathbb{N}} V_n$ and $\bigcup_{n\in\mathbb{N}} U_n$ are open and disjoint and $S \subseteq \bigcup_{n\in\mathbb{N}} U_n$, we get $\mathrm{cl} \bigcup_{n\in\mathbb{N}} V_n \cap \mathrm{int}\,\mathrm{cl}\, S = \varnothing$. Thus, $x \notin \mathrm{int}\,\mathrm{cl}\, S$, that is,

$$\mathrm{int}\,\mathrm{cl}\, S = \mathrm{int} \bigcup_{n\in\mathbb{N}} \mathrm{cl}\, U_n.$$

Let $y \in \mathrm{int} \bigcup_{n\in\mathbb{N}} \mathrm{cl}\, U_n$. We show, that $y \in \bigcup_{n\in\mathbb{N}} U_n$. Obviously, $y \in \mathrm{cl}\, W_n \subseteq K_n$, for some $n \in \mathbb{N}$. If $y \in \mathrm{cl}\,(K_n \setminus \mathrm{cl}\, W_n)$, then every neighborhood of $y$ would contain elements of the set $K_n \setminus \mathrm{cl}\, W_n$ which is disjoint from $\bigcup_{n\in\mathbb{N}} \mathrm{cl}\, U_n$; hence, $y$ could not belong to $\mathrm{int} \bigcup_{n\in\mathbb{N}} \mathrm{cl}\, U_n$—a contradiction. So, $y \in \mathrm{int}\,(-K_n \cup \mathrm{cl}\, W_n)$, and because $y \in K_n$, it implies $y \in \mathrm{int}\,\mathrm{cl}\, W_n = W_n$. By the assumption, all the sets $U_n$ and $V_n$ are pairwise disjoint, regularly open, and $U_n \cup V_n \subseteq W_n \subseteq \mathrm{cl}\,(U_n \cup V_n)$ for all $n \in \mathbb{N}$. Therefore, for all $n \in \mathbb{N}$, $(\mathrm{cl}\, U_n \setminus U_n) \cap W_n = (\mathrm{cl}\, V_n \setminus V_n) \cap W_n$. Now, since $y \in W_n$, we have $y \in \mathrm{cl}\,(U_n \cup V_n) = \mathrm{cl}\, U_n \cup \mathrm{cl}\, V_n$. Observe that $y \notin \mathrm{cl}\, U_n \setminus U_n$, because otherwise it would contradict with the fact that $y \in \mathrm{int} \bigcup_{n\in\mathbb{N}} \mathrm{cl}\, U_n$. Moreover, $y \notin \mathrm{cl}\, V_n$. Indeed, $y \notin V_n$ since $V_n \cap \mathrm{cl}\, U_n = \varnothing$, and $y \notin \mathrm{cl}\, V_n \setminus V_n$ because otherwise we would have $y \in W_n \cap (\mathrm{cl}\, V_n \setminus V_n) = W_n \cap (\mathrm{cl}\, U_n \setminus U_n)$ and hence $y \in \mathrm{cl}\, U_n \setminus U_n$ which, as we have shown, is impossible. Consequently, $y \in U_n$. And hence

$$\neg\neg S = \mathrm{int}\,\mathrm{cl}\, S = \bigcup_{n\in\mathbb{N}} U_n \supseteq Z. \tag{3}$$

Observe that

$$\begin{aligned} G[S] = \neg\neg S {\longrightarrow} S &= \mathrm{int}\,(-\bigcup_{n\in\mathbb{N}} U_n \cup \bigcup_{n\in\mathbb{N}} U_n \setminus \{z_n\}) \\ &= \mathrm{int}\,(-Z) = -\mathrm{cl}\, Z = R \setminus Z = \mathbf{T} \setminus \mathrm{cl}\, Z = Q, \end{aligned} \tag{4}$$

as we need. Hence, by (3) and (4),

$$F[S] = \neg\neg S \cup (\neg\neg S {\longrightarrow} S) = \bigcup_{n\in\mathbb{N}} U_n \cup R \setminus Z = R, \tag{5}$$

as we require. Now, we put

$$D(p,s) = (F(s) \to (\neg p \to G(s) \vee p \to G(s))) \to F(s).$$

Notice that the formula $D(p,s)$ results in substituting in the formula $H(p,q,r)$ of the proof of Theorem 6, formulas $F(s)$ for $r$ and $G(s)$ for $q$, respectively. So, since Pitts' quantifiers commute with substitution, we get $\vdash \mathsf{E} p\, D \equiv (F \to G) \to F$. So, by (1), $\vdash \mathsf{E} p\, D$ and hence

$$\mathbf{T} \models \mathsf{E} p\, D. \tag{6}$$

On the other hand, similarly as in the proof of Theorem 6, we show that

$$\exists p\, D\,[S] = F[S] = R. \tag{7}$$

Finally, by (6) and (7) we get $\mathbf{T} \not\models \mathsf{E} p\, D \to \exists p\, D$. $\square$

Let us note that the property stated in Lemma 8 is not necessary to prove Theorem 7. However, it allows us to consider a relatively simple formula as the counter-example. On the other hand, in the semantics of upward closed sets in Kripke models there is a simpler solution. Consider

$$H(p,r) = (r \to (p \vee \neg p)) \to r.$$

Notice that $\vdash \mathsf{E}p\, H \equiv \neg\neg r$, but in Kripke models $\exists p\, H$ is not generally equivalent to $\neg\neg r$. Moreover, the topological space corresponding to a frame of the required countermodel need not be $T_1$ (the two-element model with $r$ being forced only at the top node suffices). This, however, does not work when we consider the topological semantics of dense-in-themselves metric spaces. The following proposition shows this more specifically.

**Proposition 9** *The formula $\exists p\, H \equiv \neg\neg r$ is valid in every dense-in-itself metric space.*

*Proof:* Notice that the formula $\exists p\, H \to \neg\neg r$ is generally true. So, it is enough to show that for every open subset $R$ of a given dense-in-itself metric space $\mathbf{T}$,

$$\operatorname{int}\operatorname{cl} R \subseteq \bigcup\{H[P, R] \ : \ P \text{ — open}\}.$$

Let us fix $\mathbf{T}$ with a metric $\rho$ and an open subset $R$ of $\mathbf{T}$. Put $Z = \operatorname{cl}(\operatorname{int}\operatorname{cl} R \setminus R)$. We show that there is a set $T \subseteq R$ such that $T^d = Z$, that is, $Z$ is the set of the accumulation points of $T$. Of course, since $T \subseteq R$, it follows that $T$ is disjoint from $Z$.

Consider the following property which may be possessed by subsets $T$ of the set $R$:

$$(T) \qquad \rho(x,y) > \frac{1}{2}(\rho(x,Z) + \rho(y,Z)) \quad \text{for every } x, y \in T, x \neq y.$$

Notice that $(T)$ is a finite property. Hence, by virtue of Tukey's Lemma, there is a maximal set (with respect to inclusion) satisfying $(T)$. Fix such a set $T$.

First we show that $T^d \subseteq Z$. Take an arbitrary $t \in T^d$ and suppose $t \notin Z$. Then, since $Z$ is closed, $\rho(t, Z) > 0$. Consider the open ball $K = \{v \ : \ \rho(t, v) < \frac{1}{3}\rho(t, Z)\}$. By the assumption, we can find $x, y \in T$ such that $x, y \in K$. Thus we have $\rho(t, Z) \leq \frac{1}{3}\rho(t, Z) + \rho(x, Z)$ and $\rho(t, Z) \leq \frac{1}{3}\rho(t, Z) + \rho(y, Z)$ whence $\frac{2}{3}\rho(t, Z) \leq \frac{1}{2}(\rho(x, Z) + \rho(y, Z))$. Hence we get

$$\rho(x,y) \leq \rho(t,x) + \rho(t,y) \leq \frac{1}{3}\rho(t,Z) + \frac{1}{3}\rho(t,Z) \leq \frac{1}{2}(\rho(x,Z) + \rho(y,Z)),$$

a contradiction, since $x, y \in T$.

Now we prove that $Z \subseteq T^d$. Suppose that there is $z \in Z$ and $d > 0$ such that the ball $L = \{v \ : \ \rho(z, v) < d\}$ is disjoint from $T$. Since $Z \subseteq \operatorname{cl} R$, there is $y \in R$ with $\rho(y, z) < \frac{d}{4}$. Obviously, $y \notin T$ and since $T$ is a maximal set satisfying $(T)$, we have $2\rho(x, y) \leq \rho(x, Z) + \rho(y, Z)$ for some $x \in T$. Moreover, $\rho(x, Z) \leq \rho(x, z)$ and $\rho(y, Z) \leq \rho(y, z) < \frac{d}{4}$. Thus

$$2\rho(x,y) < \rho(x,z) + \frac{d}{4} \leq \rho(x,y) + \rho(y,z) + \frac{d}{4} < \rho(x,y) + \frac{d}{2},$$

which means that $\rho(x, y) < \frac{d}{2}$. Hence we get $\rho(x, z) \leq \rho(x, y) + \rho(y, z) < \frac{d}{2} + \frac{d}{4} < d$ which contradicts the supposition $L \cap T = \varnothing$. So, we have $T \subseteq R$ such that $T^d = Z$, that is, $\mathrm{cl}\, T = Z \cup T$.

Now consider $P = R \setminus T$. Notice that, since $P = R \setminus T = R \setminus \mathrm{cl}\, T$, the set $P$ is open. Moreover, $R \cap -P \cap \mathrm{cl}\, P = T$ and, consequently,

$$H[P, R] = \mathrm{int}\, (\mathrm{cl}\, (R \cap -P \cap \mathrm{cl}\, P) \cup R) = \mathrm{int}\, (\mathrm{cl}\, T \cup R) = \mathrm{int}\, (Z \cup R) \supseteq \mathrm{int}\, \mathrm{cl}\, R.$$

Since $R$ was chosen arbitrarily, we get $\mathbf{T} \models \neg\neg r \to \exists p\, H$. □

By Proposition 9, in the interpretation over any dense-in-itself metric space the formula $\exists p\, ((r \to (p \vee \neg p)) \to r)$ is equivalent to $\mathsf{E} p\, ((r \to (p \vee \neg p)) \to r)$. (The same can be asserted of any formula of the form $(r \to F(p)) \to r$ where $F$ is an arbitrary monadic formula with $\vdash p \vee \neg p \to F$.) Again, it contrasts with the case of topological semantics in general; in particular with the semantics of upward closed subsets of Kripke models.

**Corollary 10** *The semantics of upward closed sets in Kripke models differs from the topological semantics of dense-in-themselves metric spaces when restricted to the language of at least two variables.*

To conclude, we shed some more light on the topological interpretation of intuitionistic logic. The problem of new intuitionistic operators was investigated by Friedman [3], Goad [7], and Kreisel [9]. In [9] an operator $*$ defined by means of propositional quantification as

$$*(q) = \exists p\, (q \equiv \neg p \vee \neg\neg p)$$

is considered. As Kreisel shows, the operator $*$ is not definable by means of $\{\neg, \vee, \wedge, \to\}$ with regard to, for example, topological models. The investigations are continued in Troelstra [21], Wojtylak [23], and [14], [15], [16]. The main result of [23] states that when $\exists p F$ is interpreted as the disjunction

$$\bigvee \{F[p/G] : G \text{ – a monadic formula in } p\},$$

then the formula $\exists p\, (q \equiv F)$ is equivalent to a *finite* disjunction of its instances if and only if $(\neg p \vee \neg\neg p) \to F$ is not derivable. This result holds not only in the case of the standard language, but also when we consider the language extended by some nonstandard operator. In [21], it is shown that definability of the operator $*$ depends on the topological space considered. Namely, in some spaces, such as $(0, 1)$ and Cantor space, the operator $*(q)$ is definable by $\neg\neg q$ while in $[0, 1]$ $*$ is not definable with respect to the standard connectives. It should be noted that $\vdash \mathsf{E} p\, (q \equiv \neg p \vee \neg\neg p) \equiv \neg\neg q$, so definability of the operator $*$ by $\neg\neg q$ in a topological model means that the topological interpretation of $\exists p\, (q \equiv \neg p \vee \neg\neg p)$ coincides with Pitts' $\mathsf{E} p\, (q \equiv \neg p \vee \neg\neg p)$. This problem seems to require more attention. Consequently, in [16] the following is proved.

**Theorem 11** *Let* $\mathbf{T}$ *be a dense-in-itself metric space whose every open and dense subset is a union of two disjoint regularly open sets. Then all the operators of the form* $q \mapsto \exists p\, (q \equiv F(p))$, *where* $F(p)$ *is an arbitrary monadic formula of the language of Heyting calculus, are definable in* $\mathbf{T}$ *by means of* $\{\neg, \vee, \wedge, \to\}$. *Moreover,* $\mathbf{T} \models \exists p\, (q \equiv F(p)) \equiv \mathsf{E} p\, (q \equiv F(p))$.

This means that in every sufficiently regular topological space (such as Cantor space or the reals) Pitts' quantifier $\mathsf{E}$ coincides with the standard quantifier $\exists$ with respect to all the formulas $q \equiv F(p)$ in question. Note that these formulas involve two distinct variables and, according to Theorem 7, their definability by formulas of Heyting calculus depends on the properties of the space. Particularly, none of the operators of the form $q \mapsto \exists p\,(q \equiv F(p))$, for $F$ such that $\vdash \neg\neg p \vee \neg p \to F$, is definable in the standard language in the topological semantics associated with the upward closed subsets of Kripke models.

**Acknowledgments** I wish to thank the anonymous referee for the valuable comments which helped me to improve the previous version of the paper. In particular, I am grateful for the remarks and examples involving Kripke semantics which are included in the paper.

## NOTES

1. This fact can also be seen in the following two ways: (i) $q$ is maximal in the Rieger–Nishimura Lattice on $q$ below $H$; (ii) consider a Kripke model $\mathbf{K}$ (on atoms not involving $p$) with root $b$ and suppose $b \Vdash \mathsf{A}p\, H$. We show that $b \Vdash q$. Add a new root $b'$ below $\mathbf{K}$ with the same forcing as $b$. So $b'$ bisimulates with $b$. Now extend the forcing on the new model by stipulating that $x$ forces $p$ precisely if $x$ is in the old model, that is, $x \neq b'$. Since $b'$ bisimulates with $b$ (with respect to the atoms distinct from $p$) we have, by the Shavrukov-Ghilardi-Zawadowski-Visser semantics, $b' \Vdash p \vee (p \to q)$. Hence $b \Vdash q$.
2. One can also use the bisimulation semantics. Consider any Kripke model $\mathbf{K}$ for a forcing not involving $p$. Multiply $\mathbf{K}$ with the model $\mathbf{M}$ on $m_0, m_1, m_2$ generated by $m_0 \leq m_1$, $m_0 \leq m_2$ and $m_1 \Vdash p$. The forcing on the new model, say $\mathbf{N}$, is dictated by the forcing on $\mathbf{K}$ for the variables except $p$ and by the forcing on $\mathbf{M}$ as far as $p$ is concerned. It is easy to see that $\mathbf{N}$ is a bisimulation extension of $\mathbf{K}$. Moreover, for $A$ not involving $p$ and every $n$ of $\mathbf{N}$ we have: $n \Vdash A$ if and only if $n \Vdash (p \to A) \vee (\neg p \to A)$. So, for every formula $B(q)$ not involving $p$ and for any $k$ of the model $\mathbf{K}$, whenever we have $k \Vdash B(q)$, we also have $k \Vdash \mathsf{E}p\, B((p \to q) \vee (\neg p \to q))$.

## REFERENCES


[1] Bull, R. A., "On modal logic with propositional quantifiers," *The Journal of Symbolic Logic*, vol. 34 (1969), pp. 257–63. Zbl 0184.28101 1

[2] Fine, K., "Propositional quantifiers in modal logic," *Theoria*, vol. 36 (1970), pp. 331–46. Zbl 0302.02005 MR 48:5824 1

[3] Friedman, H., "Adding propositional connectives to countable infinitary logic," *Mathematical Proceedings of the Cambridge Philosphical Society*, vol. 77 (1975), pp. 1–6. Zbl 0323.02022 MR 50:6786 1

[4] Gabbay, D., *Semantical Investigations in Heyting's Intuitionistic Logic*, D. Reidel, Dordrecht, 1981. Zbl 0453.03001 MR 83b:03012 1, 1

[5] Ghilardi, S., and M. Zawadowski, "A sheaf representation and duality for finitely presented Heyting algebras," *The Journal of Symbolic Logic*, vol. 60 (1995), pp. 911–39. Zbl 0837.03047 MR 96i:03063 1, 1

[6] Ghilardi S., and M. Zawadowski, "Undefinability of propositional quantifiers in the modal system S4," *Studia Logica*, vol. 55 (1995), pp. 259–71. Zbl 0831.03008 MR 96j:03029 1

[7] Goad, C. A., "Monadic infinitary propositional logic: a special operator," *Reports on Mathematical Logic*, vol. 10 (1978), pp. 43–50. Zbl 0424.03029 MR 81e:03024 1

[8] Kaplan, D., "S5 with quantifable propositional variables," *The Journal of Symbolic Logic*, vol. 35 (1970), p. 355. 1

[9] Kreisel, G., "Monadic operators defined by means of propositional quantification in intuitionistic logic," *Reports on Mathematical Logic*, vol. 12 (1981), pp. 9–15. Zbl 0464.03051 MR 83a:03011 1, 1

[10] Kremer, P., "On the complexity of propositional quantification in intuitionistic logic," *The Journal of Symbolic Logic*, vol. 62 (1997), pp. 529–44. Zbl 0887.03002 MR 98k:03020 1, 1, 1

[11] Kremer, P., "Propositional quantification in the topological semantics for **S4**," *Notre Dame Journal of Formal Logic*, vol. 38 (1997), pp. 295–313. Zbl 0949.03020 MR 99d:03015 1

[12] Kremer, P., "Quantifying over propositions in relevance logic: non-axiomatisability of $\forall p$ and $\exists p$," *The Journal of Symbolic Logic*, vol. 58 (1993), pp. 334–49. Zbl 0786.03016 MR 94f:03024 1

[13] Pitts, A., "On an interpretation of second order quantification in the first order intuitionistic propositional logic," *The Journal of Symbolic Logic*, vol. 57 (1992), pp. 33–52. 1, 1, 1

[14] Połacik, T., "Operators defined by propositional quantification and their interpretations over Cantor space," *Reports on Mathematical Logic*, vol. 27 (1993), pp. 67–79. MR 95m:03015 1

[15] Połacik, T., "Second order propositional operators over Cantor space," *Studia Logica*, vol. 53 (1994), pp. 93–105. Zbl 0790.03005 MR 95f:03011 1

[16] Połacik, T., "Propositional quantification in the monadic fragment of intuitionistic logic," *The Journal of Symbolic Logic*, vol. 63 (1998), pp. 269–300. Zbl 0959.03005 MR 99h:03004 1, 1, 1, 1, 1, 1, 1

[17] Scedrov, A., "On some extensions of second-order intuitionistic propositional calculus," *Annals of Pure and Applied Logic*, vol. 27 (1984), pp. 155–64. Zbl 0569.03026 1

[18] Shavrukov, V. Yu., "Subalgebras of diagonalizable algebras of theories containing arithmetic," *Dissertationes Mathematicae* CCCXXIII, Polska Akademia Nauk, Mathematical Institute, Warszawa, 1993. Zbl 0803.03044 MR 94h:03131 1

[19] Skvortsov, D., "Non-axiomatizable second-order intuitionistic propositional logic," *Annals of Pure and Applied Logic*, vol. 86 (1997), pp. 33–46. Zbl 0873.03007 MR 98g:03019 1, 1

[20] Tarski, A., "Der Aussagenkalkül und die Topologie," *Fundamenta Mathemeticae*, vol. 31 (1938), pp. 103–34. Zbl 0020.33704 1, 1

[21] Troelstra, A. S., "On a second order propositional operator in intuitionistic logic," *Studia Logica*, vol. 40 (1981), pp. 113–39. Zbl 0473.03022 MR 84a:03015 1, 1

[22] Visser, A., "Uniform interpolation and layered bisimulation," pp. 139–64 in *Gödel '96, Logical Foundations of Mathematics, Computer Science and Physics—Kurt Gödel's Legacy*, edited by P. Hájek, Springer, Berlin, 1996. Zbl 0854.03026 MR 98g:03026 1, 1

[23] Wojtylak, P., "Collapse of a class of infinite disjunctions in intuitionistic propositional logic," *Reports on Mathematical Logic*, vol. 16 (1983), pp. 37–50.

Zbl 0599.03024 MR 86b:03017 1, 1

*University of Silesia*
*Institute of Mathematics*
*Bankowa 14*
*40007 Katowice*
*POLAND*
*email:* *polacik@us.edu.pl*