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An Intensional Schrödinger Logic

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Abstract We investigate the higher-order modal logic $S_\omega I$, which is a variant of the system S_ω presented in our previous work. A semantics for that system, founded on the theory of quasi sets, is outlined. We show how such a semantics, motivated by the very intuitive base of Schrödinger logics, provides an alternative way to formalize some intensional concepts and features which have been used in recent discussions on the logical foundations of quantum mechanics; for example, that some terms like 'electron' have no precise reference and that 'identical' particles cannot be named unambiguously. In the last section, we sketch a classical semantics for quasi set theory.

1 Introduction One of the most interesting debates in the field of science in our time is whether the general principles of classical logic could be viewed as necessary and permanent truths. Notwithstanding the fact that the real sciences provide numerous insights for the development of deviant logics, in general the nonclassical logical systems developed in this century did not arise from the desire to formalize the intuitions of scientists. On the contrary, the majority of them had their motivation from the pure mathematical interest of investigating, as Hilbert said, "all logically possible theories" [20]. But logic can also be viewed, in a certain sense, as "*la physique de l'objet quelconque*" (Gonseth [19], p. 155) and then, by considering this view, we may be tempted to investigate those logical systems which arise from the necessities of science, in particular by taking into account, as it was noticed by several authors, that modern physics presents strong arguments for the questioning of classical concepts like those of object and identity, which suggest revisions in standard concepts like 'set', 'individuatable thing', 'extension and intension of concepts', and so on.[1]

With regard to identity, the papers listed above present arguments for the view that in the microworld this concept does not conform with its use in the case of usual macroscopic objects, that is, with the "traditional" theory of identity of classical logic and mathematics (Krause [22]). Schrödinger, for instance, stressed that this concept simply has no meaning for elementary particles; more specifically, in discussing the

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case where one is tempted to say that one and *the same* particle was observed twice at different instants of time, he said that

> The circumstances may be such that they render it highly convenient and desirable to express oneself so, but it is only an abbreviation of speech; for there are other cases where the 'sameness' becomes entirely meaningless; and there is no sharp boundary, no clear-cut distinction between them, there is a gradual transition over intermediate cases. And I beg to emphasize this and I beg you to believe it: It is not a question of our being able to ascertain the identity in some instances and not being able to do so in others. *It is beyond doubt that the question of 'sameness', of identity, really and truly has no meaning.* [our emphasis] (Schrödinger [34], pp. 17–18; see also DaCosta [3], p. 117 and DaCosta [4])

Based on these Schrödingerian intuitions, it was outlined by da Costa in 1980 ([3], pp. 117–18) that in a two-sorted first-order logical system the principle of identity $\forall x(x = x)$ is not valid in general, since expressions like $a = b$ are atomic formulas in the case in which a and b are both terms of just one of these two species (the second, say); for the terms of the first species, that expression is simply not a formula.

A semantics for that system was also suggested, and it was noted that such a semantics, which is formulated on the usual theories of sets, does not adequately express the intuitive idea of that logic, since the constants of the first species (the "problematic" ones referred to above) were interpreted as elements of a set (as in the usual Tarskian semantics) and a set is a collection of distinguishable objects. This fact is, of course, contrary to the spirit of Schrödinger logics, since the intuitive motivation is that the objects of the first species should be considered as indistinguishable and devoid of identity.[2]

The above mentioned first-order system was extended to a higher-order logic (simple theory of types) in [4] and some additional topics were emphasized, for instance, the possibility of violating Leibniz's Principle of the Identity of Indiscernibles, which, in some sense, reflects what has been pointed out in recent literature in connection with the foundations of quantum mechanics (see French [14], French [15], and [4] for further references). From this point of view, Schrödinger logics might be viewed as alternative mathematical devices by means of which some of these fundamental intuitions can be formalized.

A "classical" semantics was also presented for the higher-order system S_ω mentioned above, that is, a semantics founded on the usual set theories, and a generalized completeness theorem was proved. But, as it was pointed out in [4], the basic problem which appears in the first-order system remains, since the concept of identity cannot be applied to some *entities*. As it was suggested in [3], to stress the intuitive idea underlying Schrödinger logics, a *quasi set theory*, in which the concept of a set is extended, should be developed. In such a theory a semantics for Schrödinger logics should be formulated. In addition, in a quasi set theory the existence of indistinguishable but nonidentical entities must be allowed. In this paper, we sketch the main features of such a semantics, using the quasi set theory developed in Dalla Chiara [13]. As an interesting consequence, we show how such an enterprise allows us to handle intensional concepts, as they have been presented in recent literature, in connection with the logical basis of quantum mechanics (Dalla Chiara [9]). As for quasi sets, we recall that quasi set theories were developed in [22] and in Krause [24]; alternative systems were proposed in Krause [26] (developed in [13]) and in Krause [25]. Independently and motivated by other insights, *quaset* theories were proposed in Dalla Chiara [10].[3] The logical system we present here, denoted by $S_\omega I$, differs from the logic S_ω of [24] in the following way: instead of considering just one type i for individuals and a two-sorted language in order to "separate" those individuals which have the usual properties from the "problematic" ones, we introduce individual types e_1 and e_2. The individuals of type e_1 are those entities to which the concept of identity does not apply (they are called *m*-objects in the quasi set terminology which will be used throughout this paper), while the individuals of type e_2 are the "classical" ones. Furthermore, $S_\omega I$ is a modal logic in which the necessity operator $\Box$ is used as a primitive symbol. The logic is based on Gallin's system $\mathrm{ML_p}$ ([21], part 2), on which our terminology is partially based. Additional explanations are introduced as we go.

2 Motivation Recent research on the foundations of physics have suggested the convenience of introducing intensional logics of some particular kind to cope with certain problems regarding the basis of quantum mechanics (QM).[4] As an example, in discussing the semantical analysis of QM, Dalla Chiara pointed out that

> Significant counterindications against the adequacy of a purely extensionalistic approach in semantics have been suggested by contemporary physics. Microphysics seems to be, from a semantical point of view, essentially a world of intensions, where individual objects and sets of individual objects appear, so to speak, as unnatural notions. The concept of *extension* (*reference*) becomes more and more blurred as we go deep into the logical intricacies of Quantum Mechanics. . . . In my opinion, some characteristic semantical features which clearly appear in the logical investigations about the microuniverse do not merely point out some fairly pathological aspects of a world that is far apart from our human world. On the contrary, they make us aware about certain oversimplifications of our usual semantical principles. ([9], see also [10])

The discussion of the use of intensional concepts in physics certainly deserves careful analysis, which we do not intend to develop here (but see the references). However, motivating the semantics presented below, we recall without details some of Dalla Chiara's examples, which illustrate from an intuitive point of view the main questions to be considered below (see Section 6). In usual (extensional) semantics, the extension of a proper name is an individual of the universe of discourse, but in microphysics the indistinguishability of elementary particles causes serious problems regarding this view. In fact, if a is a constant of the language (say, a first-order language), which should *name* an elementary particle, then QM suggests that in general a has no precise reference. In other words, what is to be $\rho(a)$, where ρ is the function corresponding to the standard denotation function in Tarskian semantics? Also predicates (it suffices to reason on one-place predicates), which usually have subsets of the domain of individuals as their extensions, may not behave semantically as expected. Take for instance the predicate 'electron'. As Dalla Chiara calls to our attention, in microphysics this concept has no precise extension, despite having a well-defined intension (characterized, roughly speaking, by the physical properties that characterize the entity 'electron').[5] That is, the extension of such a predicate cannot be identified with a well-defined subset of the domain in the standard set-theoretical sense.

Other important topics could be considered in connection with the use of intensions in physics, for instance the supposed necessity of introducing *intensional structures*, as those presented in [9]. But we hope that the few examples just mentioned are sufficient to illustrate our point.

3 The language of the system $S_\omega I$ To begin with, let us introduce the concept of type. The set of *types* is defined as the smallest collection Π such that: (a) $e_1, e_2 \in \Pi$ and (b) if $\tau_1, \dots, \tau_n \in \Pi$, then $\langle \tau_1, \dots, \tau_n \rangle \in \Pi$. e_1 and e_2 are the types of the *individuals*; the objects of type e_1 are called m-atoms and are intuitively thought of as elementary particles of modern physics (first-quantized approach). Following Schrödinger, we suppose that the concept of identity cannot be applied to them (cf. [4]). The language of the system $S_\omega I$ may be described as follows: it contains the usual connectives (we suppose that $\neg$ and $\longrightarrow$ are the primitive ones, while the others are defined as usual), the symbol of equality, auxiliary symbols, and quantifiers ($\forall$ is the primitive and $\exists$ is defined in the standard way) and the necessity operator $\Box$. With respect to variables and constants, for each type $\tau \in \Pi$ there exists a denumerably infinite collection of variables $X_1^\tau, X_2^\tau, \dots$ of type τ and a (possibly empty) set of constants $(A_1^\tau, A_2^\tau, \dots)$ of that type; we use X^τ, Y^τ, C^τ, and D^τ perhaps with subscripts as metavariables for variables and constants of type τ, respectively. The *terms* of type τ are the variables and the constants of that type; so, we have individual terms of types e_1 and e_2. We use U^τ, V^τ, perhaps with subscripts, as syntactical variables for terms of type τ. The atomic formulas are defined in the usual way: if U^τ is a term of type $\tau = \langle \tau_1, \dots, \tau_n \rangle$ and $U^{\tau_1}, \dots, U^{\tau_n}$ are terms of types $\tau_1, \dots, \tau_n$ respectively, then $U^\tau(U^{\tau_1}, \dots, U^{\tau_n})$ is an atomic formula; so is $U^\tau = V^\tau$ if τ is not of type e_1. Then, the language permits us to talk neither about the identity nor about the diversity of the individuals of type e_1. The other formulas are defined as usual. A formula containing at least $U^{\tau_1}, \dots, U^{\tau_n}$ as free variables sometimes shall be written $F(U^{\tau_1}, \dots, U^{\tau_n})$.

4 Semantics The aim of this section is to outline the tools for the proof of the "weak" completeness theorem for $S_\omega I$ which we shall sketch in the next section. We start by recalling intuitive topics concerning quasi sets which will be used below.

4.1 The quasi set theory Quasi set theories can be roughly described as mathematical devices for treating collections of indistinguishable objects (see [25]). The theory S^{**} developed in [13] (see also [24]) allows the presence of a certain kind of *Urelemente*, (the m-atoms) to which the usual concept of identity does not apply. The underlying logic of S^{**} is classical quantificational logic without identity; the specific symbols are three unary predicate letters $m(x)$ (read "x is an m-atom"), $M(x)$ (read "x is an M-atom", that is, a standard Urelement) and $Z(x)$ (read "x is a standard set"; by a *set* we mean a quasi set whose transitive closure does not contain m-atoms—these "sets" have all the properties of the ZFC-sets); two binary predicate symbols, $\in$ (membership) and $\equiv$ (indistinguishability); and a unary functional symbol qc (quasi cardinality). A quasi set (qset for short) is defined as an entity which is not an Urelement (i.e., it is anything that is neither an m-atom nor a classical atom). If x is a quasi set, we write $(Q)x$. The concept of quasi cardinality is introduced in such a way that it extends the concept of cardinality for arbitrary qsets (cf. [24]).

In what follows we talk about quasi sets in an informal way, but our exposition can be, in principle, formalized in a quasi set theory. The axioms of indistinguishability state first that $\equiv$ has the properties of an equivalence relation. Then, we define the concept of *extensional equality* $=_E$ in the following way: $x =_E y$ if and only if $(Q(x) \wedge Q(y) \wedge \forall z(z \in x \longleftrightarrow z \in y)) \vee (M(x) \wedge M(y) \wedge x \equiv y)$. That is, in the quasi set theory extensional entities are indistinguishable standard Urelemente or qsets which have exactly the "same" elements.[6] Then we postulate that the substitutivity principle is valid only to extensional entities: in symbols, $\forall x \forall y(x =_E y \longrightarrow (A(x,x) \longrightarrow A(x,y)))$ is an axiom. The extensional equality has all the formal properties of classical identity.

Furthermore, *similar* qsets are those nonempty qsets such that the elements of one of them are indistinguishable from the elements of the other. Intuitively, similar qsets are composed by elements "of the same sort." The basic axiom is the "weak" extensionality which states that similar qsets which have the same quasi cardinality are indistinguishable qsets (we also say that indistinguishable qsets are "equal").[7] Then, this equality relation is not extensional and this fact allows that intensional concepts be handled in the theory. Furthermore, the axioms entail that if the quasi cardinal of the qset x is α and if $\beta \leq \alpha$ is a quasi cardinal, then there exists a subqset $y \subseteq x$ such that the quasi cardinal of y is β. This permits us to use the concept of a *power qset* (the qset of the subqsets of x) in the next section. The *power qset axiom* states that if α is the quasi cardinal of a certain qset x, then the qset of all subquasi sets of x (the power qset of x) has quasi cardinal 2^α. This entails that the theory is consistent with the hypothesis that, for instance, x has α *singletons*, that is, α subquasi sets with just one element. A similar argument applies to subqsets with two, three, . . . , α elements. Apparently, the possibility of the existence of these singletons would entail that a kind of Leibnizian identity is automatically defined in quasi set theory. But let us remark that the quasi set theory does not permit us to distinguish between such singletons since all of them are indistinguishable from one another by the axiom of weak extensionality. The details may be found in [25], but we insist that despite the theory being compatible with the idea of α singletons, their existence cannot be proven. It should be remarked that this is precisely what happens in quantum physics, since, for example, although the hypothesis that there are exactly six electrons at the level 2p of a sodium atom does agree with experimental evidence, no one can distinguish the "unitary" subcollections of electrons. Notwithstanding, quantum physics would be contradictory if such a supposition could not be admitted. So, by means of the mentioned power qset axiom, we maintain the possibility of reasoning on absolutely indistinguishable "distinct" objects (the word 'distinct' is to be understood here as a metalinguistic expression; it cannot be formalized in quasi set theory, at least with regard to m-atoms).

Finally, let us recall here the concept of a *quasi function* which will be used below. In quasi set theory, the concept of (binary) 'relation' is defined in the standard way, as quasi sets of 'ordered pairs', by means of the weak concept of 'pair'; but there is a problem regarding the concept of function, since a function (in the standard sense) could not distinguish between possible values of a certain argument, due to their indistinguishability (when there are m-atoms involved). Then we have defined a weaker concept of a mapping which maps indistinguishable things into indistinguishable things; in the case of "macroscopic" objects (like standard Urelemente or "sets"), the concept coincides with the standard notion of function. In short, quasi functions are such that given an argument, the corresponding value is not well defined, as the formal definition below explains. For more details, see [25]. Suppose that R is the predicate for 'relation' defined in the standard sense. Let x and y be quasi sets. Then we say that f is a *q-function* from x to y if f is such that

$$R(f) \wedge \forall u(u \in x \longrightarrow \exists v(v \in y \wedge \langle u, v\rangle \in f))\wedge$$
$$\forall u \forall u' \forall v \forall v'(\langle u, v\rangle \in f \wedge \langle u', v'\rangle \in f \wedge u \equiv u' \longrightarrow v \equiv v')$$

The concepts of q-injection, q-surjection, and q-bijection can be introduced in an obvious way; for more details, see [25]. In addition, the axioms imply that the theory ZFU (Zermelo-Fraenkel with Urelemente) can be interpreted in S^{**}. Additional remarks on quasi sets will be introduced when necessary.

4.2 The generalized quasi set semantics Let $D = \langle m, M\rangle$ be an ordered pair (introduced in the usual way in the theory S^{**}), where $m \neq \varnothing$ is a finite pure qset (i.e., a finite qset which has only m-atoms as elements) and $M \neq \varnothing$ is a set.[8] Furthermore, we suppose that I is a nonempty set (whose elements are called *index* or *state of affairs*).[9] By a *frame* for $S_\omega I$ based on D and I we mean an indexed family of qsets $(\mathcal{F}_\tau)_{\tau\in\Pi}$ where

1. $\mathcal{F}_{e_1} = m$;
2. $\mathcal{F}_{e_2} = M$;
3. for each $\tau = \langle \tau_1, \ldots, \tau_n\rangle \in \Pi$, $\mathcal{F}_\tau$ is a nonempty subqset of

$$[\mathcal{P}(\mathcal{F}_{\tau_1} \times \cdots \times \mathcal{F}_{\tau_n})]^I$$

If the equality holds in (3), then the frame is *standard*. By a *general model* (*g-model* for short) for $S_\omega I$ based on D and I we understand an ordered pair

$$\mathcal{M} = \langle \mathcal{F}_\tau, \rho\rangle_{\tau\in\Pi}$$

such that

1. $(\mathcal{F}_\tau)_{\tau\in\Pi}$ is a frame for $S_\omega I$ based on D and I;
2. ρ is a quasi function which assigns to each constant C^τ an element of $\mathcal{F}_\tau$.[10]

A *standard model* for $S_\omega I$ is a g-model whose frame is standard. Before introducing other semantical concepts, let us consider some examples which illustrate the "intensional" counterpart of such a semantics. The first two examples show that the classical intensional case [21] remains valid when the entities are not of the type e_1. The last ones exemplify the specific case of Schrödinger logics.

Example 4.1 Let us consider the constant C^{e_2}. Since $\mathcal{F}_{e_2} = M$, then $\rho(C^{e_2}) \in M$, that is to say, C^{e_2} names an element of a standard set. This is not, of course, surprising, since the given constant is "classical".

Example 4.2 Now we consider the constant $C^{\langle e_2\rangle}$. In this case, $\mathcal{F}_{\langle e_2\rangle} \subseteq [\mathcal{P}(\mathcal{F}_{e_2})]^I = [\mathcal{P}(M)]^I$. Then, $\mathcal{F}_{\langle e_2\rangle}$ is a class of functions from I in $\mathcal{P}(M)$, also as in the classical case. Intuitively, $C^{\langle e_2\rangle}$ is a unary predicate (an individual property) whose arguments are individuals of type e_2 (i.e., classical individuals).[11]

Example 4.3 Let us take a constant C^{e_1}. In this case, $\mathcal{F}_{e_1} = m$ and then $\rho(C^{e_1}) \in m$, that is, the constant *names* an m-atom. Since the m-atoms cannot be individualized, counted, and so forth, the denotation of C^{e_1} is ambiguous. We can say that a constant of type e_1 plays the role of a *generalized noun* (*g-noun*).

Example 4.4 We now consider a constant $C^{\langle e_1\rangle}$. In this case, $\mathcal{F}_{\langle e_1\rangle} \subseteq [\mathcal{P}(\mathcal{F}_{e_1})]^I = [\mathcal{P}(m)]^I$. Then, $\rho(C^{\langle e_1\rangle}) \in \mathcal{F}_{\langle e_1\rangle}$, that is, it is a (quasi) function from I to $\mathcal{P}(m)$. In other words, $\rho(C^{\langle e_1\rangle})$ is a quasi function from I to $\mathcal{P}(m)$. If m is a pure qset whose elements are all indistinguishable from one another, then the denotation function does not distinguish between qsets in $\mathcal{P}(m)$. In fact, in this case the only difference among the subqsets of m is their cardinality, that is, if $\rho(C^{\langle e_2\rangle})$ is x, then every qset y such that x and y are similar (cf. Subsection 4.1) may act as the denotation of $C^{\langle e_2\rangle}$ as well (see [13]). This interpretation accommodates the intuitive idea that a predicate like ‘electron’ does not have a well-defined extension.

The last two examples suggest an interpretation of ‘quantum predicates’ such as ‘electron’, in conformity to the discussion of Section 2; such predicates, which do not have well-defined extensions (in the sense that every qset of a certain class of similar qsets may be considered as their extension) are relations-in-intension of sort $U^{\langle e_1\rangle}$. In the same way, indistinguishable elements of a pure qset are not individuatable entities; they can only be aggregated in a certain quantity (every qset has a cardinal). They accurately exemplify the ‘quanta’ in Teller and Redhead’s terminology (cf. [32], [33]). Since the m-atoms have no names, the terms of type e_1 have no precise denotation; they refer ambiguously to an arbitrary element of the domain. In other words, we may properly say that such constants do not represent anything in particular: they lack a (precise, well-defined) referent.

The set of all *assignments over* a g-model $\mathcal{M}$, denoted $As(\mathcal{M})$, is the set of all q-functions f on the set of variables of $S_\omega I$ such that $f(X^\tau) \in \mathcal{F}_\tau$, for every variable X^τ of type τ. For any $f \in As(\mathcal{M})$, we denote by $\overline{f}$ the extension of f to the set of all constants, defined by $\overline{f}(C^\tau) = \rho(C^\tau) \in \mathcal{F}_\tau$.

If $i \in I$ and $f \in As(\mathcal{M})$, then the notion

$$\mathcal{M}, i, f \ \text{sat} \ A$$

is defined by recursion on the length of the formula A as follows:

1. $\mathcal{M}, i, f$ sat $U^\tau(U^{\tau_1}, \ldots, U^{\tau_n})$ iff $\langle \overline{f}(U^{\tau_1}), \ldots, \overline{f}(U^{\tau_n})\rangle \in \overline{f}(U^\tau)(i)$;
2. $\mathcal{M}, i, f$ sat $U^\tau = V^\tau$ iff $\langle \overline{f}(U^\tau), \overline{f}(V^\tau)\rangle \in \Delta_{\equiv}(\tau)$ where $\Delta_{\equiv}(\tau)$ is the “pseudo-diagonal” of $\mathcal{F}_\tau$, which may be defined in S^{**} as the subqset of $\mathcal{F}_\tau \times \mathcal{F}_\tau$ whose elements are indistinguishable from one another (when $\tau \neq e_1$, this qset is the diagonal of $\mathcal{F}_\tau$ in the standard sense);
3. $\mathcal{M}, i, f$ sat $\Box A$ iff $\mathcal{M}, j, f$ sat A for every $j \in I$;
4. usual clauses for $\neg$, $\longrightarrow$, and $\forall$.

A formula A is *true* in a g-model $\mathcal{M}$ (denoted $\models_{\mathcal{M}} A$) if and only if $\mathcal{M}, i, f$ sat A for every $i \in I$ and $f \in As(\mathcal{M})$. A set Σ of formulas of $S_\omega I$ is *g-satisfiable* in $S_\omega I$ if and only if for some g-model $\mathcal{M}$, index i, and assignment f, $\mathcal{M}, i, f$ sat A for all $A \in \Sigma$. A formula A is a *g-semantical consequence* of a set Γ of formulas, and we write $\Gamma \models_{\mathrm{g}} A$, if and only if $\mathcal{M}, i, f$ sat A for $i \in I$, $f \in As(\mathcal{M})$ and g-model $\mathcal{M}$ whenever $\mathcal{M}, i, f$ sat B for every formula $B \in \Gamma$. If $\Gamma = \varnothing$, then we write $\models_{\mathrm{g}} A$ and say that A is *g-valid* in $S_\omega I$. In the next section we present an axiom system for $S_\omega I$ and prove a generalized completeness theorem for this logic.

5 *The theory* $S_\omega I$ The postulates of $S_\omega I$ (axiom schemata and inference rules) are the following.

- (A1) A, where A comes from a tautology in $\neg$ and $\longrightarrow$ by uniform substitution of formulas of $S_\omega I$ for the variables.
- (A2) $\forall X^\tau(A \longrightarrow B) \longrightarrow (A \longrightarrow \forall X^\tau B)$, where X does not occur free in A.
- (A3) $\forall X^\tau A(X^\tau) \longrightarrow A(U^\tau)$ where U^τ is a term free for X^τ in $A(X^\tau)$ and of the same type of X^τ.
- (A4) $X^{e_2} = X^{e_2}$.
- (A5) $X^{e_2} = Y^{e_2} \longrightarrow \Box(X^{e_2} = Y^{e_2})$.
- (A6) $\Box(U^\tau = V^\tau) \longrightarrow (A(U^\tau) \longrightarrow A(V^\tau))$, where U^τ and V^τ are free for X^τ in $A(X^\tau)$.
- (A5) $\Box A \longrightarrow A$.
- (A6) $\Box(A \longrightarrow B) \longrightarrow (\Box A \longrightarrow \Box B)$.
- (A7) $\Diamond A \longrightarrow \Box\Diamond A$.
- (R1) From A and $A \longrightarrow B$ to infer B.
- (R2) From A to infer $\forall X^\tau A$.
- (R3) From A to infer $\Box A$.

The usual syntactical notions are defined in the standard way, such as formal theorem ($\vdash$) of $S_\omega I$, the concept of the consequence of a set of formulas, and so on. A set Σ of formulas is *consistent* if and only if some formula is not derivable from Σ in $S_\omega I$. We will sketch below the proofs of the soundness and of the generalized completeness theorems for this logic.

5.1 *Soundness and generalized completeness* The soundness theorem for $S_\omega I$ is formulated as follows. If $\vdash A$ in $S_\omega I$ then $\models_{\mathrm{g}} A$ in $S_\omega I$. This result implies that if $\Gamma \vdash A$, then $\Gamma \models_{\mathrm{g}} A$ and that if a set of formulas Σ is g-satisfiable in $S_\omega I$, then Σ is consistent.

The proof is obtained by showing that all the axioms of $S_\omega I$ are g-valid and that the inference rules preserve g-validity. This follows from the fact that if $\mathcal{M}$ is a g-model for $S_\omega I$ and the term U^τ is free for the variable X^τ in $A(X^\tau)$, then for every $i \in I$ and $f \in As(\mathcal{M})$, it results that

$$\mathcal{M}; i;\ f, \overline{f}(C^\tau)\ \text{sat}\ A(X^\tau) \quad \text{iff} \quad \mathcal{M}, i, f\ \text{sat}\ A(C^\tau)$$

where the terminology has an obvious meaning ([21]).[12] The generalized completeness theorem for $S_\omega I$ is the converse of the above result; it is sufficient to prove that

Σ is consistent if and only if Σ is g-satisfiable. The implication from right to left is straightforward; we consider then only the implication from left to right.

To begin with, let us assume that the consistent set Σ omits infinitely many variables of each type, that is, there are infinitely many variables of each type which do not occur in any formula of Σ. Then there exists a sequence $\overline{\Sigma} = (\overline{\Sigma}_i)_{i\in\omega}$ of sets of formulas such that

(i) $\Sigma \subseteq \overline{\Sigma}_0$;

(ii) for each $i \in \omega$, $\overline{\Sigma}_i$ is a maximally consistent set of formulas in $S_\omega I$;

(iii) for each $i \in \omega$ and each formula $B(X^\tau)$, $\exists X^\tau B(X^\tau) \in \overline{\Sigma}_i$ iff $B(Y^\tau) \in \overline{\Sigma}_i$ for some variable Y^τ which is free for X^τ in $B(X^\tau)$: that is, $\overline{\Sigma}$ is a Henkin Theory;

(iv) for each $i \in \omega$ and each formula B, we have $\Diamond B \in \overline{\Sigma}_i$ iff $B \in \overline{\Sigma}_j$ for some $j \in \omega$;

(v) for each $i \in \omega$ and each formula $B(X^\tau)$, we have $\forall X^\tau B(X^\tau) \in \overline{\Sigma}_i$ iff $B(Y^\tau) \in \overline{\Sigma}_i$ for every variable Y^τ which is free for X^τ in $B(X^\tau)$;

(vi) for each $i \in \omega$ and each formula B, we have $\Box B \in \overline{\Sigma}_i$ iff $B \in \overline{\Sigma}_j$ for every $j \in \omega$.[13]

The g-model relative to which the formulas of Σ are g-satisfiable can be described as follows. First, we consider an equivalence relation on the collection Tr_τ of terms of $S_\omega I$ of type τ such that U^τ is equivalent to V^τ if and only if $\Box(U^\tau = V^\tau) \in \overline{\Sigma}_i$ if $\tau \neq e_1$ and, if $\tau = e_1$, then U^{e_1} is equivalent to V^{e_1} (in this case we write $U^{e_1} \equiv V^{e_1}$) if and only if for every formula F that belongs to $\overline{\Sigma}_i$, it results that $F[U^{e_1}/V^{e_1}]$ also belongs to $\overline{\Sigma}_i$. In other words, U^{e_1} is equivalent to V^{e_1} if and only if U^{e_1} and V^{e_1} can be replaced one by the other in all their occurrences in any predicate in such a way that the resulting formulas are necessarily equivalent. The defined relation does not depend on $i \in \omega$. Then, by recursion on the type τ, we define a set $\mathcal{F}_\tau$ and a mapping ρ_τ from the set of terms of type τ into $\mathcal{F}_\tau$ such that

1. ρ_τ is onto $\mathcal{F}_\tau$,
2. $\rho_\tau(U^\tau) \equiv \rho_\tau(V^\tau)$ iff $U^\tau \simeq V^\tau$.

First, let $\mathcal{F}_{e_i}$ be the quotient set $Tr_{e_i}/\simeq$ $(i = 1, 2)$[14] and $\rho(U^{e_i}) = [U^{e_i}]_\simeq$ (these are the equivalence classes of U^{e_i} by the relation $\simeq$). Then, by supposing that $\mathcal{F}_{\tau_k}$ and ρ_{τ_k} have been defined for $k < n$, we define the mapping ρ_τ from Tr_τ into $[\mathcal{P}(\mathcal{F}_{\tau_0} \times \cdots \times \mathcal{F}_{\tau_{n-1}})]^\omega$, where $\tau = \langle \tau_0, \ldots, \tau_{n-1}\rangle$, as follows:

$$\langle \rho_0(U_0^{\tau_0}), \ldots, \rho_{n-1}(U_{n-1}^{\tau_{n-1}})\rangle \in \rho_\tau(U^\tau)(i)$$

if and only if the formula $U^\tau(U^{\tau_0}, \ldots, U^{\tau_{n-1}})$ belongs to $\overline{\Sigma}_i$. If we let $\mathcal{F}_\tau$ be the range of ρ_τ, then conditions 1 and 2 above are met. The g-model based on $D = m \cup M$ and index set $I = \omega$ is the ordered pair $\mathcal{M} = \langle \mathcal{F}_\tau, \rho\rangle_{\tau\in\Pi}$, where $\rho(C^\tau) = \rho_\tau(C^\tau)$ for every constant C^τ. Then, by induction on the length of the formula A, it results that

$$\mathcal{M}, i, \mu \ \text{ sat } \ A \quad \text{iff} \quad A \in \overline{\Sigma_i}$$

for every $i \in I$, where $\mu \in As(\mathcal{M})$.[15] In the case where $i = 0$ and $\mu = f$, we obtain the desired result.

5.2 Comprehension and other axioms Our logic can be extended to a system which encompasses all the instances of the following *comprehension schema*, where $\tau = \langle \tau_1, \ldots, \tau_n \rangle$ and X^τ is the first variable of type τ which does not occur free in the formula $F(X^{\tau_1}, \ldots, X^{\tau_n})$:

$$\exists X^\tau \Box \forall X^{\tau_1}, \ldots, \forall X^{\tau_n} (X^\tau (X^{\tau_1}, \ldots, X^{\tau_n}) \longleftrightarrow F(X^{\tau_1}, \ldots, X^{\tau_n}))$$

This scheme, which is valid in all the standard models of $S_\omega I$, formalizes the principle that every formula $F(X^{\tau_1}, \ldots, X^{\tau_n})$ with free variables determines a relation-in-intension (a predicate). In considering a g-model $\mathcal{M} = \langle \mathcal{F}_\tau, \rho \rangle_{\tau \in \Pi}$ for $S_\omega I$, if $U^{\tau_1}, \ldots, U^{\tau_n}$ are respectively elements of $\mathcal{F}_{\tau_1}, \ldots, \mathcal{F}_{\tau_n}$, the predicate F being defined by

$$F(i) \stackrel{\text{def}}{=} \{(X^{\tau_1}, \ldots, X^{\tau_n}) : \mathcal{M}; i;\ f, U^{\tau_1}, \ldots, U^{\tau_n} \text{ sat } F(X^{\tau_1}, \ldots, X^{\tau_n})\}$$

for all $i \in I$ and assignment $f \in As(\mathcal{M})$ belongs to $\mathcal{F}_\tau$ (the terminology is that of [21]). Consequently, the g-model is also a g-model for $S_\omega I$ plus the comprehension schema, and the completeness theorem is also true for this extended logic.[16] The principle of *extensional comprehension*, which says that every formula with free variables determines an (extensional) n-ary relation and the axioms of infinity and choice, can be formulated in the language of $S_\omega I$ in exactly the same way as in [21], pp. 77–78; in the same way we can treat the axioms of infinity and choice (see also [4], where these last two axioms are formulated for the logic S_ω).

As occurs in Gallin's system, the principle of extensional comprehension can be proved to be independent of the axiomatic of $S_\omega I$ plus comprehension. That is, there are g-models of $S_\omega I$ plus comprehension in which the extensional comprehension principle fails. We do not present these g-models here, but due to the peculiarities of our logic (mainly regarding the failure of the general principle of identity), we guess that perhaps they are related to Takeuti's quantum set theoretical models [36], which are built starting from a complete orthomodular lattice instead of a complete Boolean algebra as usual (for this last case, see [21], chap. 4). As remarked in Dalla Chiara [8], in Takeuti's *ortho-valued* model the identity relation is *non-Leibnizian*, in the sense that the substitutivity law of identity fails. The analogies between this case and Schrödinger logics seem evident and of course deserve further attention. In future works we intend to investigate this question.

6 Concluding remarks and the classical interpretation of quasi sets At the end of the last section, we mentioned some points to be developed in connection with our logic. Here we comment on other topics we think are of interest, but once more they are referred to without details. First, taking into account Gallin's characterization of the algebra of "propositions"[17] of the g-models of his system $\mathrm{ML_p}$ plus comprehension as a subalgebra of the Boolean algebra of all subsets of I, it would be interesting to analyze the modifications to be made in the axioms and in the interpretations of what is to be considered a "proposition" (in such a way that they play the role of 'propositions' in quantum mechanics), such that the algebra be an orthomodular lattice, as occurs in QM, instead of a Boolean algebra. In this way, perhaps we can characterize a quantum logic more from a distinct point of view than the usual ones ([8], Mittelstaed [29]). We have remarked in the last section that as it occurs in the classical case (Gallin's system), the extensional comprehension fails to hold in $S_\omega I$, but in this case something stronger than in $\mathrm{ML_p}$ can be inferred. In fact, the fundamental point is not only the existence of g-models in which the extensional comprehension fails. In $S_\omega I$ we have more: there are, in fact, predicates which do not define (precise) extensions (hence relations), such as those of type $\tau = \langle \tau_1, \ldots, \tau_n \rangle$ in which at least one of the τ_i is obtained recursively from e_1. So, a formula with free variables which defines such an "ambiguous" predicate is also "ambiguous" in some sense, that is, it does not define a *precise* and well-defined relation (a well-defined subset of objects of the domain). These relations are, of course, *quasi relations*, in the sense of quasi set theories.[18]

Manin pointed out that quantum mechanics uses as a language a fragment of classical functional analysis, having not its "own language" ([28], pp. 84–85). Schrödinger himself had already felt the necessity of a radically new and different language than the classical one to speak about the fundamental entities of which matter is composed, a language which could join both the particle and the wave aspects of it (see da Costa [6]). In fact, usual quantum logics start from algebraic structures which reflect essentially the properties of operators defined on the closed subspaces of a Hilbert space, and it is not easy to recognize in what sense these formal algebras refer to the very basic ontology of quantum physics. In short, quantum logics apparently are closer to a calculus of statements about the microworld than they are to describing the very underlying logic of elementary particles. In fact, it is well known that the usual formalisms via Hilbert spaces (first-quantized approach) raise a lot of intricate "philosophical puzzles," some of them related to the birth of "surplus formal structures," that is, mathematical structures which correspond to nothing in the real world ([32], [33]). These structures are originated by the wrong implicit supposition that elementary particles are individuatable entities, that is, things which behave as classical physical objects.

The tendency to abandon labels by using (say) the Fock space formalism, which permits us to drop the labels, seems to be more adequate from the philosophical point of view, since in this case we are (apparently) formally describing no more particles but legitimate 'quanta',[19] that is, entities which cannot be strictly counted, arranged in some order, and so on, but that can only be aggregate in certain quantities. As it has been said elsewhere ([26] and also French [17]), this raises other kinds of "puzzles", since despite the fact that a vector in the Fock space is a form of description of states with a certain number of quanta without any reference whatever to "primitive thisness" (Teller [37]), we continue to talk of 'quantum entities', and it is natural to ask for the kind of logic these entities obey.[20] Since there are strong arguments against the classical theory of identity in what concerns these entities (for instance, all the arguments pointed out in recent literature which aim at showing that Leibniz's Law is not true in the quantum world ([14], [15]), we are tempted to guess that Schrödinger logics can perhaps be useful in formalizing some aspects of the behavior of elementary particles. Then, systems which combine such "deviations" from the traditional theory of identity (as it seems reasonable to admit in connection with elementary particles) with the possibility of realizing intensional concepts, a fortiori might also appear as legitimate 'quantum logics' in some sense. Further analysis may show in what sense systems of this kind should be modified in order to attain the very objective of quantum logics, as put forward in [8] and [29]. But these questions will also be postponed for future works.

6.1 Classical semantics for quasi set theory We note that it is possible to provide a "classical" interpretation of quasi sets.[21] The idea is the following one. Suppose that m is a set, that R is an equivalence relation on m, and that C_i, $i = 1, 2, \ldots$ are the corresponding equivalence classes. Then, for each $x \in m$, we define $\widehat{x} = \langle x, C_x \rangle$, where C_x is the equivalence class to which x belongs. Let us call $\widehat{m}$ the set of all such ordered pairs. Intuitively speaking, every element of m is associated with an equivalence class and $\widehat{x}$ can be thought of as "an 'individual' and a 'state' in which the individual lies on." The idea of identifying an object with an equivalence class to which it belongs is closely related to Weyl's concept of an 'aggregate of individuals' (Weyl [38], app. B; see also Krause [23]).

We may define on $\widehat{m}$ a relation $\sim$ by $\widehat{x} \sim \widehat{y}$ if and only if $C_x = C_y$. It is easy to see that $\sim$ is an equivalence relation. Then, in stating that $\widehat{x} \sim \widehat{y}$, we are "identifying" the objects denoted by x and y by the equivalence class they belong to, or by the "sort" or "state" they are in, but without direct reference to the objects themselves. One may object that this procedure is equivalent to state that xRy in the original set. Of course this is what occurs in *classical* domains. But if we are trying to approach those domains of reality in which the objects have the properties of the m-atoms mentioned in the previous sections, it would be convenient to refer to the classes or states only, and not to the objects directly.[22] The set $\widehat{m}$ may be called *Weyl's aggregate*. Now let us suppose that A is a set such that $A = M \cup \widehat{m}$ where M is a nonempty set and $\widehat{m}$ is as above.[23]

Then a translation from the language of quasi set theory to the language of ZF can be defined without difficulty (the details will be omitted since we have not described in full the language of quasi sets). The fundamental intuitive idea is to interpret the m-atoms as elements of $\widehat{m}$, that is, as ordered pairs as defined above. Then for every x and y in A, we say that x is indistinguishable from y if and only if $\widehat{x} \sim \widehat{y}$ when both x and y belong to m or both x and y belong to M and $x = y$. All other primitive symbols of the quasi set theory are translated in an obvious way and, by using this device, the translations of the axioms of that theory are true in a universe defined on the set A. In other words, we have defined a "classical model" for S^{**}. Hence, if ZF is consistent, so is S^{**}.

This fact might be interesting in the following sense. The usual formalizations of quantum mechanics, like the Copenhagen interpretation, can be formulated in a "classical" way, that is, by using the usual set theory (for instance, *via* the Hilbert-space formalism). But the Hilbert space formalism produces some "philosophical puzzles" and has motivated some authors, such as Redhead and Teller to provide arguments against the first-quantized approach and to suggest that the Fock space formalism is more convenient, as we mentioned. Nevertheless, even in the occupation number formalism one still talks of 'quantum entities' of some sort ([26], [17]) and in reality all these approaches do not lead up to the core of the (philosophical) problem concerning the individuality of elementary particles. In fact, roughly speaking, one may say that there are basically three ways of formalizing quantum mechanics: (1) by means of the usual formulations, as in the Copenhagen interpretation, where the laws of quantum mechanics turn out to be something like the laws of our macroscopic measuring instruments; (2) by means of the classical set theories, where elementary particles are considered as set-theoretical constructs (like the ordered pairs $\widehat{x}$ above); and (3) by using quasi sets, where indistinguishable but not identical entities can be considered "right from the start," as demanded, for instance, by Post [31], expressing a view which is perhaps closely related to the ontology of quantum field theories.

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NOTES

1. With regard to the concepts of physical object and sets, see [10] and [24]; concerning extensions and intensions, see Dalla Chiara [11] and [12], and Mittelstaed [30]; the view of elementary particles as individuatable entities is criticized in [32] and [33], where the authors introduced the term 'individuatable'. In these papers there are additional references.
2. We are concerned here only with showing that interesting logical systems can be developed in connection with a very plausible reading of Schrödinger's ideas (cf. [3], [4], and [6]). For a more detailed exegesis of Schrödinger's thought, see Ben-Menahem [1] and Bitbol [2].
3. In [13] a "comparative" study between these systems was outlined. (Added in proof: The fundamental distinction between quasets and quasi sets, roughly speaking, is that in quaset theory the concept of identity holds for all objects and the axiomatics entails that there is a kind of "epistemic" indeterminacy regarding the elements which belong to a quaset, while in quasi set theory the concept of identity lacks sense for some objects. In this case, there is a kind of "ontic" indeterminacy among (some of) the elements of a quasi set. See [25] for more details. The use of quasi sets to discuss the concept of vague objects was suggested in French [16] and in Krause [27].)
4. Bressan had already shown that certain higher-order modal logics are of interest in connection with the foundations of physics. See [21], p. 6.
5. Dalla Chiara and di Francia noted that, contrary to the case of the macroscopic natural-kind names, in considering particle names the intension is always represented by the conjunction of a finite number of properties. Cf. [10], p. 269. In the last section we shall provide more details about this topic.
6. It is convenient to note that the word 'extensional' is used in this definition in a quite different sense from $S_{\omega}I$ in which this word is connected with 'reference' and 'denotation'.
7. That is, the (weak) equality $=$ is introduced to abbreviate the indistinguishability relation for qsets: $x = y$ iff x and y are indistinguishable qsets. The result is that equal qsets may be extensionaly distinct in the sense that they may be equal without having "the same" elements.
8. All these concepts can be defined in quasi set theory, see [24].
9. As in Montague's approach to intensional logic, we may suppose that I is the Cartesian product $W \times T$ where W is a (quasi) set of possible worlds and T is a totally ordered set of instants of time; see Gochet [18].

10. Then, in particular, $\rho(C^{e_1}) \in m$ and $\rho(C^{e_2}) \in M$.

11. It is important to recall that, as in usual intensional logics, the word '*n*-ary predicate' is used here to mean *relation-in-intension* (cf. [21], p. 67).

12. Similar to [21], p. 74.

13. The existence of such a sequence can be proved by adapting the method presented in [21], p. 75.

14. That is to say, $m = Tr_{e_1}/\simeq$ and $M = Tr_{e_2}/\simeq$.

15. The proof is analogous to that of [21], p. 75.

16. As in [21], pp. 76ff.

17. In the language of our theory, propositions are elements of the set $\mathcal{F}_{\varnothing}5 \subseteq [\mathcal{P}(\{\varnothing\})]^I = 2^I$ in any g-model (also cf. [21], p. 72).

18. Added in proof: In a recent work [27], French and Krause developed a logic of predicates of this kind, termed 'opaque predicates', whose semantics is also founded on quasi set theory.

19. As Redhead and Teller prefer to call the basic entities, in order to avoid the 'interpretive disaster' of thinking of them as 'particles', that is, as individuatable entities (see [32], [33]).

20. Added in proof: The case of the Fock space formalism, in connection to Redhead and Teller's idea that by using it we may avoid the commitment to labels, is criticized in [17].

21. For details, see da Costa [5].

22. It seems to us that the so-called sortal logics (Stevenson [35]) may also have an interpretation in this sense. See also da Costa [7].

23. See [5].

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