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# Arithmetically Saturated Models of Arithmetic

ROMAN KOSSAK and JAMES H. SCHMERL

**Abstract** The paper presents an outline of the general theory of countable arithmetically saturated models of PA and some of its applications. We consider questions concerning the automorphism group of a countable recursively saturated model of PA. We prove new results concerning fixed point sets, open subgroups, and the cofinality of the automorphism group. We also prove that the standard system of a countable arithmetically saturated model of PA is determined by the lattice of its elementary substructures.

***1 Introduction*** Recent work on automorphisms of countable recursively saturated models of PA has revealed the importance of those models in which the standard natural numbers form a strong cut. In Kossak and Schmerl [9] we called such models arithmetically saturated, and Proposition 2.4 below explains why.

The aim of this paper is to outline the basic theory of arithmetically saturated models of PA and to present new results illustrating the special character of these structures. The special character of arithmetically saturated models was first noted in Kossak [6]. It was shown there that arithmetic saturation for models of PA can be characterized in terms of the existence of elementary initial segments with some special properties. Later in Kaye, Kossak, and Kotlarski [3] it was proved that if $\mathcal{M}$ is a countable recursively saturated model of PA then $\mathcal{M}$ is arithmetically saturated iff there exists an automorphism of $\mathcal{M}$ that moves all nondefinable elements. Other results where the assumption of arithmetic saturation, instead recursive saturation, is needed can be found in Kossak, Kotlarski, and Schmerl [8].

One of the important, and somewhat surprising, results is the theorem of Lascar [11], saying that countable arithmetically saturated models PA have the small index property.

In the next three sections of this paper we consider automorphisms and automorphism groups of recursively saturated models of PA. The last section is devoted to lattices of elementary substructures of arithmetically saturated models of PA.

In Section 3 we prove that, if $\mathcal{M}$ is a countable recursively saturated model of PA, then $\mathcal{M}$ is arithmetically saturated if and only if $\mathrm{Aut}(\mathcal{M})$ is finitely generated over each of its open subgroups. Then, as a corollary of the results from Hodges et al. [1] and [11], we show that the cofinality of the automorphism group of a recursively saturated model of PA is uncountable iff the model is arithmetically saturated. It is an interesting open problem whether recursively saturated models of PA that are not arithmetically saturated have the small index property. The above result seems to suggest that they might not.

In Section 4 we consider automorphisms moving all nondefinable elements. We show that if $\mathcal{M}$ is countable and arithmetically saturated model of PA, then there is an automorphism $f$ of $\mathcal{M}$ such that $f(x) > x$, for every $x$ greater than all definable elements of $\mathcal{M}$. This is a strengthening of previously known results, and it leads to some interesting open questions.

Kotlarski has asked in [10] if either the automorphism group or the elementary substructure lattice of a countable recursively saturated model of PA determines the model. We have shown in [9] that this is the case for the automorphism group of arithmetically saturated models for any fixed complete extension of PA. More precisely, if $\mathcal{M}$ and $\mathcal{N}$ are countable arithmetically saturated models of PA, then $\mathrm{Aut}(\mathcal{M}) \cong \mathrm{Aut}(\mathcal{N})$ implies $\mathrm{SSy}(\mathcal{M}) = \mathrm{SSy}(\mathcal{N})$. In the last section we will prove that the same is true for lattices: the standard system of an arithmetically saturated model of PA (of arbitrary cardinality) is determined by the lattice of elementary substructures of the model.

## 2 Preliminaries

Let us fix some notation and terminology. The set of standard natural numbers will be denoted by $\omega$, $\mathbb{N} = (\omega, +, \cdot, 0, 1)$ is the standard model, and TA is $\mathrm{Th}(\mathbb{N})$. The *standard system* of a model $\mathcal{M}$, $\mathrm{SSy}(\mathcal{M})$, is the family of those $X \subseteq \mathcal{M}$ for which there is an $Y$ definable in $\mathcal{M}$ with parameters, such that $X = \omega \cap Y$. $L_{\mathrm{PA}}$ will denote the language of PA.

We will say that a type $p(\bar{v}, \bar{a})$ in variables $\bar{v} = v_1, \ldots, v_n$, and parameters $\bar{a} = a_1, \ldots, a_m \in \mathcal{M}$ is recursive, arithmetic, etc., if the set of Gödel numbers of formulas $\varphi(\bar{v}, \bar{w}) \in p(\bar{v}, \bar{w})$, where $\bar{w} = w_1, \ldots, w_m$, is recursive, arithmetic, etc. In the same sense we will speak of types as being subsets of $\omega$.

The notion of $\mathcal{A}$-saturation was introduced by Wilmers in [16]. Let $\mathcal{A}$ be a family of subsets of $\omega$. We say that a model $\mathcal{M}$ is *$\mathcal{A}$-saturated* if the following two conditions are satisfied: (i) for every $\bar{a} \in [\mathcal{M}]^{<\omega}$, the type of $\bar{a}$, $\mathrm{tp}(\bar{a})$, is in $\mathcal{A}$; (ii) for every type $p(\bar{v}, \bar{a})$ in $\mathcal{A}$, if $p(\bar{v}, \bar{a})$ is realized in some elementary extension of $\mathcal{M}$, then it is realized in $\mathcal{M}$.

A *Scott set* is an $\omega$-model of $\mathrm{WKL}_0$. The standard system of a model of PA is a Scott set; moreover, every countable Scott set is the standard system of a model of PA. If $T$ is a completion of PA, $\mathcal{X}$ is a countable Scott set and $T \in \mathcal{X}$, then there is a recursively saturated countable model $\mathcal{M} \models T$ such that $\mathrm{SSy}(\mathcal{M}) = \mathcal{X}$. Proofs of the above statements can be found in Kaye [2].

The next proposition shows that a recursively saturated model of PA is much more than just recursively saturated.

**Proposition 2.1** (Smoryński [15]) *A model $\mathcal{M} \models \mathrm{PA}$ is recursively saturated iff $\mathcal{M}$ is $\mathrm{SSy}(\mathcal{M})$-saturated.*

We will use a fixed arithmetical coding of finite sequences. If $\mathcal{M}$ is a model of PA and $a, i \in \mathcal{M}$, then $(a)_i$ denotes the $i$-th term of the sequence coded by $a$, and $\mathrm{len}\, a$ is the length of the sequence coded by $a$.

The standard cut $\omega$ is *strong* in $\mathcal{M}$ if for every $a$ in $\mathcal{M}$ there is $c > \omega$ such that for every $i \in \omega$, $(a)_i \in \omega \leftrightarrow (a)_i < c$.

Strong cuts were introduced and studied by Kirby in [4]. In particular Kirby proved the following (see the the first section of [5] for a discussion and references).

**Proposition 2.2** *The standard cut is strong in a model* $\mathcal{M} \models \mathrm{PA}$ *iff* $(\omega, \mathrm{SSy}(\mathcal{M})) \models \mathrm{ACA}_0$.

**Definition 2.3** ([9]) For an arbitrary structure $\mathcal{M}$ in a recursive language $\mathcal{L}$, we say that $\mathcal{M}$ is *arithmetically saturated* if whenever $\Sigma(v, u_0, \dots, u_n)$ is a set of $\mathcal{L}$-formulas which is arithmetic in the type of $(a_0, \dots a_n)$ and $a_0, \dots, a_n \in \mathcal{M}$ are such that $\Sigma(v, \bar{a})$ is realized in some elementary extension of $\mathcal{M}$, then $\Sigma(v, \bar{a})$ is realized in $\mathcal{M}$.

Our main characterization of arithmetic saturation follows directly from Propositions 2.1 and 2.2

**Proposition 2.4** *If $\mathcal{M}$ is a model of* PA*, then the following are equivalent:*

1. *$\mathcal{M}$ is arithmetically saturated;*
2. *$\mathcal{M}$ is recursively saturated and $\omega$ is strong in $\mathcal{M}$;*
3. *$\mathcal{M}$ is recursively saturated and* $(\omega, \mathrm{SSy}(\mathcal{M})) \models \mathrm{ACA}_0$.

From the definition of arithmetic saturation it follows easily that for every model $\mathcal{M}$ there is an arithmetically saturated model $\mathcal{N}$ such that $\mathcal{M} \prec \mathcal{N}$ and $\mathrm{card}\mathcal{M} = \mathrm{card}\mathcal{N}$. Also from remarks preceding Proposition 2.1 and from Proposition 2.4 we have the following corollary.

**Corollary 2.5** *If $(\omega, \mathcal{X})$ is a countable model of* $\mathrm{ACA}_0$*, and $T \in \mathcal{X}$ is a completion of* PA*, then there is a countable arithmetically saturated model $\mathcal{M} \models T$ such that* $\mathrm{SSy}(\mathcal{M}) = \mathcal{X}$.

Corollary 2.5 and well-known facts concerning $\omega$-models of $\mathrm{ACA}_0$ imply that every completion of PA has continuum many nonisomorphic countable arithmetically saturated models.

There are many differences between the class of arithmetically saturated models and the class of recursively saturated models. Now let us just note the following two observations. It is easy to see that every countable arithmetically saturated model of PA has a countable cofinal extension that is not arithmetically saturated; thus, the Smoryński-Stavi theorem on cofinal extensions does not hold when recursive saturation is replaced by arithmetical saturation. If $\mathcal{M}$ is a cofinal extension of an arithmetically saturated model, then $\mathcal{M}$ realizes all arithmetic pure types consistent with $\mathrm{Th}(\mathcal{M})$ (in fact all consistent with $\mathrm{Th}(\mathcal{M})$ types with parameters in $\mathcal{M}$). This, together with the previous remark shows that in Definition 2.3, we need to assume that $\Sigma(u, u_0, \dots, u_n)$ is arithmetic in a type of a tuple of elements of $\mathcal{M}$, rather than just arithmetic; compare this statement with Proposition 2.1. These deficiencies are compensated for by many structural properties enjoyed only by arithmetically saturated models. Theorem 2.6 below presents a list of such properties.

If $\mathcal{M}$ is a model of PA and $\bar{a} \in [M]^{<\omega}$, then $\mathrm{K}(\mathcal{M}; \bar{a})$ is the Skolem closure of $\bar{a}$ in $\mathcal{M}$. In particular, $\mathrm{K}(\mathcal{M}; 0)$ is the set of definable elements of $\mathcal{M}$. $\mathcal{M}(\bar{a})$ will denote the smallest initial segment of $\mathcal{M}$ containing $\mathrm{K}(\mathcal{M}; \bar{a})$.

$\mathrm{Aut}(\mathcal{M})$ is the automorphism group of $\mathcal{M}$, and for an automorphism $f$, $\mathrm{fix}(f)$ is the set of fixed points of $f$.

A submodel $K \prec \mathcal{M} \models \mathrm{PA}$ is *small* (cf. [11]) if, for some $a \in \mathcal{M}$, $K = \{(a)_n : n \in \omega\}$. A straightforward argument shows that if $\mathcal{M} \models \mathrm{PA}$ is recursively saturated then, for every $\bar{a} \in [M]^{<\omega}$, $\mathrm{K}(\mathcal{M}; \bar{a})$ is a small submodel of $\mathcal{M}$. One can also show that every such $\mathcal{M}$ has small submodels that are not finitely generated.

A subgroup $H$ of the automorphism group of a model $\mathcal{M}$ is called *basic open* if it is a pointwise stabilizer of a finite subset of $\mathcal{M}$, and $H$ is *open* if it contains a basic open subgroup.

For every $f \in \mathrm{Aut}(\mathcal{M})$, $(\mathrm{fix}(f), +, \cdot, 0, 1)$ is an elementary substructure of $\mathcal{M}$. In the next theorem and in the following discussion we will identify $\mathrm{fix}(f)$, with $(\mathrm{fix}(f), +, \cdot, 0, 1)$.

**Theorem 2.6** *Let $\mathcal{M}$ be a countable recursively saturated model of* PA, *and let $G = \mathrm{Aut}(\mathcal{M})$. Then the following are equivalent:*

1. *$\mathcal{M}$ is arithmetically saturated;*
2. *There is $f \in G$ such that $\mathrm{fix}(f) = K(\mathcal{M}; 0)$;*
3. *There are $f \in G$ and a small $K \prec \mathcal{M}$ such that $\mathrm{fix}(f) = K$;*
4. *For every small $K \prec \mathcal{M}$ there is $f \in G$ such that $\mathrm{fix}(f) = K$;*
5. *There is $f \in G$ such that $\mathrm{fix}(f)$ is not isomorphic to $\mathcal{M}$;*
6. *There are $f, g \in G$ such that $\mathrm{fix}(f)$ is not isomorphic to $\mathrm{fix}(g)$;*
7. *There if $f \in G$ such that $\mathrm{fix}(f) \subseteq \mathcal{M}(0)$;*
8. *There exist $g \in G$ and an open subgroup $H < G$ such that for every $f \in G$, $f^{-1}gf \notin H$.*

Proofs of (1) $\leftrightarrow$ (4) and (1) $\leftrightarrow$ (8) are given in [3] (see Corollary 5.4. and Theorem 5.7. there). Obviously (2) implies (5). All other implications follow from the next proposition. The role of equivalence (1) $\leftrightarrow$ (8) will be discussed in the next section.

**Proposition 2.7** ([3]) *If $\mathcal{M}$ is countable recursively saturated model of* PA *and $\mathcal{M}$ is not arithmetically saturated, then, for every $f \in \mathrm{Aut}(\mathcal{M})$, $\mathrm{fix}(f)$ is isomorphic to $\mathcal{M}$.*

Proposition 2.7 was not stated explicitly in [3], but from a slight modification of the proof of Proposition 5.2 (ii) of [3] it follows that, under the assumptions of Proposition 2.7, $\mathrm{fix}(f)$ is recursively saturated and $\mathrm{SSy}(\mathrm{fix}(f)) = SSy(\mathcal{M})$, proving that $\mathrm{fix}(f)$ is isomorphic to $\mathcal{M}$.

In Sections 3 and 4 we will add further properties to the list in Theorem 2.6.

Theorem 2.6 suggests the following question: For $\mathcal{M} \models \mathrm{PA}$ countable and arithmetically saturated, what is the set $I(\mathcal{M})$ of isomorphism types of structures of the form $\mathrm{fix}(f)$, $f \in \mathrm{Aut}(\mathcal{M})$? We know that if $\mathcal{M}$ is recursively saturated but not arithmetically saturated than $I(\mathcal{M})$ consists of one element: the isomorphism type of $\mathcal{M}$.

**Problem 2.8** Let $\mathcal{M}$ be a countable arithmetically saturated model of PA. What is the cardinality of $I(\mathcal{M})$?

Let us note that, under the assumptions of the problem, not every $\mathcal{N} \prec \mathcal{M}$ is of the form $\mathrm{fix}(f)$. Since every countable model has only countably many small substructures, the result will follow from the next two propositions.

**Proposition 2.9** *If $\mathcal{M} \models$ PA is recursively saturated and $\mathcal{N} \prec \mathcal{M}$ is small in $\mathcal{M}$, then, for every $f \in \mathrm{Aut}(\mathcal{M})$, $\mathrm{fix}(f) \cap \mathcal{N}$ is small.*

*Proof:* Let $\mathcal{N} = \{(c)_i : i \in \omega\}$, for some $c \in \mathcal{M}$. Let $f \in \mathrm{Aut}(\mathcal{M})$ be given and let $c' = f(c)$. Consider the type

$$\{(v)_i = (c)_i : \ i \in \omega \text{ and } (c)_i = (c')_i\} \cup \{(v)_i = 0 : \ i \in \omega \text{ and } (c)_i \neq (c')_i\}.$$

The type is finitely realizable in $\mathcal{M}$, and it is recursive in $\mathrm{tp}(c, c')$. Hence, it is realized in $\mathcal{M}$. If $b$ realizes the type in $\mathcal{M}$, then $\mathrm{fix}(f) \cap \mathcal{N} = \{(b)_i : \ i \in \omega\}$; hence, $\mathrm{fix}(f) \cap \mathcal{N}$ is small in $\mathcal{M}$. □

**Proposition 2.10** *If $\mathcal{M} \models$ PA is recursively saturated, then there exists a small $\mathcal{N} \prec \mathcal{M}$ such that $\{K : \ K \prec \mathcal{N}\}$ is uncountable.*

*Proof (Sketch):* Without loss of generality we can assume that $\mathcal{M}$ is countable. Let $S$ be a partial inductive satisfaction class for $\mathcal{M}$ such that $(\mathcal{M}, S)$ is recursively saturated. Let $\mathcal{N}$ consists of the points definable in $(\mathcal{M}, S)$. Then $\mathcal{N}$ is small in $\mathcal{M}$, and $\mathcal{N}$ is recursively saturated; hence $\mathcal{N}$ has $2^{\aleph_0}$ elementary submodels. □

A major result concerning arithmetically saturated models of PA is due to Lascar:

**Theorem 2.11** ([11]) *If $\mathcal{M}$ is a countable arithmetically saturated model of* PA *and $G = \mathrm{Aut}(\mathcal{M})$, then, for every $H < G$, the index of $H$ in $G$ is countable iff $H$ is open.*

The property stated in Theorem 2.11, known as the *small index property*, can be used to reduce problems concerning the automorphism group of a model to a priori easier problems concerning automorphism groups equipped with the topology whose basic open subgroups are the stabilizers of finite subsets of the model. Lascar's theorem was used in [9] in proving that the isomorphism type of a countable arithmetically saturated model of PA is determined uniquely by its complete theory and its automorphism group.

The "arithmetical" part Lascar's proof of Theorem 2.11 is Lemma 2.13 below. We will give a short proof of the lemma to illustrate the power of the concept of arithmetic saturation. Our proof, although essentially the same as Lascar's, is much shorter, as we are using arithmetic saturation directly rather than various coding techniques based on part (2) of Proposition 2.4. We need one more definition.

**Definition 2.12** ([11]) If $K \prec \mathcal{M} \models$ PA and $f \in \mathrm{Aut}(K)$, then $f$ is *existentially closed* if $f = g \restriction K$ for some $g \in \mathrm{Aut}(\mathcal{M})$ and for every formula $\varphi(x, y)$ with parameters in $K$ and for every $h \in Aut(\mathcal{M})$, if $h$ extends $f$ and, for some $x \in \mathcal{M}$, $\mathcal{M} \models \varphi(x, h(x))$, then, for some $x \in K$, $\mathcal{M} \models \varphi(x, f(x))$.

**Lemma 2.13** ([11]) *Let $\mathcal{M}$ be a countable arithmetically saturated model of* PA*. Suppose $c, d \in \mathcal{M}$ are such that $\mathrm{tp}(c) = \mathrm{tp}(d)$. Then there is a small $K \prec \mathcal{M}$ and an $f \in \mathrm{Aut}(K)$ such that: $f(c) = d$, $c, d \in K$, and $f$ is an existentially closed automorphism of $K$.*

*Proof:* Our task is to find $a, b \in \mathcal{M}$ coding sequences of nonstandard length, and such that the following conditions are satisfied:

1. $(a)_0 = c$, $(b)_0 = d$, and $\mathrm{tp}(a) = \mathrm{tp}(b)$;

2. $\{(a)_i : i \in \omega\} = \{(b)_i : i \in \omega\}$;
3. $K = \{(a)_i : i \in \omega\} \prec \mathcal{M}$, and $f$, defined by $f((a)_i) = (b)_i$, is an existentially closed automorphism of $K$.

Notice that conditions (1) and (2) imply that $f$, as defined in (3), can be extended to an automorphism of $\mathcal{M}$.

We will complete our task by first defining a sequence of finite approximations of $f$, $f_0 \subset f_1 \subset f_2 \ldots$ such that $f_n = ((a)_0, \ldots, (a)_n) \mapsto ((b)_0, \ldots, (b)_n)$, and then by observing that the approximations yield the required $a$ and $b$ by arithmetic saturation. The details will be left to the reader. In the descripition of the $f_n$ we have to require that: $\mathrm{tp}((a)_0, \ldots, (a)_n) = \mathrm{tp}((b)_0, \ldots, (b)_n)$ and that for some recursive increasing sequence $n \mapsto k_n$ we have

$$\{(a)_i : i < k_n\} = \{(b)_i : i < k_n\}.$$

This will guarantee (1) and (2). To guarantee (3) we need a sequence of formulas $\varphi_n(\bar{w}, x, y)$ in which every formula of $L_{\mathrm{PA}}$ occurs infinitely often and such that for some increasing sequence $n \mapsto l_n$ if there are $x, y \in \mathcal{M}$ such that

$$\mathrm{tp}((a)_0, \ldots, (a)_{l_n}, x) = \mathrm{tp}((b)_0, \ldots, (b)_{l_n}, y)$$

and $\mathcal{M} \models \varphi_n((a)_0, \ldots, (a)_{l_n}, x, y)$ then

$$\mathcal{M} \models \varphi_n((a)_0, \ldots, (a)_{l_n}, (a)_{l_n+1}, (b)_{l_n+1}).$$

Let us now consider the type $\Sigma(v, w, c, d)$ expressing that the map $(v)_i \mapsto (w)_i :$ $i \leq n$ has the properies of $f_n$ described above. The definition of this type depends on the sequence $l_n : n \in \omega$. Now to finish the proof we must select a sequence $l_n$ that will guarantee (3) and the consistency of $\Sigma(v, w, c, d)$. But since these conditions are arithmetic in the type of $c$ and $d$, the sequence $l_n$ can be chosen to be arithmetic in $\mathrm{tp}(c, d)$ and the result follows. □

**3 *Open subgroups of the automorphism group*** Equivalence (1) ↔ (8) of Theorem 2.6 characterizes arithmetic saturation in terms of open subgroups of the automorphism group of the model, and it was used in [3] to prove that there is no bicontinuous isomorphism between the automorphism groups of two countable recursively saturated models of PA of which only one is arithmetically saturated. Then, by Lascar's theorem, it follows that these groups cannot be isomorphic as abstract groups (see [11]). In this section we will exhibit another characterization of arithmetic saturation in terms of open subgroups. Roughly speaking it says that $\mathcal{M}$ is arithmetically saturated iff every open subgroup of $\mathrm{Aut}(\mathcal{M})$ is "large" in the sense that it generates the whole group together with just one additional automorphism.

For the rest of this section let $G = \mathrm{Aut}(\mathcal{M})$. For $a \in \mathcal{M}$, $G_{(a)} = \{f \in G : f(a) = a\}$ is the stabilizer of $a$.

**Definition 3.1** Let $a$ and $b$ be elements of $\mathcal{M} \models \mathrm{PA}$. We will say that $*(a, b)$ holds in $\mathcal{M}$ if for every formula $\varphi(x, y)$ of $L_{\mathrm{PA}}$ if $\mathcal{M} \models \varphi(a, b)$ then $\mathcal{M} \models \varphi(a, c)$ for some $c \in \mathrm{K}(\mathcal{M}; 0)$.

In the standard model theoretic terminology $*(a, b)$ means that $\mathrm{tp}(a, b)$ is an heir of $\mathrm{tp}(a)$ over $\mathrm{K}(\mathcal{M}; 0)$.

The next lemma is due to Lascar (personal communication); the proof is ours.

**Lemma 3.2** *If $\mathcal{M} \models \mathrm{PA}$ is arithmetically saturated, then for all $a, b \in \mathcal{M}$ there is $b' \in \mathcal{M}$ such that $\mathrm{tp}(b') = tp(b)$ and $*(a, b')$.*

*Proof:* Let $p(v) = \mathrm{tp}(b)$. Consider the type $\Sigma(a, v)$:

$$p(v) \cup \{\varphi(a, v) : \forall k \in \mathrm{K}(\mathcal{M}; 0)\ \mathcal{M} \models \varphi(a, k)\}.$$

Clearly, $\Sigma(a, v)$ is arithmetic in $p(v)$ and is consistent. It is easy to verify that if $b'$ realizes $\Sigma(a, v)$ in $\mathcal{M}$ then $*(a, b')$. □

**Lemma 3.3** *If $\mathcal{M} \models \mathrm{PA}$ is countable and recursively saturated and $a, b \in \mathcal{M}$ satisfy $*(a, b)$, then $G_{(a)} \cup G_{(b)}$ generates $G$.*

*Proof:* Consider some $f \in G$ and suppose that $f(a) = c$. Consider the recursive set of formulas $\Gamma(a, b, c, y)$ expressing that the pairs $(a, b)$, $(a, y)$, and $(c, y)$ each realize the same type. To see that $\Gamma(a, b, c, y)$ is consistent, consider the sentence $\varphi(a, b)$ for which $\mathcal{M} \models \varphi(a, b)$. Since $*(a, b)$ holds, there is a constant term $d$ such that $\mathcal{M} \models \varphi(a, d)$. But since $a$ and $c$ realize the same type, it follows that $\mathcal{M} \models \varphi(c, d)$. This proves consistency of $\Gamma(a, b, c, y)$.

Now let $d \in \mathcal{M}$ realize $\Gamma(a, b, c, y)$. Let $\alpha \in G_{(a)}$ be such that $\alpha(b) = d$, and let $h \in G_{(d)}$ be such that $h(a) = c$. Then $\alpha^{-1}h\alpha \in G_{(b)}$, so that $h = \alpha\beta\alpha^{-1}$ for some $\beta \in G_{(b)}$. Then $h^{-1} f \in G_{(a)}$, so that $f = h\gamma = \alpha\beta\alpha^{-1}\gamma$ for some $\gamma \in G_{(a)}$. Therefore, $f$ is in the group generated by $G_{(a)} \cup G_{(b)}$. □

**Lemma 3.4** *Let $\mathcal{M}$ be a recursively saturated model of* PA. *Then for every $a \in \mathcal{M}$ and every $f \in G$ there is nonstandard $d \in \mathcal{M}$ such that for all $i \in \omega$ if $(a)_i < d$, then $f((a)_i) = (a)_i$.*

*Proof:* Let $a' = f(a)$. For every $n < \omega$

$$\mathcal{M} \models \forall i < n\ (a')_i \neq (a)_i \rightarrow (a)_i > n.$$

Hence for some nonstandard $d$,

$$\mathcal{M} \models \forall i < d\ (a')_i \neq (a)_i \rightarrow (a)_i > d,$$

and the result follows. □

**Theorem 3.5** *Let $\mathcal{M} \models \mathrm{PA}$ be countable and recursively saturated with automorphism group $G = \mathrm{Aut}(\mathcal{M})$. Then $\mathcal{M}$ is arithmetically saturated iff whenever $H < G$ is open, then there is $g \in G$ such that $H \cup \{g\}$ generates $G$.*

*Proof:* Assume $\mathcal{M}$ is arithmetically saturated, and let $H < G$ be open. Without loss of generality we can assume that $H = G_{(a)}$ for some $a \in \mathcal{M}$. Let $b \in \mathcal{M}$ be such that $\mathrm{tp}(a) = \mathrm{tp}(b)$ and $*(a, b)$. Let $g \in G$ be such that $g(b) = a$, and therefore $g^{-1}Hg = G_{(b)}$, and therefore $G$ is generated by $H \cup \{g\}$ by Lemma 3.3.

Next, suppose that $\mathcal{M}$ is not arithmetically saturated. Let $a \in \mathcal{M}$ be a witness to the failure of the standard cut being strong; that is there is no $d \in \mathcal{M}$ such that for $i \in \omega,\ (a)_i \in \omega \leftrightarrow (a)_i < d$.

Let $\langle b_j : j \in \omega \rangle$ be a recursive list of all constant terms. By recursive saturation there is $b \in \mathcal{M}$ such that $(b)_j = b_j$ whenever $j < \omega$. We now claim that we can impose an additional requirement upon $a$: if $(a)_i = (b)_j$ and $i, j < \omega$, then $(b)_j < \omega$. If $a$ does not already have this property, then replace $a$ with an element satisfying the recursive and consistent set of formulas, the $i$-th formula of which asserts the following: $(v)_i$ is the least element not in the set $\{(b)_j : j \leq (a)_i\}$.

We can set $H = G_{(a)}$ and consider arbitrary $g \in G$, intending to show that $H \cup \{g\}$ does not generate $G$. Let $d > \omega$ be such that for all $i < \omega$ if $(a)_i < d$ then $g((a)_i) = (a)_i$ (Lemma 3.4).

Let $i < \omega$ be such that $\omega < (a)_i < d$, and let $c = (a)_i$. Thus $g(c) = c$. By the additional requirement imposed on $a$, $c \notin \{(b)_j : j < \omega\}$, so $c$ realizes a nonprincipal type; therefore $G_{(c)} < G$. But on the other hand, $H = G_{(a)} \leq G_{(c)}$ and $g \in G_{(c)}$ so that $H \cup \{g\}$ does not generate $G$. □

Incidentally, Theorem 3.5 implies the following converse to Lemma 3.2: if $\mathcal{M} \models \mathrm{PA}$ is recursively saturated and if for all $a, b$ in $\mathcal{M}$ such that $\mathrm{tp}(a) = \mathrm{tp}(b)$ there is $b'$ such that $\mathrm{tp}(b') = \mathrm{tp}(b)$ and $*(a, b)$, then $\mathcal{M}$ is arithmetically saturated.

As a corollary of Theorem 3.5 we have a simple proof of the following.

**Corollary 3.6** ([11]) *If $\mathcal{M}_1$ and $\mathcal{M}_2$ are countable recursively saturated models of* PA *and only one of the models is arithmetically saturated, then $G_1 = \mathrm{Aut}(\mathcal{M}_1)$ and $G_2 = \mathrm{Aut}(\mathcal{M}_2)$ are nonisomorphic.*

*Proof:* Suppose $F : G_1 \to G_2$ is an isomorphism and assume that $\mathcal{M}_1$ is arithmetically saturated. We will show that $\mathcal{M}_2$ is arithmetically saturated as well. Let $H < G_2$ be open, and let $H_1 = F^{-1}[H]$. Then $[G_2 : H] \leq \aleph_0$, and therefore $[G_1 : H_1] \leq \aleph_0$. Now since, by Lascar's theorem, $H_1$ is open in $G_1$, by Theorem 3.5, there is $g_1 \in G_1$ such that $H \cup \{g_1\}$ generates $G_1$. Letting $g_2 = F[g_1]$, we see that $H \cup \{g_2\}$ generates $G_2$. Since $H$ was an arbitrarily chosen open subgroup of $G_2$ it follows from Theorem 3.5 that $\mathcal{M}_2$ is arithmetically saturated. □

Theorem 3.5 has also the following corollary.

**Corollary 3.7** *If $\mathcal{M} \models \mathrm{PA}$ is countable arithmetically saturated and $\mathrm{Aut}(\mathcal{M})$ is the union of a chain of proper subgroups, then none of the subgroups in the chain is open.*

For an arbitrary group $H$ which is not finitely generated, the cofinality of $H$, written $c(H)$, is the least cardinal $\lambda$ such that $H$ can be expressed as the union of a chain of $\lambda$ proper subgroups (cf. MacPherson and Neuman [12]). Equipped with Corollary 3.7 and the results of [11], one can repeat the proof of Theorem 6.1 of [1] to show that, for every countable arithmetically saturated model $\mathcal{M}$ of PA, $c(\mathrm{Aut}(\mathcal{M})) > \aleph_0$. Here we will show the converse.

**Theorem 3.8** *If $\mathcal{M} \models \mathrm{PA}$ is countable recursively saturated but not arithmetically saturated, then $\mathrm{Aut}(\mathcal{M})$ is the union of a countable chain of proper basic open subgroups.*

*Proof:* Let $a \in \mathcal{M}$ be the witness to the failure of the standard cut being strong in $\mathcal{M}$, such that for every $f \in \mathrm{Aut}(\mathcal{M})$ there is $i < \omega$ for which $(a)_i$ is nondefinable and $f((a)_i) = (a)_i$. We have defined such an $a$ the proof of Theorem 3.5.

Let us define $\langle c_j : j < \omega \rangle$ as follows:

$$\mathrm{len}(c_j) = \mathrm{card}\{(a)_k : (a)_k \le (a)_j\},$$

where card$X$, for $X$ coded in $\mathcal{M}$ is the cardinality of $X$ in the sense of $\mathcal{M}$, and, for all $l < \mathrm{len}(c_j)$

$$(c_j)_l = \begin{cases} (a)_l & \text{if } (a)_l \le (a)_j \\ 0 & \text{otherwise.} \end{cases}$$

Thus $\langle c_j : j \in \omega \rangle$ is a coded sequence, and, for all $i < \omega$, $c_i < \omega \leftrightarrow (a)_i < \omega$. Also, for $i, j < \omega$, if $c_j > \omega$ and $a_i < a_j$ then, for all $l < \mathrm{card}\{(a)_k : (c_j)_k \le (c_j)_i\} = \mathrm{len}(c_i)$, we have

$$(c_i)_l = \begin{cases} (c_j)_l & \text{if } (c_j)_l \le (c_j)_i \\ 0 & \text{otherwise.} \end{cases}$$

Hence $c_i$ is definable from $c_j$.

It follows that, if $(a)_{k_0} > (a)_{k_1} > \dots$ is a decreasing sequence of terms of the sequence coded by $a$, where all $k_i$'s are standard, then $G_{(c_{k_0})} < G_{(c_{k_1})} < \dots$.

It is easy to see that $\langle c_j : j \in \omega \rangle$ is a witness to the failure of the standard cut being strong in $\mathcal{M}$, having the property of the sequence coded by $a$ mentioned above. Hence, if $\inf_{i<\omega}(c_{k_i}) = \omega$, then for every $f \in \mathrm{Aut}(\mathcal{M})$ there is $i < \omega$ such that $f \in G_{(c_{k_i})}$, and the result follows. □

**Corollary 3.9** *If $\mathcal{M} \models$ PA is countable and recursively saturated, then $\mathcal{M}$ is arithmetically saturated iff $c(\mathrm{Aut}(\mathcal{M}))$ is uncountable.*

Let us note that Corollary 3.6 is an immediate consequence of Corollary 3.9.

We finish this section with another characterization of arithemetic saturation that is an immediate corollary of Theorems 3.5 and 3.8.

**Corollary 3.10** *If $\mathcal{M} \models$ PA is countable and recursively saturated, then $\mathcal{M}$ is arithmetically saturated iff Aut$(\mathcal{M})$ is finitely generated over each of its open subgroups.*

***4 Moving all nondefinable elements*** The equivalence (1) $\leftrightarrow$ (2) of Theorem 2.6 was proved in [3] by a back-and-forth argument that left open the following question: if $\mathcal{M}$ is a countable arithmetically saturated model of TA, is there an $f \in \mathrm{Aut}(\mathcal{M})$ such that, for every nonstandard $x \in \mathcal{M}$, $f(x) < x$? The answer is affirmative, and this section is devoted to the proof of it. We will formulate our result in a more general form; a preliminary version of it has been published in Kossak [7], where it was motivated by an attempt to classify some conjugacy classes in Aut$(\mathcal{M})$. Here we want to add another equivalence to the list in Theorem 2.6.

We will need a strengthening of the condition "$\omega$ is strong in $\mathcal{M}$."

**Proposition 4.1** *Let $\mathcal{M}$ be a recursively saturated model of* PA, *and let $I = \sup\{(a)_n : n \in \omega\}$ for some $a \in \mathcal{M}$ be a cut of $\mathcal{M}$. Then $\mathcal{M}$ is arithmetically saturated iff for every $b \in \mathcal{M}$ there is $d \in \mathcal{M}$ such that for all $i \in \omega$ $(b)_i \in I \leftrightarrow (b)_i < d$.*

*Proof:* Assume $\mathcal{M}$ is arithmetically saturated. For $I = \sup_{n\in\omega}(a)_n$, and $b \in \mathcal{M}$ consider the type $\Gamma(v, a, b)$

$$\{(a)_n < v;\ n \in \omega\} \cup \{v < (b)_n; \forall i \in \omega \mathcal{M} \models (a)_i < (b)_n\}.$$

$\Gamma(v, a, b)$ is arithmetic in $\mathrm{tp}(a, b)$, and it is finitely realizable in $\mathcal{M}$. If $d \in \mathcal{M}$ realizes $\Gamma(v, a, b)$, then, for all $i \in \omega$, $(b)_i \in \omega$ iff $(b)_i < d$.

To prove the converse, observe that without loss of generality we can assume that, for all $n \in \omega$, $(a)_n < (a)_{n+1}$. Then it is easy to see that the condition in the lemma implies that $\omega$ is strong in $\mathcal{M}$. □

The reader might find it helpful to know that the names of elements in the next two lemmas are chosen with a back-and-forth construction in mind. We will be constructing partial finite automorphisms $\bar{a} \mapsto \bar{b}$.

If $\mathcal{M}$ is a model of PA and $\bar{a} \in [M]^{<\omega}$, then $\mathrm{K}^*(\mathcal{M}; \bar{a}) = \mathrm{K}(\mathcal{M}; \bar{a}) \setminus \mathcal{M}(0)$. Recall that $\mathcal{M}(0) = \sup \mathrm{K}(\mathcal{M}; 0)$.

**Lemma 4.2** *Let $\mathcal{M}$ be a recursively saturated model of* PA, $\bar{a}, \bar{b} \in [M]^{<\omega}$, $\mathrm{tp}(\bar{a}) = \mathrm{tp}(\bar{b})$, *and $d \in \mathcal{M}$ be such that $\mathcal{M}(0) < d < K^*(\mathcal{M}; \bar{a})$. Then for every $\alpha > \mathcal{M}(0)$ there is $d' \in \mathcal{M}$ such that $d' < \alpha$ and* $\mathrm{tp}(\bar{a}, d) = \mathrm{tp}(\bar{b}, d')$.

*Proof:* Consider the type $\Delta(v, \bar{a}, \bar{b}, d, \alpha)$:

$$\{\varphi(\bar{a}, d) \leftrightarrow \varphi(\bar{b}, v) : \varphi \in L_{\mathrm{PA}}\} \cup \{v < \alpha\}.$$

If $\Delta(v, \bar{a}, \bar{b}, d, \alpha)$ were inconsistent, then for some $\Phi(\bar{w}, v)$, such that $\mathcal{M} \models \Phi(\bar{a}, d)$, we would have $\mu(\bar{b}) = \min\{v : \Phi(\bar{b}, v)\} \geq \alpha > \mathcal{M}(0)$, hence $\mathcal{M}(0) < \mu(\bar{a}) \leq d$, a contradiction. □

**Lemma 4.3** *If $\mathcal{M}$ is a recursively saturated model of* PA $\bar{a}, \bar{b} \in [M]^{<\omega}$, $\mathrm{tp}(\bar{a}) = \mathrm{tp}(\bar{b})$, *and $c', d, d'$ are such that*

$$\mathcal{M}(0) < d < K^*(\mathcal{M}; \bar{a}) \text{ and } \mathcal{M}(0) < d' < K^*(\mathcal{M}; \bar{b}, c'),$$

*then there is $c \in \mathcal{M}$ such that* $\mathrm{tp}(\bar{a}, c) = \mathrm{tp}(\bar{b}, c')$ *and $d < K^*(\mathcal{M}; \bar{a}, c)$.*

*Proof:* By Lemma 4.2 we can assume that $\mathrm{tp}(\bar{a}, d) = \mathrm{tp}(\bar{b}, d')$. Then notice that any $c \in \mathcal{M}$ such that $\mathrm{tp}(\bar{a}, c, d) = \mathrm{tp}(\bar{b}, c', d')$ satisfies the requirements of the lemma. □

Now everything is prepared for the main theorem of this section.

**Theorem 4.4** *A countable recursively saturated model $\mathcal{M} \models$* PA *is arithmetically saturated iff there is an automorphism $f$ of $\mathcal{M}$ such that, for every $x \in \mathcal{M} \setminus \mathcal{M}(0)$, $f(x) < x$.*

*Proof:* If there is such an $f$ then $\mathcal{M}$ must be arithmetically saturated by Theorem 2.6. To prove the converse, let us assume that $\mathcal{M}$ is arithmetically saturated and let us fix an enumeration of $\mathcal{M} \setminus \mathcal{M}(0)$. Also, let $\langle \alpha_n : n \in \omega \rangle$ be a decreasing sequence of elements of $\mathcal{M}$ such that $\inf_{n\in\omega} \alpha_n = \mathcal{M}(0)$. At the $n$-th stage of the construction of $f$ we will have $\bar{a} = (a_0, \ldots, a_{2n+1})$, $\bar{b} = (b_0, \ldots, b_{2n+1})$, $\bar{d} =$

$(d_0, \ldots, d_n)$, $\bar{d}^+ = (d_1, \ldots, d_{n+1})$ such that the following inductive assumptions hold:

$$\begin{aligned} \mathrm{tp}(\bar{a}, \bar{d}) &= \mathrm{tp}(\bar{b}, \bar{d}^+); \\ \mathcal{M}(0) < d_{n+1} &< \mathrm{K}^*(\mathcal{M}; \bar{a}, \bar{d}); \\ d_{n+1} &< \alpha_{n+1}. \end{aligned}$$

Assume that we have $\bar{a}, \bar{b}, \bar{d}, d_{n+1}$ as above. Let $a = a_{2n+2}$ be the first element in the enumeration of $\mathcal{M} \setminus \mathcal{M}(0)$ not in $\bar{a}, \bar{d}$. Using Lemma 4.2 we can find $d_{n+2}$ such that $\mathcal{M}(0) < d_{n+2} < \mathrm{K}^*(\mathcal{M}; \bar{a}, a, \bar{d}, d_{n+1})$, $d_{n+2} < \alpha_{n+2}$ and

$$\mathrm{tp}(\bar{a}, \bar{d}, d_{n+1}) = \mathrm{tp}(\bar{b}, \bar{d}^+, d_{n+2}).$$

Now, let $b = b_{2n+2} \in \mathcal{M}$ be such that

$$\mathrm{tp}(\bar{a}, a, \bar{d}, d_{n+1}) = \mathrm{tp}(\bar{b}, b, \bar{d}^+, d_{n+2}).$$

This finishes the "forth" step.

To do the "back" step first take $b' = b_{2n+3}$ to be the first element in the enumeration of $\mathcal{M} \setminus \mathcal{M}(0)$ not among $\bar{b}, \bar{d}^+, d_{n+2}$, and then use Lemma 4.3 to find $a' = a_{2n+3}$ such that

$$\begin{aligned} \mathrm{tp}(\bar{a}, a, a', \bar{d}, d_{n+1}) &= \mathrm{tp}(\bar{b}, b, b', \bar{d}^+, d_{n+2}), \\ d_{n+2} &< \mathrm{K}^*(\mathcal{M}; \bar{a}, a, a', \bar{d}, d_{n+1}). \end{aligned}$$

To initiate the construction start with an arbitrary nonstandard $d_0 < \alpha_0$ and $d_1$ such that $\mathcal{M}(0) < d_1 < \mathrm{K}^*(\mathcal{M}; d_0)$ and $d_1 < \alpha_1$. Notice that we are using arithmetic saturation, as we need Lemma 4.1, where $I = \mathcal{M}(0)$, and $b$ codes $\mathrm{K}(\mathcal{M}; \bar{a})$, to make sure that the assumptions of Lemmas 4.2 and 4.3 are satisfied at every stage of the construction.

Now we can define $f$ by letting $f(a_i) = b_i$ and $f(d_i) = d_{i+1}$, for $i \in \omega$. Then, $f$ determines an automorphism of $\mathcal{M}$ and, for every $x \leq d_0$, such that $\mathcal{M}(0) < x$, $f(x) < x$. To complete the proof notice that we must also have $f(x) < x$ for every $x$ such that $\mathcal{M}(0) < x < f^{-n}(d_0) : n \in \omega$. But since $\mathrm{K}(\mathcal{M}; d_0) < f^{-1}(d_0)$, the model $K = \sup_{n\in\omega} f^{-n}(d_0)$ is recursively saturated and it is elementary in $\mathcal{M}$. Hence, it is isomorphic to $\mathcal{M}$ and the result follows. □

Regarding Theorem 4.4 let us note the following. If $\mathcal{M} \models \mathrm{PA}$ is a model with nonstandard definable elements, then for every $f \in \mathrm{Aut}(\mathcal{M})$, such that, for some $x_1 < a \in \mathrm{K}(\mathcal{M}; 0)$, $f(x_1) < x_1$, we have: for $x_2 = a - x_1$, $f(x_2) = a - f(x_1) > x_2$. But still a problem remains open. For a model $\mathcal{M} \models \mathrm{PA}$ define the equivalence relation $R$ by: $R(x, y)$ iff there are no definable $z$ such that $x < z < y$ or $y < z < x$. Let $\Omega(x)$ be the equivalence class of $x$.

**Problem 4.5** Suppose that $\mathcal{M} \models \mathrm{PA}$ is countable and arithmetically saturated. Is there an $f \in \mathrm{Aut}(\mathcal{M})$ such that for every $a \in \mathcal{M} \setminus \mathrm{K}(\mathcal{M}; 0)$ either, for every $x \in \Omega(a)$, $f(x) < x$ or for every $x \in \Omega(a)$ $f(x) > x$?

The following generalization of Theorem 4.4 was proved in [7]. If $I$ is an elementary initial segment of a countable recursively saturated model $\mathcal{M} \models \mathrm{PA}$, then there is $f \in \mathrm{Aut}(\mathcal{M})$ such that $\mathrm{fix}(f) = I$, and for all $x > I$, $f(x) < x$ iff $I$ is strong in $\mathcal{M}$. The proof is a slightly more elaborate version of the proof of Theorem 4.4 here.

***5 Lattices of elementary submodels*** For a model $\mathcal{M} \models \mathrm{PA}$ let $\mathrm{Lt}(\mathcal{M})$ be the lattice of elementary substructures. Kotlarski [10] asks if $\mathrm{Lt}(\mathcal{M})$ determines $\mathcal{M}$ when $\mathcal{M}$ is countable and recursively saturated. The following theorem gives a partial answer to this in the case of arithmetically saturated models.

**Theorem 5.1** *If $\mathcal{M}$, $\mathcal{N}$ are arithmetically saturated models of* PA *and $Lt(\mathcal{M}) \cong Lt(\mathcal{N})$, then* $\mathrm{SSy}(\mathcal{M}) = \mathrm{SSy}(\mathcal{N})$.

The theorem will follow from a series of lemmas. For each $X \subseteq \omega$ we are going to define a countably infinite, distributive lattice $\mathcal{D}(X) = (D(X), \wedge, \vee)$. First we will show in Lemma 5.2 that if an arithmetically saturated model $\mathcal{M}$ of PA has an element $b$ such that $\mathrm{Lt}(\mathrm{K}(\mathcal{M}; b))$ is isomorphic to $\mathcal{D}(X)$, then $X \in \mathrm{SSy}(\mathcal{M})$. The proof of this seems to require the full strength of arithmetic saturation and we do not now whether the result is true for arbitrary recursively saturated models. We will give this proof in detail. The rest of the paper is devoted to an outline of the proof showing that if $\mathcal{M} \models \mathrm{PA}$ is recursively saturated and $X \in \mathrm{SSy}(\mathcal{M})$, then there is $b \in \mathcal{M}$ such that $\mathcal{D}(X) \cong \mathrm{Lt}(\mathrm{K}(\mathcal{M}; b))$. The proof is based on techniques of constructing models of PA with prescribed elementary substructure lattices, that were developed in Schmerl [13],[14]. Our task will be to show that for given $X \subseteq \omega$ the type of the element $b$ above can be constructed effectively in $X$. Our arguments in this part of the proof will be more sketchy, and the reader is adviced to consult [13] or [14] first.

We will arrange for $D(X) = \omega \cup \{\infty\}$. The lattice will have minimum and maximum elements 0 and 1 respectively. We first define some finite lattices. If $Y \subseteq n < \omega$, we define a finite distributive lattice $\mathcal{D}(Y, n) = (D(Y, n), \wedge, \vee)$ so that $D(Y, n) = 2 + m + |Y|$, and 0 and 1 are the minimum and maximum element respectively. We will use $\trianglelefteq$ to denote the partial order of a lattice: $x \trianglelefteq y \leftrightarrow x \vee y = y$.

The following is the definition, by recursion, of $\mathcal{D}(Y, n)$ for $Y \subseteq n < \omega$.

- If $n = 0$ and $Y = \varnothing$, then $D(\varnothing, 0) = \{0, 1\}$ and $0 \triangleleft 1$.
- Suppose $n \notin Y \subseteq n + 1$. Then let $\mathcal{D}(Y, n + 1) = (D(Y, n + 1), \wedge, \vee)$ be such that $\mathcal{D}(Y, n) \subseteq \mathcal{D}(Y, n + 1)$, and if $m = D(Y, n)$, then $D(Y, n + 1) = m + 1$ and $x \triangleleft m \triangleleft 1$ whenever $1 \neq x < m$.
- Suppose $n \in Y \subseteq n + 1$. Then let $\mathcal{D}(Y, n + 1) = (D(Y, n + 1), \wedge, \vee)$ be such that $\mathcal{D}(Y \cap n, n) \subseteq \mathcal{D}(Y, n + 1)$, and if $m = D(Y \cap n, n)$, then $D(Y, n + 1) = m + 3$, and $x \triangleleft m \triangleleft m + 2 \triangleleft 1$ and $x \triangleleft m + 1 \triangleleft m + 2$ whenever $1 \neq x < m$, and $m \vee (m + 1) = m + 2$.

For $Y \subseteq n < \omega$, let $\mathcal{D}'(Y, n) = (D'(Y, n), \wedge, \vee)$ be the lattice where $D'(Y, n) = D(Y, n) \cup \{\infty\}$, $\mathcal{D}(Y, n) \subseteq \mathcal{D}'(Y, n)$ and $x \triangleleft \infty \triangleleft 1$ whenever $1 \neq x \in D(Y, n)$. For illustrative purposes lattices $\mathcal{D}(Y, n)$ and $\mathcal{D}'(Y, n)$, for $Y = \{2, 4, 5\}$ and $n = 7$, are presented in Figure 1.

Clearly, if $X \subseteq \omega$ then $\mathcal{D}(X \cap 0, 0) \subseteq \mathcal{D}(X \cap 1, 1) \subseteq \mathcal{D}(X \cap 2, 2) \subseteq \dots$ and $\mathcal{D}'(X \cap 0, 0) \subseteq \mathcal{D}'(X \cap 1, 1) \subseteq \mathcal{D}'(X \cap 2, 2) \subseteq \dots$.

Let $\mathcal{D}(X) = \bigcup_{n<\omega} \mathcal{D}'(X \cap n, n)$.

**Lemma 5.2** *Suppose $\mathcal{M} \models$* PA *is arithmetically saturated and $Lt(\mathcal{M})$ has an ideal isomorphic to $\mathcal{D}(X)$. Then $X \in$* $\mathrm{SSy}(\mathcal{M})$.

*Proof:* Notice that, since $\mathcal{D}(X)$ is isomorphic to an ideal of $\mathrm{Lt}(\mathcal{M})$, $\mathcal{D}(X) \cong \mathrm{Lt}(\mathcal{N})$, for some $\mathcal{N} \prec \mathcal{M}$. Now, because $\mathcal{D}(X) \setminus \{1\}$ has a maximum element $\infty$, $\mathcal{N}$ must

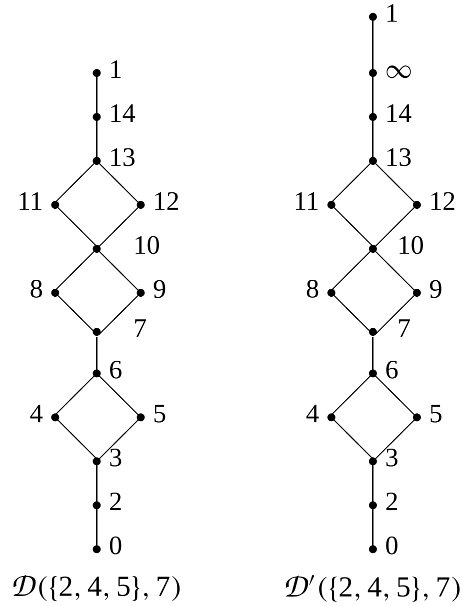


Figure 1: $\mathcal{D}$ and $\mathcal{D}'$

be finitely generated. Let $b \in \mathcal{M}$ be such that $\mathcal{N} = \mathrm{K}(\mathcal{M}; b)$. Since $\mathcal{M}$ is recursively saturated, the type $p(x)$ of $b$ is in $\mathrm{SSy}(\mathcal{M})$. We will show that $X$ is arithmetic in $p(x)$, thereby showing that $X \in \mathrm{SSy}(\mathcal{M})$. The definition of $X$ is inductive. Suppose that we already know $X \cap n$. Then $n \in X$ iff the lattice $\mathcal{D}((X \cap n) \cup \{n\}, n+1)$ is isomorphic to the corresponding initial sublattice of $\mathrm{Lt}(\mathrm{K}(\mathcal{M}; b))$. Formally: $n \in X$ iff there are terms $t_i(x)$, $i \in m$ where $m = D((X \cap n) \cup \{n\}, n+1)$ such that:

1. $i \trianglelefteq j$ iff there is a term $t(x)$ such that the formula $t(t_j(x)) = t_i(x)$ is in $p(x)$;
2. for any term $t(x)$ there is $i$ such that $1 \neq i < m$ and there are terms $t'(x)$ and $t''(x)$ such that the formulas $t'(t(t_{m-1}(x))) = t_i(x)$ and $t''(t_i(x)) = t(t_{m-1}(x))$ are in $p(x)$. □

**Lemma 5.3** *Suppose $\mathcal{M} \models \mathrm{PA}$ is recursively saturated and $X \in \mathrm{SSy}(\mathcal{M})$. Then $\mathrm{Lt}(\mathcal{M})$ has an ideal isomorphic to $\mathcal{D}(X)$.*

Lemma 5.3 follows from the lemmas below. It suffices to show that there is a set $\Sigma(x)$ of formulas consistent with each completion of PA, such that $\Sigma(x)$ is recursive in $X$ and whenever $\mathcal{N} \models \mathrm{PA}$ is generated by an element realizing $\Sigma(x)$, then $\mathrm{Lt}(\mathcal{N}) \cong \mathcal{D}(X)$. For any set $A$, let $\Pi(A)$ be the lattice of partitions of $A$, where $\pi_1 \trianglelefteq \pi_2$ whenever $\pi_2$ refines $\pi_1$. Let $\mathbf{1}_A$ be the partition into singletons and $\mathbf{0}_A = \{\{A\}\}$. Then $\mathbf{0}_A \trianglelefteq \pi \trianglelefteq \mathbf{1}_A$ for any $\pi \in \Pi(A)$.

See [13] and [14] for more on the relationship between $\Pi(A)$ and $\mathrm{Lt}(\mathcal{N})$.

**Definition 5.4** If $Y \subseteq n < \omega$, then an embedding $\alpha : \mathcal{D}(Y,n) \to \Pi(A)$ will be called a *standard representation* of $\mathcal{D}(Y,n)$ if the following three conditions hold.

1. if $x, y \in D(Y,n)$ and $x \triangleleft y$, and if $E \in \alpha(x)$, then there are infinitely many $F \in \alpha(y)$ for which $F \subseteq E$;

2. if $x, y \in D(Y, n)$ and $x \wedge y \lhd x, y \lhd x \vee y$, and if $E \in \alpha(x \wedge y)$, $E_1 \in \alpha(x)$, $E_2 \in \alpha(y)$ and $E_1, E_2 \subseteq E$, then $E_1 \cap E_2 \neq \varnothing$;
3. $\alpha(0) = \mathbf{0}_A$ and $\alpha(1) = \mathbf{1}_A$.

Notice that (1) implies that if $\alpha : \mathcal{D}(Y, n) \to \Pi(A)$ is a standard representation, then $A$ is infinite. We observe that if $A$ and $B$ are countable and $\alpha : \mathcal{D}(Y, n) \to \Pi(A)$ and $\beta : \mathcal{D}(Y, n) \to \Pi(B)$ are standard representations, then $\alpha$ and $\beta$ are isomorphic (that is, there is a bijection $\gamma : A \to B$ such that whenever $x \in D(Y, n)$ and $a, b \in A$, then $\{a, b\} \subseteq \alpha(x)$ iff $\{\gamma(a), \gamma(b)\} \subseteq \beta(x)$).

The following lemma is not difficult to prove by induction on $n$, using several applications of Ramsey's Theorem. The proof of Lemma 5.6 is easy.

**Lemma 5.5** *Suppose that $Y \subseteq n < \omega$, that $\alpha : \mathcal{D}(Y, n) \to \Pi(A)$ is a standard representation, and that $\pi \in \Pi(A)$. Then there is $B \subseteq A$ such that $\alpha \mid B : \mathcal{D}(Y, n) \to \Pi(B)$ is a standard representation and $(\alpha \mid B)(x) = \pi \mid B$ for some $x \in D(Y, n)$.*

**Lemma 5.6** *Suppose that $Y \subseteq n + 1 < \omega$ and that $\alpha : \mathcal{D}(Y \cap n, n) \to \Pi(A)$ is a standard representation. Then there is a standard representation $\beta : \mathcal{D}(Y, n + 1) \to \Pi(A)$ which extends $\alpha$.*

Both of these lemmas have appropriate formalized versions which are provable in PA. Suppose $m = D(Y, n)$ and $\langle \varphi_i(x, y) : i < m \rangle$ is an $m$-tuple of formulas with free variables $x$ and $y$. Then we say that PA proves that $\langle \varphi_i(x, y) : i < m \rangle$ is a standard representation of $\mathcal{D}(Y, n)$ if $m = D(Y, n)$ and, letting $\theta(x) = \varphi_0(x, x)$, then PA proves each of the following.

1. each $\varphi_i(x, y)$ is an equivalence relation on the set $\theta(x)$;
2. if $i \lhd j < m$, then each equivlence class of $\varphi_i(x, y)$ contains unboundedly many equivalence classes of $\varphi_j(x, y)$;
3. if $i, j < m$ and $i \wedge j \lhd i, j \lhd i \vee j$, then $\varphi_{i \vee j}(x, y) \leftrightarrow \varphi_i(x, y) \wedge \varphi_j(x, y)$ and $\varphi_{i \wedge j}(x, y) \leftrightarrow \exists z(\varphi_i(x, z) \wedge \varphi_j(z, y))$.
4. $\varphi_0(x, y) \leftrightarrow \theta(x) \wedge \theta(y)$ and $\varphi_1(x, y) \leftrightarrow \theta(x) \wedge x = y$.

Lemmas 5.7 and 5.8 are the formalizations in PA of Lemmas 5.5 and 5.6 respectively. It is these lemmas that we actually use.

**Lemma 5.7** *Suppose that $Y \subseteq n < \omega$ and that* PA *proves that $\langle \varphi_i(x, y) : i < m \rangle$ is a standard representation of $\mathcal{D}(Y, n)$. Suppose that $\psi(x, y)$ is a formula. Then there is a formula $\theta(x)$ such that* PA *proves that $\langle \varphi_i(x, y) \wedge \theta(x) \wedge \theta(y) : i < m \rangle$ is a standard representation of $\mathcal{D}(Y, n)$, and* PA *also proves: if $\psi(x, y)$ defines an equivalence relation on the universe, then for some $i < m$,*

$$\forall x \forall y[\theta(x) \wedge \theta(y) \wedge \varphi_0(x, y) \to (\psi(x, y) \leftrightarrow \varphi_i(x, y))].$$

*Moreover $\theta(x)$ can be effectively obtained from $Y, n$ and $\langle \varphi_i(x, y) : i < m \rangle$.*

**Lemma 5.8** *Suppose that $Y \subseteq n + 1 < \omega$ and that* PA *proves $\langle \varphi_i(x, y) : i < m \rangle$ is a standard representation of $\mathcal{D}(Y \cap n, n)$. Then there are $\varphi_i(x, y)$ for $m \leq i < k$ (where $k = m + 3$ if $n \in Y$, and $k = m + 1$ if $n \notin Y$) such that* PA *proves $\langle \varphi_i(x, y) : i < k \rangle$ is a standard representation of $\mathcal{D}(Y, n + 1)$. Moreover $\varphi_m(x, y), \ldots, \varphi_{k-1}(x, y)$ can be effectively obtained from $Y, n + 1$ and $\langle \varphi_i(x, y) : i < m \rangle$.*

Now, given $X \subseteq \omega$, we construct the set $\Sigma(x)$ of formulas. Let $\langle \psi_i(x, y) : i < \omega \rangle$ be a recursive list of all formulas in the language of PA. We construct, recursively in $X$, a doubly indexed sequence $\langle \varphi_{in}(x, y) : i \in D(X \cap n, n),\ n < \omega \rangle$ of formulas such that, for each $n < \omega$, PA proves that $\langle \varphi_{in}(x, y) : i \in\ D(X \cap n, n) \rangle$ is a standard representation of $\mathcal{D}(X \cap n, n)$. Let $\varphi_{00}(x, y)$ be the formula $x = x \wedge y = y$ and $\varphi_{10}(x, y)$ be the formula $x = y$. Then PA proves that $\langle \varphi_{00}(x, y), \varphi_{10}(x, y) \rangle$ is a standard representation of $\mathcal{D}(\varnothing, 0)$. At stage $n$ we will have $\langle \varphi_{in}(x, y) : i < m \rangle$, which PA proves is a standard representation of $\mathcal{D}(X \cap n, n)$. By Lemma 5.7 let $\theta(x)$ be such that $\langle \varphi_{in}(x, y) \wedge \theta(x) \wedge \theta(y) : i < m \rangle$ is proved by PA to be a standard representation of $\mathcal{D}(X \cap n, n)$ and such that for some $i < m$

$$\mathrm{PA} \vdash \forall x \forall y [\theta(x) \wedge \theta(y) \wedge \varphi_{0n}(x, y) \rightarrow (\varphi_{in}(x, y) \leftrightarrow \psi_n(x, y))].$$

Then by Lemma 5.8, let $\langle \varphi_{i,n+1}(x, y) : i < k \rangle$ be such that PA proves that it is a standard representation of $\mathcal{D}(X \cap (n + 1), n + 1)$ and that $\varphi_{i,n+1}(x, y) = \varphi_{in}(x, y) \wedge \theta(x) \wedge \theta(y)$ for $i < m$.

Now let $\Sigma(x) = \{\varphi_{0n}(x, x) : n < \omega\}$. It is easily seen that $\mathrm{Th}(\mathcal{M}) \cup \Sigma(x)$ generates a unique type $p(x)$ and there is $b \in \mathcal{M}$ realizing $p(x)$. It is also easily shown that $\mathrm{Lt}(\mathrm{K}(\mathcal{M}; b)) \cong \mathcal{D}(X)$. This completes the proof of Lemma 5.3 and finishes the proof of Theorem 5.1.

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*Department of Mathematics, Box G–0930*
*Baruch College/CUNY*
*17 Lexington Avenue*
*New York, NY 10010*
*email: rkobb@cunyvm.cuny.edu*

*Department of Mathematics U-9*
*University of Connecticut*
*Storrs, CT 06269*
*email: schmerl@uconnvm.uconn.edu*