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Naive Set Theory with Extensionality in Partial Logic and in Paradoxical Logic

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Abstract Two distinct and apparently "dual" traditions of non-classical logic, three-valued logic and paraconsistent logic, are considered here and a unified presentation of "easy-to-handle" versions of these logics is given, in which full naive set theory, i.e. Frege's comprehension principle + extensionality, is not absurd.

1 Introduction We consider here two types of nonclassical logics. The logics of the first type will be called "partial" and are inspired by the work concerning "partial set theory" of Gilmore [5]. They are clearly "three-valued logics" but we prefer to give them a different name in order to distinguish them from the already existing three-valued logics. For an analogous reason, we will call the logics of the second type "paradoxical", although they are clearly paraconsistent logics, at least if one accepts the following definition of Arruda [1]: "Loosely speaking, a paraconsistent logic is a logic in which a contradiction, $A \;\&\; \neg A$, is not in general an antinomy." (An antinomical theory is simply a trivial theory, i.e., one in which everything is provable).

We will only consider first-order languages, with equality as a primitive symbol and with $\exists, \forall, \vee, \wedge, \neg, \rightarrow$ as primitive quantifiers and connectives.

We introduce, for convenience, two notions of "implication" which are equivalent to the primitive implication "$\rightarrow$" in classical logic, but not necesarily in our non-classical logics:

$A \overset{\neg\vee}{\longrightarrow} B$ is the abbreviation of $(\neg A) \vee B$
$A \overset{s}{\rightarrow} B$ is the abbreviation of $(A \rightarrow B) \wedge (\neg B \rightarrow \neg A)$.
(the "s" means "strong").

Naturally, "$A \leftrightarrow B$" will be the abbreviation of "$(A \rightarrow B) \wedge (B \rightarrow A)$", for each of the three notions of implication.

Starting with the language L, we define a new language $L^{\pm}$:
$L^{\pm}$ has "$=$" as a primitive symbol;

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$L^{\pm}$ has "$\neq$" as a (new) primitive symbol. Further, the relation symbols of $L^{\pm}$ are exactly the (new) symbols "R^{+}" and "R^{-}", for each n-ary symbol "R" in L. For the equality symbol, we consider that "$=^{+}$" is exactly "$=$" and that "$=^{-}$" is exactly "$\neq$". Finally, the function and constant symbols of $L^{\pm}$ are exactly those of L.

Now, for any formula φ in L there corresponds in a natural way a pair of formulas in $L^{\pm}$, φ^{+} and φ^{-}.

The inductive definition of φ^{+} and φ^{-} is:

(1) If φ is an atomic formula of type $R(\vec{x})$ (where R is a relation-symbol in L) then φ^{+} is $R^{+}(\vec{x})$, and φ^{-} is $R^{-}(\vec{x})$.

(2) $(\varphi \vee \psi)^{+}$ is $\varphi^{+} \vee \psi^{+}$ and $(\varphi \vee \psi)^{-}$ is $\varphi^{-} \wedge \psi^{-}$.

(3) $(\varphi \wedge \psi)^{+}$ is $\varphi^{+} \wedge \psi^{+}$ and $(\varphi \wedge \psi)^{-}$ is $\varphi^{-} \vee \psi^{-}$.

(4) $(\neg\varphi)^{+}$ is φ^{-} and $(\neg\varphi)^{-}$ is φ^{+}.

(5) $(\varphi \to \psi)^{+}$ is $\varphi^{+} \to \psi^{+}$ and $(\varphi \to \psi)^{-}$ is $\varphi^{+} \wedge \psi^{-}$.

(6) $(\exists x\varphi)^{+}$ is $\exists x\varphi^{+}$ and $(\exists x\varphi)^{-}$ is $\forall x\varphi^{-}$.

(7) $(\forall x\varphi)^{+}$ is $\forall x\varphi^{+}$ and $(\forall x\varphi)^{-}$ is $\exists x\varphi^{-}$.

(8) If $\varphi(x_1, x_2, \ldots, x_n)$ is a formula in L and $\tau_1, \tau_2, \ldots, \tau_n$ are terms in L, then $(\varphi(\tau_1, \tau_2, \ldots, \tau_n))^{+}$ is $\varphi^{+}(\tau_1, \tau_2, \ldots, \tau_n)$.

Remark 1.1 The rules for the implications $\overset{\neg\vee}{\longrightarrow}$ and $\overset{s}{\to}$ are:

$(\varphi \overset{\neg\vee}{\longrightarrow} \psi)^{+}$ is $\varphi^{-} \vee \psi^{+}$
$(\varphi \overset{s}{\longrightarrow} \psi)^{+}$ is $(\varphi^{+} \to \psi^{+}) \wedge (\psi^{-} \to \varphi^{-})$.

Note that $(\varphi \to \psi)^{-}$ is equivalent to $\varphi^{+} \wedge \psi^{-}$ for the three versions of "$\to$" so that $(\varphi \leftrightarrow \psi)^{-}$ is equivalent to $(\varphi^{+} \wedge \psi^{-}) \vee (\varphi^{-} \wedge \psi^{+})$.

Remark 1.2 When we say that "$\to$" does not occur in a formula φ of L, we mean that neither the primitive "$\to$", nor "$\overset{s}{\to}$", occur in φ; so "$\overset{\neg\vee}{\longrightarrow}$" can occur.

Remark 1.3 It should be noted that φ^{+} and φ^{-} are positive formulas of $L^{\pm}$ when "$\to$" does not occur in φ.

The axioms of Partial Logic are:

$$Ax\ Pt\ Log \equiv \neg(S^{+}(\vec{x}) \wedge S^{-}(\vec{x}))$$

(for every relation symbol S, including the equality symbol). The axioms of Paradoxical Logic are:

$$Ax\ Pd\ Log \equiv (S^{+}(\vec{x}) \vee S^{-}(\vec{x}))$$

(for every relation symbol S, including the equality symbol).

Clearly, Partial Logic and Paradoxical Logic appear as "dual" weakenings of classical logic. While in the classical situation the "collections" $\{\vec{x}|S^{+}(\vec{x})\}$ and $\{\vec{x}|S^{-}(\vec{x})\}$ (where S^{+} is interpreted as S and S^{-} as $\neg S$) correspond to a partition of the "universe", they correspond to disjointed collections in the partial case and to a covering of the "universe" in the paradoxical case.

A routine induction now shows that this situation is still true for any formula $\varphi(x_1, \ldots, x_n)$ in L, i.e. that $\neg(\varphi^{+}(\vec{x}) \wedge \varphi^{-}(\vec{x}))$ can be (classically) deduced from

Ax Pt Log and that $\varphi^+(\vec{x}) \vee \varphi^-(\vec{x})$ can be (classically) deduced from *Ax Pd Log*. The word "classically" refers to ordinary classical logic with equality. We will see later that the "duality" between *Pt* Logic and *Pd* Logic is not as perfect as one could expect.

We need some more definitions:

Definition 1.4 An axiomatic system (or, for short, a system) in L is any set of sentences in L.

Definition 1.5 If Σ, Σ', are systems in L, then $\Sigma \vdash_{Pt} \Sigma'$ means that *Ax Pt Log* $+$ $\Sigma^+ \vdash_{Class} (\Sigma')^+$ (where "$\vdash_{Class}$" is the usual symbol for provability in classical logic with equality). We adopt the obvious similar definition for "$\vdash_{Pd}$".

Definition 1.6 A system Σ is said Partially Inconsistent (for short, *Pt*-inconsistent) iff $\Sigma \vdash_{Pt} \sigma$ for any sentence σ in L.

"Consistent" will mean "not inconsistent". We adopt the obvious similar definition for "*Pd*-inconsistent".

What is the relation of *Pt*- and *Pd*-consistency to classical consistency?

Remark 1.7 (The *Pt*-case) Σ is *Pt*-consistent $\Leftrightarrow$ $\Sigma^+ +$ *Ax Pt Log* is (classically) consistent. This is due to the fact that, in the *Pt*-case, there exists a false sentence of type σ^+. For example $\sigma \equiv \forall x \;\; \neg x = x$.

As our metatheory is classical (for example *ZF*), we get Σ is *Pt*-consistent $\Leftrightarrow \exists M$ (a model for the language $L^\pm$) such that $M \vDash_{Class} (\Sigma^+ + \textit{Ax Pt Log})$. Naturally, we define "$M \vDash_{Pt} \sigma$" as meaning $M \vDash_{Class} \sigma^+$, where "$\vDash_{Class}$" is the classical symbol for "is a model of".

Remark 1.8 (The *Pd*-case) This case is (slightly) less simple as we don't have a "false positive sentence" ($x \neq x$ is not forbidden here). Actually we have:

> Σ is *Pd*-consistent $\Leftrightarrow$ $\Sigma^+ +$ *Ax Pd Log* does not prove (classically) any σ^+ (for σ a sentence in L) $\Leftrightarrow$ $\exists M$ (M a model for $L^\pm$) and $\exists \sigma$ (a sentence in L) such that $M \vDash_{Class} (\Sigma^+ + \textit{Ax Pd Log})$ and $\neg M \vDash_{Class} \sigma^+$.

Thus the implication

> "Σ is *Pd*-consistent $\Rightarrow$ $\Sigma^+ +$ *Ax Pd Log* is (classically) consistent"

is true, while in general the converse is not. For example take L to be $\{=\}$; M to be $\langle \{a\}, =_M, \neq_M \rangle$; $=_M$ to be the "real" equality on M (i.e.: $\{(a, a)\}$); $\neq_M$ to be $\{(a, a)\}$; and Σ to be $\{\forall x \; \forall y \;\; x = y\}$.

An easy induction on the length of the formula φ in L shows that $\Sigma \vdash_{Pd} \forall \vec{x} \;\; \varphi(\vec{x})$. So Σ is *Pd*-inconsistent. However, M is (classically) a model for $\Sigma^+ +$ *Ax Pd Log*.

Actually, as our main concern is set theory, we will only be interested in infinite models. And the existence of such models will guarantee *Pd*-consistency, because "$\Sigma^+ +$ *Ax Pd Log* $+ \exists a \exists b \;\; \neg a = b$ is (classically) consistent" implies "Σ is *Pd*-consistent" (as $\neg \Sigma \vdash_{Pd} \forall a \forall b \;\; a = b$).

Note that the *Pd*-consistency does not imply the existence of a model of cardinality ≥ 2. For example take $L \equiv \{=\}$; $\Sigma \equiv \{\forall x \forall y \;\; x = y\}$; $M \equiv \langle \{a\}, =_M, \neq_M \rangle$; $=_M \equiv \{(a, a)\}$; and $\neq_M$ to be empty. This M is a *Pd*-model for Σ; as $\neg M \vDash_{Pd} \neg x = x$, Σ is *Pd*-consistent. However $\Sigma^+ +$ *Ax Pd Log* $+ \exists a \exists b \;\; \neg a = b$ is (classically) inconsistent.

Let us now introduce the definition of the notion of "syntactical variant of a system Σ". First we define the equivalence "φ' is a syntactical variant of φ" (we introduce the notation $\varphi' \sim \varphi$) inductively by the following rules ($\sim$ is an equivalence on the set of the formulas of L):

(1) $\varphi \sim \neg\neg\varphi$.

(2) $\varphi \vee \psi \sim \neg((\neg\varphi) \wedge (\neg\psi))$.

(3) $\varphi \to \psi \sim (\neg\varphi) \vee \psi$.

(4) If ψ is a subformula of φ and $\psi \sim \psi'$, then $\varphi' \sim \varphi$, where φ' is obtained from φ by replacing ψ by ψ'.

(5) If $\varphi \sim \varphi'$, then $\varphi \sim (\varphi \vee \varphi')$.

(6) If $R(x_1, \ldots, x_n)$ is an atomic formula (i.e. R is a primitive relation symbol in L) then $\exists x_i' \ (x_i' = x_i \wedge R(x_1, x_2, \ldots, x_i', \ldots, x_n)) \sim R(x_1, x_2, \ldots, x_i, \ldots, x_n)$ (where "x_i'" is any variable, distinct from "x_1", "x_2", ..., "x_i", ..., "x_n").

Obviously, $\varphi \sim \varphi'$ implies that φ and φ' are equivalent in classical logic and that they have exactly the same free variables.

Definition 1.9 If Σ, Σ', are systems Σ' is a syntactical variant of Σ iff $((\forall\sigma \in \Sigma \ \exists\sigma' \in \Sigma' \ \sigma \sim \sigma') \ \& \ (\forall\theta \in \Sigma' \ \ \exists\,\theta' \in \Sigma \ \ \theta \sim \theta'))$.

Syntactical variants of a system Σ are obviously classically equivalent, i.e. they prove the same theorems (in classical logic). However they are not necessarily *Pt*- or *Pd*-equivalent, as is shown in this paper for naive set theory with extensionality.

2 Duality It will become clear below that the "duality" $Pt - Pd$ works in a satisfying way only when the $\neq$ relation is classical. So we introduce two new types of non-classical logic:

$$Pt^{\neq}\text{-logic} \qquad \text{and} \qquad Pd^{\neq}\text{-logic.}$$

The axioms of $Pt^{\neq}$-logic (i.e. Pt-logic with a classical $\neq$) are:

$$Ax\ Pt^{\neq}\ Log \equiv Ax\ Pt\ Log + \text{the axiom: } (x = y \leftrightarrow \neg x \neq y).$$

$Pd^{\neq}$-logic is defined in the obvious similar way. Naturally a $Pt^{\neq}$-model (f.ex.) will just be a model for the language $L^{\pm}$ satisfying (classically) Ax $\text{Pt}^{\neq}$ Log, etc...

Consider a model M for $L^{\pm}$.

The <u>dual</u> of M is the structure $\mathring{M}$, having the same universe, but where the relation symbols S^{+} and S^{-} of $L^{\pm}$ are interpreted by $S^{\oplus}$ and $S^{\ominus}$, and where these are defined by:

$$S^{\oplus}(\vec{a}) \Leftrightarrow \neg M \vDash S^{-}(\vec{a})$$

$$S^{\ominus}(\vec{a}) \Leftrightarrow \neg M \vDash S^{+}(\vec{a})$$

(with $a_1, a_2, \ldots, a_n$ in M).

So clearly, $\mathring{\mathring{M}}$ is M and M is a $Pt^{\neq}$-model iff $\mathring{M}$ is a $Pd^{\neq}$-model.

Remark 2.1 The problem, when $\neq$ is <u>not</u> classical, is that $=^{\oplus}$ (defined by $x =^{\oplus} y \leftrightarrow \neg x \neq_M y$) is not even necessarily an equivalence relation!

This problem cannot be avoided, even in very natural situations. Suppose, for example, that M is a model for $\mathcal{L}^{\pm}$, with $\mathcal{L} = \{\in, =\}$, and that $\neq_M$ has the property:

$$x \neq_M y \quad \leftrightarrow \quad \exists t \in M((t \in_M^+ x \wedge t \in_M^- y) \vee (t \in_M^- x \wedge t \in_M^+ y)).$$

Note that this expresses exactly a syntactical variant of the axiom of extensionality:

$$EXT \overset{def}{\equiv} (\forall t(t \in x \leftrightarrow t \in y)) \leftrightarrow x = y.$$

It is easy to find (even finite) models M where such an $\neq_M$ does <u>not</u> induce a transitive relation $=^{\oplus}$. It suffices to have a situation as follows: $(\neg x \neq_M y) \wedge (\neg y \neq_M z) \wedge (x \neq_M z)$, for some x, y, z in M. The topological models introduced below in Section 5 will contain such situations.

So, let us assume that M is a model (for $L^{\pm}$) where $\neq_M$ is classical.
We distinguish the cases for *Pt* and *Pd*.

First case: $M \vDash_{Class} Ax\ Pt^{\neq}\ Log$.

So, if $M \vDash_{Pt} \varphi(\vec{a})$, we have $M \vDash_{Class} \varphi^+(\vec{a})$ and so (as we are in the *Pt*-situation) $M \vDash_{Class} \neg\varphi^-(\vec{a})$. Now, it is easy to verify that, for formulas φ (in L) in which "$\rightarrow$" does not occur

$$M \vDash_{Class} \neg\varphi^-(\vec{a}) \Leftrightarrow \quad \mathring{M} \vDash_{Class} \varphi^+(\vec{a})$$

and

$$M \vDash_{Class} \neg\varphi^+(\vec{a}) \Leftrightarrow \quad \mathring{M} \vDash_{Class} \varphi^-(\vec{a}).$$

So we conclude; $\mathring{M} \vDash_{Pd} \varphi(\vec{a})$ <u>if</u> $M \vDash_{Pt} \varphi(\vec{a})$ and "$\rightarrow$" does not occur in φ.

The fact that we don't have this for any φ in L suggests the following definition. A system Σ is called "$Pt^{\neq}$-classical" iff:

$\forall M$ model of $L^{\pm}$ (M is a $Pt^{\neq}$-model for $\Sigma \Rightarrow \mathring{M}$ is a $Pd^{\neq}$-model for Σ)

or, equivalently:

$\forall M$ model of $L^{\pm}$ ($M \vDash_{Class}$ (Ax Pt$^{\neq}$ Log $+ \Sigma^+) \Rightarrow \mathring{M} \vDash_{Class} \Sigma^+$).

So, clearly, any system whose axioms do not contain "$\rightarrow$" (that is the primitive symbol "$\rightarrow$"; the axioms *may* contain "$\xrightarrow{\neg v}$") <u>is</u> $Pt^{\neq}$-classical.

Second case: Suppose $M \vDash_{Class} Ax\ Pd^{\neq}\ Log$ and $M \vDash_{Class} \varphi^+(\vec{a})$. Here the first step, i.e. getting $M \vDash_{Class} \neg\varphi^-(\vec{a})$ from $M \vDash_{Class} \varphi^+(\vec{a})$ is already problematic. And the second step; i.e. getting $\mathring{M} \vDash_{Class} \varphi^+(\vec{a})$ from $M \vDash_{Class} \neg\varphi^-(\vec{a})$, presents the same difficulties as in our first case above. So we introduce a definition (similar to the one we gave in our first case). A system Σ is called "$Pd^{\neq}$-classical" iff

$\forall M$ model of $L^{\pm}$ ($M \vDash_{Class} (Ax\ Pd^{\neq}\ Log + \Sigma^+) \Rightarrow \quad \mathring{M} \vDash_{Class} \Sigma^+$).

Clearly, if Σ is $Pd^{\neq}$-classical and $Pd^{\neq}$-consistent, then Σ is $Pt^{\neq}$-consistent.

The general impression we get from these 2 cases is that it will generally be harder to get a $Pt^{\neq}$ model $\mathring{M}$ (for Σ) from a $Pd^{\neq}$ model M (for Σ) than the converse. Anyhow, the "duality" *Pt*–*Pd* is far from perfect. This has also been noted by Crabbé in a slighlty different context in his [3].

Here follow some set-theoretical examples for the language $\mathcal{L} \equiv \{\in, =\}$.

Example 2.2 Let us call the following syntactical variant of EXT (the axiom of extensionality); "EXT1":

$$(\forall t\,(t \in x \overset{s}{\leftrightarrow} t \in y)) \leftrightarrow x = y.$$

One can easily check that EXT1 (i.e. the system {EXT1}) is $Pd^{\neq}$- and $Pt^{\neq}$-classical.

Example 2.3 Let us call the following syntactical variant of EXT; "EXT2":

$$(\forall t\,(t \in x \overset{s}{\leftrightarrow} t \in y)) \overset{s}{\leftrightarrow} x = y.$$

It is a natural version of EXT, both in the *Pt* and *Pd*-cases; it gives a nice characterization of $\neq$: $x \neq y \leftrightarrow \exists t\,((t \in^{+} x \wedge t \in^{-} y) \vee (t \in^{-} x \wedge t \in^{+} y))$. Furthermore, in the *Pd*-case it gives an interesting sense to $x \neq x$, as $x \neq x \leftrightarrow x$ has $\in^{+}$ and $\in^{-}$ members, i.e. x is not a classical set.

But EXT2 is *not* adapted to set theory in the $Pd^{\neq}$-case. For there $x \neq x$ is excluded, so that all the sets are classical (i.e. $M \vDash_{Class} \forall t \forall x (t \in^{+} x \leftrightarrow \neg t \in^{-} x))$. So if one assumes comprehension, one gets the Russell-paradox. However uninteresting it may be for set theory, EXT2 is actually $Pd^{\neq}$-classical. Indeed, if M is a $Pd^{\neq}$-model for EXT2, then the sets in M are classical and so $\mathring{M}$ is exactly M itself!

Note that EXT2 is not $Pt^{\neq}$-classical, as is shown by the structure:

$$M = \langle \{0, 1\}, \in^{+}_{M}, \in^{-}_{M}, =_M, \neq_M \rangle$$

where $=_M$ and $\neq_M$ are the "real" $=$ and $\neq$ on $\{0, 1\}$, and where $\in^{+}_{M} \equiv \{(0, 0)\}$, $\in^{-}_{M} \equiv \{(0, 1)\}$. M is a $Pt^{\neq}$-model for EXT2. But $\mathring{M}$ cannot be a $Pd^{\neq}$-model for EXT2 because otherwise (as we remarked previously) $\mathring{M}$ would be $\mathring{M}$ itself and so M would be $\mathring{M}$. And this is not the case.

Definition 2.4 We will adopt the following usefull notation: Comp($\leftrightarrow$) is the axiom schema $\equiv \forall \vec{y}\ \exists x\ \forall t\,(t \in x \leftrightarrow \varphi(t, \vec{y}))$ for φ in $\mathcal{L} \equiv \{\in, =\}$, such that "$x$" is not free in φ.

Comp($\overset{\neg v}{\leftrightarrow}$) is the schema $\equiv \forall \vec{y}\ \exists x\ \forall t\,(t \in x \overset{\neg v}{\leftrightarrow} \varphi(t, \vec{y}))$ for φ as above.

Comp($\overset{s}{\leftrightarrow}$) is the schema $\equiv \forall \vec{y}\ \exists x\ \forall t\,(t \in x \overset{s}{\leftrightarrow} \varphi(t, \vec{y}))$ for φ as above.

Actually, any axiom of Comp($\overset{s}{\leftrightarrow}$) is $Pt^{\neq}$-and $Pd^{\neq}$-classical. This fact will be used later.

3 The system of Frege in $\mathcal{L} \equiv \{\in, =\}$ We take the system of Frege simply to be $F \equiv$ extensionality + the full comprehension scheme, i.e.: for any formula φ in $\mathcal{L}$ (where "x" is not free),

$$\forall \vec{y}\ \exists x\ \forall t\,(t \in x \leftrightarrow \varphi(t, \vec{y})).$$

We will consider different syntactical variants of this basic system F:

$$F1 \equiv EXT1 + \text{Comp}(\overset{s}{\leftrightarrow})$$

$$F2 \equiv EXT2 + \text{Comp}(\overset{s}{\leftrightarrow})$$

$$F3 \;\equiv\; EXT1 + \text{Comp}^{\star},$$

where "Comp" is defined at the end of Section 2, and where Comp* is the schema:

$$\forall \vec{y}\ \ \exists x\ \forall t\ (t \in x \ \ \overset{s}{\leftrightarrow}\ \ \varphi^{\star}(t, \vec{y})),$$

for φ in $\mathcal{L}$, without "$\to$" and "x" not free in φ. Further $\varphi^{\star}$ is obtained from φ by replacing in φ any subformula of type "$z \in t$" by "$\exists t'\ (z \in t' \wedge t = t')$" (where "$z$", "$t$", and "$t'$" are *distinct* variables) and "$t \in t$" by "$\exists t'\ (t \in t' \wedge t = t')$".

We can discuss now briefly the main results and conjectures about *F1*, *F2*, *F3*.

Remark 3.1 The system *F1*, seen in *Pt*-logic, corresponds exactly to the "3-valued Frege" already discussed in Hinnion [7]. We conjecture that *F1* is *Pt*-consistent. We also conjecture that *F2* is *Pt*-consistent. In Sections 5 and 6 we suggest paths which will perhaps lead to solutions for these open questions.

Remark 3.2 The system *F1*, seen in *Pd*-logic, as well as the system *F2*, are *Pd*-consistent. In Section 5 we construct topological models for it in ZF. These models are very similar to the Scott-models for lambda-calculus.

Remark 3.3 In Section 5 we construct a topological model in ZF for *F3* in *Pt*-logic. So *F3* is *Pt*-consistent relative to ZF. This shows that at least one syntactical variant of *F* is *Pt*-consistent.

Remark 3.4 The system *F1*, seen in $Pt^{\neq}$-logic, corresponds exactly to the system "SF_3" ("strong 3-valued Frege") studied in Lenzi [10]. Lenzi proves that SF_3 cannot have a recursive term model. The $Pt^{\neq}$-consistency of *F1* is still an open question.

In Section 6 we construct interpretations of *F1* for $Pt^{\neq}$, $Pd^{\neq}$, *Pt*, and *Pd*-logic in "positive" set theories (in classical logic). So the corresponding consistency problems are reduced to the classical consistency of these theories. These are open questions.

Note that $Pd^{\neq}$- and $Pt^{\neq}$-consistency are equivalent for *F1* as *F1* is $Pt^{\neq}$ and $Pd^{\neq}$-classical.

In order to show that *Pt*- or *Pd*-logic does <u>not</u> automatically guarantee that the Russell-like paradoxes disappear, let us give here a small list of syntactical variants of *F* which are not *Pd*- or *Pt*- consistent.

(a) Even without any form of EXT, Comp($\overset{\neg v}{\leftrightarrow}$) is *Pt*-inconsistent. To see this, try this comprehension schema for $\varphi(t) \equiv \neg t \in t$.

(b) Take $F' \equiv F1$ where one replaces, in EXT, any "$\to$" by "$\overset{\neg v}{\longrightarrow}$". F' is *Pt*-inconsistent.

Indeed for this form of EXT, the corresponding EXT^{+} implies that (for $x = y$):

$$\forall t (t \in^{+} x \ \vee \ t \in^{-} x),$$

so that any x is a classical set, and one gets the Russell paradox.

Actually, it is not really the full EXT that is responsible, but only:

$$x = y \to \forall t (t \in x \leftrightarrow t \in y).$$

(c) Even without any form of EXT, the full comprehension scheme of type: $\forall \vec{y}\ \exists x\ \forall t\ (t \in x \overset{s}{\leftrightarrow} \varphi(t, \vec{y}))$, for φ in $\mathcal{L}$ (so that "$\rightarrow$" may occur in φ) is *Pd*-inconsistent. Indeed take any formula ψ in $\mathcal{L}$ and define:

$$\varphi_\psi \overset{def}{\equiv} (t \in t \rightarrow \psi) \wedge \neg((\neg t \in t) \rightarrow \psi).$$

The + of the corresponding comprehension axiom produces something like this:

$$\exists x\ \{[\ t \in^+ x \leftrightarrow ((t \in^+ t \rightarrow \psi^+) \wedge (\ldots)\ ))\ ]$$

$$\wedge\ [t \in^- x \leftrightarrow ((t \in^+ t \wedge \psi^-) \vee (t \in^- t \rightarrow \psi^+))\ ]\}$$

and this implies (replacing "t" by "x"):

$$[x \in^+ x \rightarrow (x \in^+ x \rightarrow \psi^+)] \wedge [x \in^- x \rightarrow ((x \in^+ x \wedge \psi^-) \vee (x \in^- x \rightarrow \psi^+))]$$

As we are in the *Pd*-case $x \in^+ x \vee x \in^- x$ is true, so that we can conclude that ψ^+ is true. As this happens for any ψ in $\mathcal{L}$, this theory is indeed *Pd*-inconsistent.

(d) *F2* is $Pd^{\neq}$-inconsistent. As mentioned in Section 2, EXT2 is not adapted to $Pd^{\neq}$-logic. Actually one simply gets the Russell-paradox because in *F2* for $Pd^{\neq}$-logic all the sets are classical.

4 Extensions of the Frege system The "Super-Frege" systems introduced here are directly inspired by the "partial set" theories of Gilmore [5].

These systems make no sense in classical logic. They permit to define the elements ($\in^+$) and the anti-elements ($\in^-$) of a set by two formulas φ and ψ, with ψ not necessarily equivalent to $\neg\varphi$.

We will consider 2 variants, adapted respectively to *Pt*-logic and to *Pd*-logic.

PtSF ("Super-Frege for the *Pt*-case") is the system (in $\mathcal{L} \equiv \{\in, =\}$), EXT + the following "comprehension" schema:
For any pair of formulas in $\mathcal{L}$, $\varphi(t, \vec{z})$ and $\psi(t, \vec{z})$, in which "x" is not free:

$$[\forall t\ \forall t'\ ((\varphi(t, \vec{z}) \wedge \psi(t', \vec{z})) \rightarrow \neg t = t')] \rightarrow$$

$$\exists x\ \ \forall t\ [(t \in x \leftrightarrow \varphi(t, \vec{z})) \wedge (\neg t \in x \leftrightarrow \psi(t, \vec{z}))\ ].$$

PdSF is the system: EXT + the following "comprehension" schema (with φ, ψ as above):

$$[\forall t (\varphi(t, \vec{z}) \vee \psi(t, \vec{z}))] \rightarrow \exists x\ \forall t\ [(t \in x \leftrightarrow \varphi(t, \vec{z})) \wedge (\neg t \in x \leftrightarrow \psi(t, \vec{z}))].$$

In Section 5 we construct in ZF "topological" models for suitable syntactical variants of Super-Frege.

One can also extend Frege or Super-Frege by adopting a language $\mathcal{L}_{\langle\rangle}$, whose primitive symbols are $\in$, $=$, and a specific abstraction operator $\langle | | \rangle$ which defines terms in the following manner: $\langle t | \varphi(t, \vec{z}) | \psi(t, \vec{z}) \rangle$. The idea is that the elements of this term should be in $\{t | \varphi(t, \vec{z})\}$, while the anti-elements should be in $\{t | \psi(t, \vec{z})\}$.

Here are the inductive rules defining $\mathcal{L}_{\langle\rangle}$:

(1) Any variable is a term of $\mathcal{L}_{\langle\rangle}$.

(2) Any formula of $\mathcal{L}$ is a formula of $\mathcal{L}_{\langle\rangle}$.

(3) If $\varphi(t, \vec{z})$ and $\psi(t, \vec{z})$ are formulas of $\mathcal{L}_{\langle\rangle}$, then $\langle t|\varphi(t, \vec{z})|\psi(t, \vec{z})\rangle$ is a term of $\mathcal{L}_{\langle\rangle}$.

(4) If $\tau_1 \ldots \tau_m$ are terms of $\mathcal{L}_{\langle\rangle}$ and $\varphi(x_1, \ldots, x_m)$ is a formula of $\mathcal{L}_{\langle\rangle}$, then $\varphi(\tau_1, \tau_2, \ldots, \tau_m)$ is a formula of $\mathcal{L}_{\langle\rangle}$.

If "$\{t|\varphi(t, \vec{z})\}$" is the notation for "$\langle t|\varphi(t, \vec{z})|\neg\varphi(t, \vec{z})\rangle$", and if we restrict rule (3) to the pairs φ, ψ such that ψ is $\neg\varphi$, we get back the language $\mathcal{L}_\tau$ already defined in Hinnion [6], and in Forti and Hinnion [4].

We can define natural versions of Frege and Super-Frege, adapted to the language $\mathcal{L}_{\langle\rangle}$.

Gilmore has studied two of these in the $Pt^{\neq}$-case. His theory PST ("partial set theory") is exactly the following version of Super-Frege without extensionality, seen in $Pt^{\neq}$-Logic. (Naturally the terms should be seen as functional symbols, so that (f.ex) $(x \in \langle t|\varphi|\psi\rangle)^+$ is simply $x \in^+ \langle t|\varphi|\psi\rangle$). We have the "comprehension" schema (for φ, ψ in $\mathcal{L}_{\langle\rangle}$, where "$\rightarrow$" does not occur):

$$[\forall t \ \ \forall t' ((\varphi(t, \vec{y}) \wedge \psi(t', \vec{y})) \rightarrow \neg t = t')] \rightarrow$$

$$\forall z[(z \in \langle t|\varphi(t, \vec{y})|\psi(t, \vec{y})\rangle \leftrightarrow \varphi(z, \vec{y})) \wedge (\neg z \in \langle t|\varphi(t, \vec{y})|\psi(t, \vec{y})\rangle \leftrightarrow \psi(z, \vec{y}))].$$

Gilmore showed that this system is $Pt^{\neq}$-consistent (he did not use this terminology however!) but incompatible with EXT. His theory PST$^+$ is the following comprehension schema:

$$\forall z(z \in \{t|\varphi(t, \vec{y})\} \overset{s}{\leftrightarrow} \varphi(z, \vec{y}))$$

(for any φ in $\mathcal{L}_\tau$, where "$\rightarrow$" does not occur), seen in $Pt^{\neq}$-Logic. PST$^+$ is also incompatible with EXT (see Hinnion [6]).

We are not strongly interested here in these systems, as they don't admit EXT. However, to conclude this section, let us mention that Gilmore's construction can be adapted (in the obvious "dual" way) to prove that the natural $Pd^{\neq}$-versions of the two preceeding systems are $Pd^{\neq}$-consistent. This was noticed independently by Crabbé, in [3]. We don't know what these systems become in *Pt*- and *Pd*-logic (i.e. when $\neq$ is not necessarily classical), with respect to EXT.

5 Topological models We will prove below our:

Theorem 5.1 *In ZF there exist models for:*

(1) SF in the Pt and the Pd-case,

(2) F2 in the Pd-case,

(3) F3 in the Pt-case.

5.1 Topological models for set theories can be found in Forti and Hinnion [4], Hinnion [11], Hinnion [8], and Boffa [2]. The difficulties here arise because one has to deal with $\in^+$, $\in^-$, $=$, $\neq$ instead of simply $\in$, $=$. The principle we employ however is the one already used by Hinnion in [8].

5.2 The *Pd*-case is the easiest one, so let us start with it. Take a finite, non-empty set X, and define:

$$X_0 \stackrel{def}{=} X$$

$$X_{n+1} \stackrel{def}{=} \{(A, B) | A \cup B = X_n\} \quad (\text{for } n \in \omega).$$

Let s be any surjection $X_1 \to X_0$. One easily extends s to the higher levels, by the rule:

$$s_1 = s$$

$$s_{n+1} : X_{n+1} \to X_n : (A, B) \mapsto (im_{s_n} A, im_{s_n} B)$$

(where $im_s Y \stackrel{def}{=} \{s(y) | y \in Y\}$). Any s_{n+1} is again a surjection. For simplicity we write "s" instead of "s_{n+1}".

The universe of our model M is:

$$X_\omega \stackrel{def}{=} \{x \in \prod_{i \in \omega} X_i | \forall j \in \omega \;\; s(x_{j+1}) = x_j\}$$

(this is the "projective limit" of the X_i).
The relations $\in^+_\omega$, $\in^-_\omega$, $=_\omega$, $\neq_\omega$ (on X_ω) are defined by:

$$x \in^+_\omega y \leftrightarrow \forall i \in \omega \; (x_i \in^+_i y_{i+1})$$

$$x \in^-_\omega y \leftrightarrow \forall i \in \omega \; (x_i \in^-_i y_{i+1})$$

$$x =_\omega y \leftrightarrow x = y$$

$$x \neq_\omega y \leftrightarrow \exists t \in X_\omega((t \in^+_\omega x \wedge t \in^-_\omega y) \vee (t \in^-_\omega x \wedge t \in^+_\omega y))$$

where "x_i" is the component "i" of x; $x_i \in^+_i (A, B)$ means $x_i \in A$; and $x_i \in^-_i (A, B)$ means $x_i \in B$, (for $(A, B) \in X_{i+1}$).

Our model is: $M \stackrel{def}{=} \langle X_\omega, \in^+_\omega, \in^-_\omega, =_\omega, \neq_\omega \rangle$.

If we put the discrete (compact) topology on X_i and the natural topology on X_ω (induced by the product topology), we get a compact X_ω. The notion of convergence for X_ω is obviously:

$$\lim_{n \to \infty} x^{(n)} = x \quad (\text{in } X_\omega)$$

iff

$$\forall i \in \omega \quad \lim_{n \to \infty} \left(x^{(n)}\right)_i = x_i \quad (\text{in} X_i).$$

As the topology on each X_i is discrete, this reduces to:

$$\forall i \in \omega \;\; \exists n \in \omega \;\; \forall n \geq N \quad \left(x^{(n)}\right)_i = x_i.$$

Here follows a list of properties of M:

Lemma 5.2 The Extension Lemma

$$\forall v \in X_i \;\; \forall x \in X_\omega \;\; (v \in^+_i x_{i+1} \to \exists z \in X_\omega \, (z_i = v \wedge z \in^+_\omega x))$$

(There is a similar result for "$\in^-$").

Proof: The proof is trivial.

Lemma 5.3 *s is a morphism for $\in^+$ and $\in^-$, i.e.:*

$$\forall u \in X_{i+1}\ \ \forall v \in X_{i+2}\ \ u \in^+_{i+1} v \rightarrow s(u) \in^+_i s(v)$$

(There is a similar result for "$\in^-$").

Proof: Trivial.

Lemma 5.4 $\forall x, y \in X_\omega\ (x \in^+_\omega y \vee x \in^-_\omega y)$.

Proof: Assume $\neg x \in^+_\omega y$. Then by 5.3 $\neg x_i \in^+_i y_{i+1}$ is true for infinitely many $i \in \omega$. So $x_i \in^-_i y_{i+1}$ for infinitely many $i \in \omega$ (by the definition of X_{i+1}). Then, by Lemma 5.3; $x \in^-_\omega y$.

Lemma 5.5 $\forall x, y \in X_\omega\ (x =_\omega y \vee x \neq_\omega y)$.

Proof: Assume $\neg x =_\omega y$. Then $\neg x_{i+1} = y_{i+1}$ for infinitely many $i \in \omega$. By the definition of X_{i+1}, this implies that, for infinitely many $i \in \omega$:

$$\exists a_i \in X_i\ \ ((a_i \in^+_i x_{i+1} \wedge a_i \in^-_i y_{i+1}) \vee (a_i \in^-_i x_{i+1} \wedge a_i \in^+_i y_{i+1})).$$

Then, by Lemma 5.3 this is true $\forall i \in \omega$. So one of the formulas:

$$\exists a_i \in X_i\ \ (a_i \in^+_i x_{i+1} \wedge a_i \in^-_i y_{i+1})$$

or

$$\exists a_i \in X_i\ \ (a_i \in^-_i x_{i+1} \wedge a_i \in^+_i y_{i+1})$$

is true for infinitely many $i \in \omega$, and so also $\forall i \in \omega$. Suppose it is the first one (f.ex.). By the Extension Lemma we get:

$$\forall i \in \omega\ \ \exists z^{(i)} \in X_\omega \qquad \left(z^{(i)}\right)_i \in^+_i x_{i+1} \wedge \left(z^{(i)}\right)_i \in^-_i y_{i+1}.$$

Take a convergent subsequence $\left(z^{(i_k)}\right)_{k\in\omega}$, with limit $z^\star \in X_\omega$. Fix a level $j \in \omega$. For k large enough $\left(z^{(i_k)}\right)_j = (z^\star)_j$. Take $i_k \geq j$ and use Lemma 5.3. Clearly $(z^\star)_j \in^+_j x_{j+1} \wedge (z^\star)_j \in^-_j y_{j+1}$. As this is true for any $j \in \omega$

$$z^\star \in^+_\omega x \wedge z^\star \in^-_\omega y,$$

and so $x \neq_\omega y$.

Lemma 5.6 *Lemmas 5.4 and 5.5 show that M is a Pd-model. Furthermore $M \vDash_{Pd}$ EXT2.*

Proof: We have to prove two things:

(a) $[\forall t \in X_\omega((t \in^+_\omega x \leftrightarrow t \in^+_\omega y) \wedge (t \in^-_\omega x \leftrightarrow t \in^-_\omega y))] \leftrightarrow x =_\omega y$

(b) $(\exists t \in X_\omega((t \in^+_\omega x \wedge t \in^-_\omega y) \vee (t \in^-_\omega x \wedge t \in^+_\omega y))) \leftrightarrow x \neq_\omega y$.

The proof of (b) follows trivially from our definition of $\neq_\omega$. So let us prove the non-trivial direction in (a). Suppose x and y in X_ω have the same $\in^+_\omega$- and $\in^-_\omega$-elements. Fix any level $j \in \omega$. Suppose $(A, B) = x_{j+1} \in X_{j+1}$ and take $a \in A$. By the Extension Lemma $\exists t \in X_\omega\ (t_i = a \wedge t \in^+_\omega x)$. So, by our initial assumption $t \in^+_\omega y$. If y_{j+1} is (A', B'), we get $a \in A'$. The situation is symmetric in x, y; so we see that $A = A'$.

One proves that $B = B'$ in the same way, so that in fact:

$$\forall j \in \omega \quad x_{j+1} = y_{j+1}.$$

And this is equivalent to $x =_\omega y$.

Lemma 5.7 The Coding Lemma *Suppose $A \cup B = X_\omega$; then (A, B) is* **coded** *in M (that is $\exists x \in X_\omega \forall t \in X_\omega[(t \in_\omega^+ x \leftrightarrow t \in A) \wedge (t \in_\omega^- x \leftrightarrow t \in B)]$) iff A and B are* **closed** *subsets in X_ω.*

Proof: The direction $\Downarrow$ is easy to prove, as $\in_\omega^+$ and $\in_\omega^-$ are closed in $(X_\omega)^2$. So let us check the direction $\Uparrow$. Take for x (supposed to code (A, B)) the sequence such that:

$$x_{i+1} = (\{y_i | y \in A\}, \{z_i | z \in B\}).$$

Obviously $s(x_{j+1}) = x_j$ for any $j \in \omega$, so $x \in X_\omega$. And clearly $t \in A \to t \in_\omega^+ x$ & $t \in B \to t \in_\omega^- x$. Let us show now that (f.ex.):

$$t \in_\omega^+ x \to t \in A.$$

By the Extension Lemma, $t \in_\omega^+ x$ implies that $\forall n \in \omega\ \exists y \in A\ t_n = y_n$. Take such an y for each $n \in \omega$, and call it $y^{(n)}$. The sequence $\left(y^{(n)}\right)_{n\in\omega}$ has a convergent subsequence $\left(y^{(n_k)}\right)_{k\in\omega}$ with a limit $y^\star$. As any $y^{(n)}$ is in A and A is closed, $y^\star$ is in A. Further $t_n = \left(y^{(n)}\right)_n$, so $t_i = \left(y^{(n)}\right)_i$ for $i \leq n$. So, for any fixed $i \in \omega$:

$$t_i = \left(y^{(n_k)}\right)_i \quad \text{for } n_k \geq i.$$

Take the limit for $k \to \infty$:

$$t_i = (y^\star)_i.$$

As this is true for any $i \in \omega$:

$$t = y^\star \in A.$$

Definition 5.8 Let us call "pseudo-positive formulas in $\mathcal{L}^\pm$" the formulas in the class C defined inductively by the rules:

(i) The atomic formulas of $\mathcal{L}^\pm$ are in C,

(ii) C is closed under $\wedge, \vee, \forall, \exists$

(iii) If φ is in C, $\theta(\vec{x}) \equiv \theta(x_1, \ldots, x_m)$ is any formula in $\mathcal{L}^\pm$ (with the usual convention that the notation $\theta(x_1, x_2, \ldots, x_m)$ indicates that the set of the free variables of θ is a subset of $\{x_1, x_2, \ldots, x_m\}$) and "$x$", "$y$" are distinct variables, then:

$$\forall \vec{x}\ (\theta(\vec{x}) \to \varphi), \forall x \in^+ y\ \ \varphi, \forall x \in^- y\ \ \varphi, \forall x\ (y \in^+ x \to \varphi)$$

and

$$\forall x\ (y \in^- x \to \varphi)$$

are in C.

The class C plays here the role of the class GPF ("generalized positive formulas") in Forti and Hinnion [4], Hinnion [8], and Weydert [11].

Lemma 5.9 *The formulas in C define <u>closed</u> subsets in X_ω, i.e., if $\varphi(x, y, z, \ldots)$ is in C and $b, c, \ldots$ are in X_ω then $\{a \in X_\omega | M \vDash_{Class} \varphi(a, b, c, \ldots)\}$ is a <u>closed</u> subset of X_ω.*

Proof: We will prove this by induction on the length of φ. If $\left(x^{(n)}\right)_{n\in\omega}$, $\left(y^{(n)}\right)_{n\in\omega}$, $\left(z^{(n)}\right)_{n\in\omega}$,... are convergent sequences of elements of X_ω, with respective limits $x^\star$, $y^\star$, $z^\star$, ... then:

$$[\exists N \in \omega \ \forall n \geq N \quad M \vDash \varphi(x^{(n)}, y^{(n)}, z^{(n)}, \ldots)] \Rightarrow \quad M \vDash \varphi(x^\star, y^\star, z^\star, \ldots).$$

Let us consider only the non-trivial inductive steps, namely the "$\exists$" case and the "$\forall x \in^+ y$", etc... cases.

(1) The "$\exists$" case. Suppose that $\forall n \in \omega \quad M \vDash \exists x \quad \varphi(x, y^{(n)}, z^{(n)}, \ldots)$ and $y^{(n)} \to y^\star$, $z^{(n)} \to z^\star$, etc... For any $n \in \omega$, take such an $x \in X_\omega$ and call it "$x^{(n)}$". Further, take a convergent subsequence $\left(x^{(n_k)}\right)_{k\in\omega}$ with limit $x^\star \in X_\omega$. So $\forall k \in \omega, \quad M \vDash \varphi(x^{(n_k)}, y^{(n_k)}, z^{(n_k)}, \ldots)$. Now by the inductive hypothesis:

$$M \vDash \varphi(x^\star, y^\star, z^\star, \ldots).$$

And so $M \vDash \exists x \ \ \varphi(x, y^\star, z^\star, \ldots)$. Note that the compacity of X_ω is essential here.

(2) The "$\forall x \in^+ y$", etc... cases. These cases depend on the following "approximation Lemmas" (and their similar variants for "$\in^-$"):

Lemma 5.10 *If $x \in^+_\omega y$ and y is the limit of the sequence $\left(y^{(n)}\right)_{n\in\omega}$ then there exists a sequence $\left(x^{(n)}\right)_{n\in\omega}$ such that x is the limit of $\left(x^{(n)}\right)_{n\in\omega}$ and, for large enough n:*

$$x^{(n)} \in^+_\omega y^{(n)}$$

(i.e., $\exists N \in \omega \ \ \forall n \geq N \ \ x^{(n)} \in^+_\omega y^{(n)}$).

Proof: Trivial.

Lemma 5.11 *If $x \in^+_\omega y$ and x is the limit of $\left(x^{(n)}\right)_{n\in\omega}$ then there exists a sequence $\left(y^{(n)}\right)_{n\in\omega}$ such that y is the limit of $\left(y^{(n)}\right)_{n\in\omega}$ and, for large enough n:*

$$x^{(n)} \in^+_\omega y^{(n)}.$$

Proof: Trivial.

Let us consider (f.ex.) the case: "$\forall x \in^+ y$".

Suppose $\forall n \in \omega \quad M \vDash \forall x \in^+ y^{(n)} \ \varphi(x, y^{(n)}, z^{(n)}, \ldots)$ and $y^{(n)} \to y$, $z^{(n)} \to z$, etc... Take $x \in^+_\omega y$. By Lemma 5.10 there exists a sequence $x^{(n)}$ s.t. $x^{(n)} \to x$ and $x^{(n)} \in^+_\omega y^{(n)}$ for large enough n. So $M \vDash \varphi(x^{(n)}, y^{(n)}, z^{(n)}, \ldots)$ for large enough n. And so, by our induction hypothesis:

$$M \vDash \varphi(x, y, z, \ldots)$$

So we get: $M \vDash \forall x \in^+ y \ \ \varphi(x, y, z, \ldots)$.

Lemma 5.12 *M is a Pd-model for PdSF (the Pd-version of Super Frege defined in Section 4), with the EXT2 variant for EXT.*

Actually we have an even stronger result, as our "Coding Lemma" guarantees that any (A, B), with A, B closed and $A \cup B = X_\omega$, will be coded in M and Lemma 5.12 shows that formulas in C define closed sets in X_ω.

One can easily verify that we can allow some occurences of "$\rightarrow$" in the formulas φ, ψ appearing in the comprehension schema of *PdSF*. More precisely, we can admit formulas φ, ψ in the class C' defined inductively by:

(i) Any formula (of $\mathcal{L}$) which does not contain "$\rightarrow$" is in C'.

(ii) C' is closed under $\vee$, $\wedge$, $\exists$, $\forall$.

(iii) If $\theta(\vec{x}) \equiv \theta(x_1 \ldots x_m)$ is any formula in $\mathcal{L}$ (with (at most) $x_1 \ldots x_m$ as free variables), "x" "y" are distinct variables and φ is in C', then $\forall\vec{x}(\theta(\vec{x}) \rightarrow \varphi)$; $\forall x \in y\ \varphi$; $\forall x(\neg x \in y \rightarrow \varphi)$; $\forall x(y \in x \rightarrow \varphi)$; and $\forall x(\neg y \in x \rightarrow \varphi)$ are also in C'.

It is easy to verify that, when φ is in C', then φ^+ and φ^- are in C (i.e. they are pseudo-positive).

Lemma 5.13 *M is a Pd-model for F2.*

Here also, we have a stronger result. Actually, the comprehension schema can be extended (for the same reasons as in Lemma 5.12) to the formulas φ of the class C'', defined inductively by:

(i) Any formula (of $\mathcal{L}$), where "$\rightarrow$" does not occur, is in C''.

(ii) C'' is closed under $\vee$, $\wedge$, $\exists$, $\forall$.

(iii) If $\theta(\vec{x}) \equiv \theta(x_1, \ldots, x_m)$ is a formula in C'' (with at most $x_1, \ldots, x_m$ as free variables), "x", "y" are distinct variables, and φ is in C'', then $\forall\vec{x}(\theta(\vec{x}) \rightarrow \varphi)$; $\forall x(x \in y \rightarrow \varphi)$; $\forall x(\neg x \in y \rightarrow \varphi)$; $\forall x(y \in x \rightarrow \varphi)$; and $\forall x(\neg y \in x \rightarrow \varphi)$ are in C''.

Note that C'' is a subclass of C'. The reason for the restriction $\ll \theta$ is in $C'' \gg$ is that, for $\varphi \equiv \forall z(\theta(z) \rightarrow \psi)$ (f.ex.), φ^- is $\exists z\,(\theta^+(z) \wedge \psi^-)$, so that we need to be sure that θ^+ itself is also in the class we use.

Remark 5.14 The model M indeed proves *Pd*-consistency, as $M \vDash_{Class} \neg(\forall x\ x \in x)^+$. Take for example $x =$ the element coding (ξ, X_ω).

Remark 5.15 Topological models like M will never present a classical $\neq$. This is due to the possibility of bounded quantification (see the definition of the classes C' and C'' in Lemmas 5.12 and 5.13). Consider for example:

$A = \{t \in X_\omega | \forall z \in^+_\omega t \quad z \neq_\omega t\}$

$B = \{t' \in X_\omega | \exists z \in^+_\omega t' \quad z \neq_\omega t'\}$.

This pair is coded in M. But, when $\neq_\omega$ is classical, (A, B) is exactly

$$(\{t \in X_\omega | \neg t \in^+_\omega t\}\,,\ \{t' \in X_\omega | t' \in^+_\omega t'\}).$$

If $x \in X_\omega$ is the element which codes (A, B) we get the Russell-paradox, $x \in^+_\omega x \leftrightarrow \neg x \in^+_\omega x$.

5.3 Topological models for the Pt-case If we want to adapt the construction of 5.2 to the *Pt*-case, some modifications will be necessary. Clearly we should start with $X_1 \stackrel{def}{=} \{(A, B) | A \cup B \subset X_0 \wedge A \cap B = \varnothing\}$, where X_0 is some finite, non-empty set X. Again s will simply be a surjection $X_1 \rightarrow X_0$. The problem which arises here is that, even if $A \cup B \subset X$ and $A \cap B = \varnothing$, we don't get necessarily $im_s A \cap im_s B = \varnothing$ and so $(im_s A, im_s B)$ is not necessarily an element of X, (in 5.2 we used the fact that $A \cup B = X_1$ implies $im_s A \cup im_s B = X_0$). So our inductive definition of the X_i will be:

$$X_{i+1} \stackrel{def}{=} \{(A, B) | A \cup B \subset X_i \wedge (im_s A, im_s B) \in X_i\}$$

(for $i \geq 1$). We extend s to the higher levels as in 5.2. For $(A, B) \in X_{i+1}$, $s(A, B) \overset{def}{=} (im_s A, im_s B)$.
The definitions of X_ω, $\in^+_\omega$, $\in^-_\omega$, $=_\omega$ and of the topology are those of 5.2.
Again X_ω is compact and versions of Lemmas 5.2 and 5.3 can be proved as in 5.2.

The next problem is the definition of $\neq_\omega$. To discuss this, let us introduce the notion of "distinction on level $i+1$". For $a, b \in X_{i+1}$,

$$a\ D_{i+1} b \overset{def}{\leftrightarrow} \exists t \in X_i\ ((t \in^+_i a \wedge t \in^-_i b) \vee (t \in^-_i a \wedge t \in^+_i b)).$$

Note that this can also be expressed as follows: for (A, B) and (A', B') elements of X_{i+1},

$$(A, B) D_{i+1} (A', B') \overset{def}{\leftrightarrow} (A \cap B') \cup (A' \cap B) \neq \varnothing\,.$$

The "distinction on level ω" is exactly $\neq_\omega$ as defined in Section 5.2 (we use the notation "D_ω").

Now it is easy to prove that for $x, y \in X_\omega$,

$$x\ D_\omega y \leftrightarrow \forall i \in \omega \quad x_{i+1} D_{i+1} y_{i+1}.$$

(Again we use the compactness of X_ω and the Extension Lemma).

But then, if one considers the comprehension case $\{x | a = x\}$, one expects $(\{x \in X_\omega | a =_\omega x\}, \{y \in X_\omega | a \neq_\omega y\})$ to be coded in X_ω. If $\neq_\omega$ is D_ω (as in 5.2), then

$$(\{a_1\}, \{x_1 \in X_1 | x_1 D_1 a_1\})$$

should be an element of X_1. And this supposes at least that

$$\{s(a_1)\} \cap \{s(x_1) | x_1 D_1 a_1\} = \varnothing.$$

But this is <u>never</u> realised, for all the elements $a_1 \in X_1$, when X_0 is finite. For there are too many pairs $\{u, v\} \subset X_1$ such that $u D_1 v$, consider for example; $X = \{1, 2, 3\}$.

So we have to adopt another definition for $\neq_\omega$. A natural one is this:

$$x \neq_\omega y \overset{def}{\leftrightarrow} \neg x_0 = y_0.$$

One can easily check that, with this definition, $M \overset{def}{=} \langle X_\omega, \in^+_\omega, \in^-_\omega, =_\omega, \neq_\omega \rangle$ is a *Pt*-model for EXT1.

Further we get properties of M, corresponding to those obtained in 5.2. We obtain a "Coding Lemma;"

Lemma 5.16 *Suppose $A \cup B \subset X_\omega$ and $A \cap B = \varnothing$. Then (A, B) is* **coded** *in M iff A and B are* **closed** *subsets of X_ω and $\forall a \in A\ \forall b \in B\ a \neq_\omega b$.*

Note that this last additional condition has no counterpart in the Coding Lemma in Section 5.2. A version of Lemma 5.11 can again be proved here, as well as the obvious adaptation of Lemma 5.12, namely: M is a *Pt*-model for *PtSF* (the Super Frege version for *Pt*-logic defined in Section 4), but with EXT1 this time.

What does Lemma 5.13 become here, i.e. which version of Frege is modeled by M? By Lemma 5.9, any φ^+ and φ^- (even for φ in C'') will define closed subsets of

X_ω. Further $\{t \in X_\omega | M \vDash \varphi^+(t, \vec{y})\} \cap \{z \in X_\omega | M \vDash \varphi^-(z, \vec{y})\} = \varnothing$ (for $\vec{y}$ in X_ω). But there is a problem for the additional condition in the Coding Lemma, namely:

$$\forall a \in A \quad \forall b \in B \quad a \neq_\omega b.$$

The problem is this. Does $M \vDash \varphi^+(t, \vec{y}) \wedge \varphi^-(z, \vec{y})$ imply $t \neq_\omega z$? When one tries to prove this by induction on the length of φ, all the $\vee, \wedge, \neg, \exists, \forall$ cases and the atomic cases work, except for:

$$\varphi(t, y_1) \equiv y_1 \in t$$

and

$$\varphi(t) = t \in t.$$

This is due to the fact (already mentioned in our discussion concerning the definition of $\neq_\omega$) that one does <u>not</u> have, for all the elements a in X_0 and b, b', in X_1:

$$a \in_0^+ b \wedge a \in_0^- b' \rightarrow \neg s(b) = s(b')$$

or (equivalently):

$$b\ D_1\ b' \rightarrow \neg s(b) = s(b').$$

So we will proceed as follows. We will replace in the formula $\varphi(t, y_1, \dots, y_n)$ (in $\mathcal{L}$, where "$\rightarrow$" does not occur); (i) any occurence of a formula of type "$z \in t$" by "$\exists t' \ (z \in t' \wedge t = t')$" where "$z$", "$t$" are distinct variables; and (ii) any occurence of a formula of type "$t \in t$" by "$\exists t' \ (t \in t' \wedge t = t')$" (where "$t'$" is a new variable, distinct from any variable in φ).

The result of this operation is called $\varphi^\star$. Note that $(\varphi^\star)^+$ is (classically) equivalent to φ^+, but that $(\varphi^\star)^-$ is not, in general, equivalent to φ^-.

However, $\varphi^\star$ and φ *are* (classically) equivalent. This gives us the syntactical variant *F3* of Frege, modeled by M. Again, we have a slightly stronger result, for we can admit some specific occurences of "$\rightarrow$" in φ (as we could in 5.2). In particular, we can show that $\neq_\omega$ can never be classical in this kind of model.

5.4 Topological models with an infinite X_0 One can hope that a more general construction then the one of Section 5.3 will perhaps furnish a *Pt*-model for *F2* (defined in Section 3). The idea is to start with X_0 = a compact metric space X, and X_1 = a compact set of pairs (A, B), where A, B are disjoint, closed subsets of X_0. Further one needs a continuous surjection $s : X_1 \rightarrow X_0$.

However this time the situation is slightly more complicated. Most of the arguments used in 5.3 can be used here. So we will essentially focus on the differences from the preceeding situation.

It is well-known (see [9]) that, if X is a compact metric space, then $\boldsymbol{P}_c(X) \setminus \{\varnothing\}$ (where $\boldsymbol{P}_c(X)$ is the set of the closed subsets of X) is again a compact metric space, when the distance of Hausdorff is used:

$$d_H(A, B) \stackrel{def}{=} \max\{\sup_{b \in B} d(A, b), \sup_{a \in A} d(a, B)\}$$

$$\text{where} \qquad A, B \in \mathcal{P}_c(X) \setminus \{\varnothing\}$$

$$\text{and} \qquad d(x, Y) \stackrel{def}{=} \inf_{y \in Y} d(x, y).$$

The corresponding notion of limit is:

$$\lim_{n\to\infty} A_n = A \qquad \text{iff} \qquad \lim_{n\to\infty} d(A_n, A) = 0 \quad .$$

Further, one can introduce a "set-theoretical" notion of convergence. For A_n a sequence of non-empty closed subsets of X:

$$\overline{\lim} A_n \stackrel{def}{=} \{ \lim_{k\to\infty} a_{n_k} \mid (n_k)_{k\in\omega} \text{ is a strictly increasing sequence in } \omega \text{ and } \forall k \in \omega \; a_{n_k} \in A_{n_k} \text{ and } \left(a_{n_k}\right)_{k\in\omega} \text{ is a convergent sequence}\}$$

$$\underline{\lim} A_n \stackrel{def}{=} \{ \lim_{\substack{n\to\infty \\ n\geq N}} a_n \mid N \in \omega \text{ and } \forall i \geq N \;\; a_i \in A_i \text{ and } (a_i)_{i\geq N} \text{ is a convergent sequence}\}$$

$$\lim_S A_n = A \quad \text{iff}_{def} \quad \overline{\lim} A_n = \underline{\lim} A_n = A.$$

Actually, both notions of limit ("lim" and "$\lim_S$") coincide (see [9]).

Now it is easy to extend this to $\mathcal{P}_c(X)$. We just define:

$$d(\xi, \xi) \stackrel{def}{=} 0$$

and

$$d(\xi, A) \stackrel{def}{=} \mathcal{E} + 1$$

$$(\text{where } A \neq \varnothing \text{ and } \mathcal{E} = \sup_{x,y\in X} d(x, y)).$$

Then ξ is an isolated point in $\mathcal{P}_c(X)$ and one can easily check that the two notions of limit (for $\mathcal{P}_c(X)$) still coincide. Further $\mathcal{P}_c(X)$ is also compact.

So let us start with X_0, a non-empty metric compact space. Then $(\mathcal{P}_c(X_0))^2$ is again a metric compact space. Take $X_1 =$ a *closed* subset $\mathcal{F}$ of $(\mathcal{P}_c(X_0))^2$. So $\mathcal{F}$ is itself is a metric compact space. Now, if s is a continuous surjection: $\mathcal{F} \to X_0$, we can reproduce the construction of 5.3: $X_{n+1} \stackrel{def}{=} \{(A, B) | A$ and B are closed subsets of X_n & $(im_s A, im_s B) \in X_n\}$.

Note that this time we have to add a condition, namely "A, B are closed". This condition was trivially satisfied when X_0 was finite.

One can extend s to the higher levels and define X_ω, $\in^+_\omega$, $\in^-_\omega$, $=_\omega$ as in 5.3. Naturally, in order to get the desired result, we have to put conditions on s and $\mathcal{F}$. These conditions will permit us to define $\neq \omega$ as in 5.2. and to get the strong extensionality EXT2 (for *Pt*-logic).

The conditions are:

(1) $(A, B) \in \mathcal{F} \;\to\; (A \cap B = \xi \;\;\&\;\; (B, A) \in \mathcal{F})$

(2) $[(A, B) \in \mathcal{F} \;\&\; A', B' \in \mathcal{P}_c(X_0) \;\&\; A' \subset A \;\&\; B' \subset B] \;\to\; (A', B') \in \mathcal{F}$

(3) $A \in \mathcal{P}_c(X) \to (A, \varnothing) \in \mathcal{F}$

(4) $[(\forall i \in I \; (A_i, B_i) \in \mathcal{F}) \;\&\; (\bigcup_{i\in I} A_i \in \mathcal{P}_c(X))] \;\to\; (\bigcup_{i\in I} A_i \,,\, \bigcap_{i\in I} B_i) \in \mathcal{F}$ (for *any* set I and *arbitrary* families $(A_i)_{i\in I}$, $(B_i)_{i\in I}$).

(5) $(\{s(t) | x \in^+_0 t \wedge t \in \mathcal{F}\}, \{s(z) | x \in^-_0 z \wedge z \in \mathcal{F}\}) \in \mathcal{F}$ (for any $x \in X_0$; let us recall that $x \in^+_0 (A, B)$ means $x \in A$, etc. . .)

(6) $(\{s(t) | s(t) \in^+_0 t \wedge t \in \mathcal{F}\}, \{s(z) | s(z) \in^-_0 z \wedge z \in \mathcal{F}\}) \in \mathcal{F}$.

Theorem 5.17 *If a compact metric space X_0 ($\neq \varnothing$), a closed subset $\mathcal{F}$ of $(\mathcal{P}(X_0))^2$ and a continuous surjection $s : \mathcal{F} \to X_0$ realize the conditions (1) $\to$ (6), then $M = \langle X_\omega, \in_\omega^+, \in_\omega^-, =_\omega, \neq_\omega \rangle$ is a Pt-model for F2, (with $x \neq_\omega y$ iff$_{(def)}$ $x\ D_\omega y$, see 5.3).*

Remark 5.18 We conjecture that such X_0, $\mathcal{F}$, and s, exist. At present we do not know any examples of them. However the conditions (1), (2), (3), and (4), are easy to satisfy. Here, for example, is a uniform construction permitting to realize these conditions; X_0 is any metric, compact, non-empty space. We define

$$\mathcal{F} = \{(A, \xi) | A \in \mathcal{P}_c(X_0)\}$$

$$\cup\ \{(\xi, B) | B \in \mathcal{P}_c(X_0)\} \cup \{(A, B) | A, B \in \mathcal{P}_c(X_0)\}\ \&\ \delta(A, B) \geq \tfrac{\mathcal{E}}{2}\},$$

where $\quad \mathcal{E} = \sup\limits_{x,y \in X} d(x, y) \quad$ and $\quad \delta(A, B) \overset{def}{=} \inf\limits_{\substack{x \in A \\ y \in B}} d(x, y).$

Remark 5.19 In any case, no finite X_0 can admit such $\mathcal{F}$ and s. One can easily check that condition (5) implies that $tD_1 z \to \neg s(t) = s(z)$ (for $t, z \in \mathcal{F} = X_1$), and that there are always too many pairs $\{t, z\}$ with $tD_1 z$ when X_0 is finite, and X_0, $\mathcal{F}$, s, satisfy the conditions (1) $\to$ (6).

Proof: (Sketch) As any X_i is compact, $\prod\limits_{i \in \omega} X_i$ is compact and so will be X_ω (X_ω is closed). Further one can easily verify that M is indeed a *Pt*-model for EXT2. The problem we met in 5.3 disappears here because in this case $x \neq_\omega y \to \neg x_0 = y_0$, by condition (5). The "Coding Lemma" becomes, in this context:

If A, B are subsets of X_ω and $A \cap B = \varnothing$ then (A, B) is coded in M iff
(i) A, B are closed
(ii) $(\{a_0 | a \in A\}, \{b_0 | b \in B\}) \in \mathcal{F}$.

One further proves that, for φ in $\mathcal{L}$, where "$\to$" does not occur, the sets $A = \{t \in X_\omega | M \models \varphi^+(t, \vec{b})\}$ and $B = \{z \in X_\omega | M \models \varphi^-(z, \vec{b})\}$ satisfy the conditions (i) and (ii) of this Coding Lemma. For condition (i) the proof is the same as in 5.2. So let us just briefly discuss the proof for condition (ii). This proof is by induction on φ:

(1) The cases $\varphi \equiv y_1 \in y_2$ and $\varphi \equiv y_1 \in y_1$ are trivial.

(2) The case $\varphi \equiv y_1 \in t$. Here $A = \{t \in X_\omega | y \in_\omega^+ t\}$ and $B = \{z \in X_\omega | y \in_\omega^- t\}$ (with y fixed in X_ω). We should just verify that $(\{t_0 | t \in A\}, \{z_0 | z \in B\}) \in \mathcal{F}$. Obviously $t \in A \to y \in_\omega^+ a \to y_0 \in_0^+ t_1$ and $z \in B \to y \in_\omega^- b \to y_0 \in_0^- z_1$. So $\{t_0 | t \in A\} = \{s(t_1) | t \in A\} \subset \{s(t_1) | y_0 \in_0^+ t_1 \wedge t_1 \in \mathcal{F}\}$ and $\{z_0 | z \in B\} \subset \{s(z_1) | y_0 \in_0^- z_1 \wedge z_1 \in \mathcal{F}\}$. Combining conditions (2) and (5) we get the desired result:

$$(\{t_0 | t \in A\}\,,\ \{z_0 | z \in B\}) \in \mathcal{F}.$$

Note that in 5.3. condition (2) was automatically satisfied as we took for X_{n+1} the set of <u>all</u> the disjoint pairs of (closed) subsets of X_n. Here we have to verify that the pair which interests us has indeed been selected.

(3) The case $\varphi \equiv t \in t$ follows from condition (6). This is very similar to the preceeding case.

(4) The case $\varphi \equiv t = t$ follows from condition (3).

(5) $\varphi \equiv y_1 = t$ is not really atomic. It corresponds to:

$$A = \{y\}$$

$$B = \{z | z\ D_\omega\ z\} = \{z | \exists t \in X_\omega((t \in^+_\omega y \wedge t \in^-_\omega z) \vee (t \in^-_\omega y \wedge t \in^+_\omega z)\}$$

(for a fixed $y \in X_\omega$). So this case follows once the $\exists, \vee$, and $\wedge$-cases have been established. The same is true for the cases $\varphi \equiv y_1 = y_2$, $\varphi \equiv y_1 = y_1$.

(6) The $\wedge, \vee$-cases are trivial. One has mainly to use condition (4).

(7) The $\forall$-case is the dual of the $\exists$-case and this last one also follows from condition (4).

(8) The $\neg$-case follows trivially from condition (1).

Remark 5.20 Again one can admit specific occurences of "$\rightarrow$" in φ, i.e. φ can be taken in the class C'' (see Section 5.2). To prove this the "set-theoretical" notion of limit is very helpfull (inter alia for the cases "$\forall x \in y$", etc. . .).

Remark 5.21 This construction never produces a classical $\neq$, so that we can't get from it $Pt^{\neq}$-consistency (see Sections 5.2 and 5.3).

Remark 5.22 Conditions (5) and (6) do not seem to be easy to realize. But at least we have a path here which will (perhaps) lead to the proof of the *Pt*-consistency of *F2*.

5.5 Other Topological models One of the initial problems in the construction in Section 5.4 is that the images of disjoint sets are not necessarily disjoint, at least for surjections. However they are disjoint for injections. So one might imagine that a similar construction, using an initial *injection* $i : X_1 \rightarrow X_0$ (instead of a surjection s) could work. This is indeed the case, modulo suitable conditions on X_1, X_0 and i. We give here a slightly different (but equivalent) presentation, which is easier to handle and which is possible because (by the injection i) X_1 can be seen as a "subset" of X_0.

Take X_0 as a topological compact space where any finite subset is closed. Further, suppose that $\in^+_0, \in^-_0$ are closed subsets of X_0^2, which have the "approximation property" (see Lemma 5.10), i.e.: if $y^{(n)}$ is a sequence with limit y, and $x \in^+_0 y$ (respectively: $x \in^-_0 y$), then there exists a sequence $x^{(n)}$, with limit x, such that, for large enough n, $x^{(n)} \in^+_0 y^{(n)}$ (respectively: $x^{(n)} \in^-_0 y^{(n)}$).

Definition 5.23 U = the class of the "urelemente" (atoms) of $(X_0, \in^+_0, \in^-_0)$

$$\stackrel{def}{=} \{x \in X_0 | x^+ \cup x^- = \varnothing\},$$

where $x^+ \stackrel{def}{=} \{t \in X_0 | t \in^+_0 x\}$ and $x^- \stackrel{def}{=} \{t \in X_0 | t \in^-_0 x\}$ (for $x \in X_0$).

Further, suppose U is open and $(X_0, \in^+_0, \in^-_0)$ is extensional for the "sets" (i.e. the non-urelemente) $(x \notin U\ \&\ y \notin U\ \&\ x^+ = y^+\ \&\ x^- = y^-) \rightarrow x = y$. So $X_1 \stackrel{def}{=} X_0 \setminus U$ is closed in X_0. Finally suppose that the following conditions (corresponding to the conditions (1) $\rightarrow$ (6) already met in Section 5.4) hold:

(1) $x \in X_1 \rightarrow (x^+ \cap x^- = \varnothing \quad \wedge \quad \exists y \in X_1(x^+ = y^- \wedge x^- = y^+))$

(2) If A, B are closed subsets of X_0, $A \subset x^+$, $B \subset x^-$ and $x \in X_1$ then (A, B) is "coded" in $(X_0, \in^+_0, \in^-_0)$, i.e. : $\exists y \in X_1 \quad (y^+ = A \quad \wedge \quad y^- = B)$.

(3) If A is a closed subset of X_0, then $\exists x \in X_1\ (x^+ = A \quad \wedge \quad x^- = \varnothing)$.

(4) For any set I and any family $(x_i)_{i \in I}$ of elements of X_0: if $\bigcup_{i \in I} x_i^+$ is closed in X_0, then $(\bigcup_{i \in I} x_i^+, \bigcap_{i \in I} x_i^-)$ is coded in $(X_0, \in_0^+, \in_0^-)$.

(5) $\forall x \in X_0\ (\{z \in X_1 | x \in_0^+ z\}, \{t \in X_1 | x \in_0^- t\})$ is coded in $(X_0, \in_0^+, \in_0^-)$.

(6) $(\{z \in X_1 | z \in_0^+ z\}, \{t \in X_1 | t \in_0^- t\})$ is coded in $(X_0, \in_0^+, \in_0^-)$.

Our construction is this:

$$X_{n+1} \stackrel{def}{=} \{x \in X_n | x^+ \cup x^- \subset X_n\} \quad (\text{for } n \geq 1).$$

Obviously $(X_n)_{n \in \omega}$ is a decreasing chain. Take $X_\omega \stackrel{def}{=} \bigcap_{n \in \omega} X_n$ and define:

$$M = \langle X_\omega, \in_\omega^+, \in^-\omega, =_\omega, \neq_\omega \rangle$$

where $\in_\omega^+, \in_\omega^-, =_\omega, \neq_\omega$ are the restrictions of (respectively) $\in_0^+, \in_0^-, =, \neq$ to X_ω, and where

$$x \neq y \stackrel{def}{\leftrightarrow} (x^+ \cap y^-) \cup (x^- \cap y^+) \neq \varnothing \quad (\text{for } x, y \in X_0).$$

Obviously, X_ω is $\in_0^+$ and $\in_0^-$-transitive, i.e. $x \in X_\omega \to x^+ \cup x^- \subset X_\omega$.
So $x \neq y$ is also equivalent to $\exists t \in X_\omega\ ((t \in_\omega^+ x \wedge t \in_\omega^- y) \vee (t \in_\omega^- x \wedge t \in_\omega^+ y))$.

Theorem 5.24 *M is a Pt-model for F2.*

Proof: The proof is in the manner of that in Section 5.2. One easily checks that any X_n is closed in X_0 and so X_ω is closed in X_0. Note that the inductive proof for "X_n is closed" uses the "approximation property" of $\in_0^+, \in_0^-$. Further one gets a (trivial) "Coding Lemma:" If (A, B) is coded in $(X_0, \in_0^+, \in_0^-)$ and $A \cup B \subset X_\omega$, then (A, B) is coded in M. This can be formulated as follows; $(x \in X_0 \wedge x^+ \cup x^- \subset X_\omega) \to x \in X_\omega$.

Again one can easily prove (as in 5.2) that, for any φ in $\mathcal{L}$, where "$\to$" does not occur, φ^+ and φ^- define closed subsets in M.

Remark 5.25 This construction allows non-metric compact spaces X_0. In 5.2. we took metric ones to be sure that $X_1, X_2, \ldots$ will again be compact. The situation here is simpler with regard to that problem. The $X_1, X_2, \ldots$ are closed in X_0, and so are compact because X_0 is compact.

6 *Interpretations of non-classical Frege in (classical) "positive" theories*

6.1 The language $\mathcal{L}_\tau$ was defined in Section 4.3. The theories we want to consider here have as axioms; extensionality + specific comprehension axioms. The "positive" set theory in $\mathcal{L}_\tau$ studied in [6] and [4] is incompatible with extensionality. We hope that the comprehension principles presented here are not. Naturally these theories are considered in classical logic.

Let us first define inductively the "admissible" terms. Naturally, we adopt the usual conventions about terms in $\mathcal{L}_\tau$, namely: $\tau(z_1, z_2, \ldots, z_n)$ indicates that the set of the free variables of τ is a subset of $\{z_1, z_2, \ldots, z_n\}$, i.e. that τ is of the form $\{t | \varphi(t, z_1, \ldots, z_n)\}$ for some formula φ in $\mathcal{L}_\tau$. Further, a term is said "closed" if it does not contain any free variable.

Inductive rules for A.T. ($\equiv$ admissible terms):

(R1) Any variable is an A.T.

(R2) $\xi \overset{def}{=} \{x|x \neq x\}$ is an A.T.

(R3) If "x_i", "x_j", "x_k" are (not necessarily distinct) variables, then $\{x_i|x_j = x_k\}$ is an A.T. In particular $V \overset{def}{=} \{x|x = x\}$ is an A.T.

(R4) If τ is a *closed* A.T. and "x", "y", "z" are <u>distinct</u> variables, then $\{x|y \in z\}$, $\{x|\tau \in x\}$, $\{x|y \in z\}$, $\{x|y \in \tau\}$ are A.T.

(R5) If τ, τ' are A.T., then so are:

$$\{\tau\} \overset{def}{=} \{x|x = \tau\}$$

$$\tau \cup \tau' \overset{def}{=} \{x|x \in \tau \vee x \in \tau'\}$$

$$\tau \cap \tau' \overset{def}{=} \{x|x \in \tau \wedge x \in \tau'\}.$$

In particular

$$(\tau, \tau') \overset{def}{=} \{\{\tau\}, \{\tau, \tau'\}\}$$

(Kuratowski's ordered pair) is an A.T.

(R6) If τ is a <u>closed</u> A.T. and τ' is any A.T., then the following terms are A.T.:

$$\mathcal{P}\tau \overset{def}{=} \{x|x \subset \tau\}$$

$$\tau^2 \overset{def}{=} \{z|\exists x \in \tau \quad \exists y \in \tau \;\; z = (x, y)\}$$

$$\bigcup_{x\in\tau} \tau' \overset{def}{=} \{t|\exists x \in \tau \;\; t \in \tau'\}$$

$$\bigcap_{x\in\tau} \tau' \overset{def}{=} \{t|\forall x \in \tau \;\; t \in \tau'\}.$$

In particular, for τ a <u>closed</u> A.T. and any variable "x", the following terms are A.T.:

$$\cup\tau \overset{def}{=} \bigcup_{x\in\tau} x$$

$$\cap\tau \overset{def}{=} \bigcap_{x\in\tau} x$$

$$\cup x \overset{def}{=} \bigcup_{z\in V} (\{t|z \in x\} \cap z)$$

$$\cap x \overset{def}{=} \bigcap_{z\in V} (\{t|z \in x\} \cap z).$$

(R7) If $\tau(y)$ is an A.T. (with at most 1 free variable "y", distinct from the variable "x"), then $\{x|\forall y \in x \quad \tau(y) \in x\}$ is an A.T.

(R8) The following are A.T.:

$$\pi_1(x) \overset{def}{=} \{t|\exists a \exists b (x = (a, b) \wedge t \in a)\}$$

$$\pi_2(x) \stackrel{def}{=} \{t|\exists a\exists b(x = (a, b) \wedge t \in b)\}$$

In our theories, these terms are the obvious projections (i.e. $\pi_1((a, b)) = a$; $\pi_2((a, b)) = b$; $x \notin V^2 \rightarrow \pi_1(x) = \pi_2(x) = \varnothing$).

(R9) If "x_i", "x_j", "x_ℓ" are (not necessarily distinct) variables, then $\{x_i|x_j \in \pi_k(x_\ell)\}$ is an A.T. (for $k = 1, 2$).

(R10) If "x", "y" are distinct variables, then $\{x|x \in y \wedge \pi_k(x) \subset y\}$ is an A.T. (for $k = 1, 2$).

(R11) $\{x|\pi_1(x) \cap \pi_2(x) = \varnothing\}$ is an A.T. Our theory T has as axioms the usual extensionality:

$$(\forall t(t \in x \leftrightarrow t \in y)) \leftrightarrow x = y,$$

plus the obvious comprehension for the A.T., i.e.: if $\tau = \{x|\varphi(x, \vec{y})\}$ is an A.T., then

$$\forall\vec{y}\,\forall x(x \in \tau \leftrightarrow \varphi(x, \vec{y})).$$

Remark 6.1 Note that any sub-term of an admissible term is itself admissible.

Remark 6.2 The situation here is slightly unusual, in the sense that the replacement of variables in an admissible term by admissible terms does not necessarily produce an admissible term (i.e. the class of the A.T. is not closed under replacements); f.ex.: $\{x|y \in z\}$ is an A.T. but $\{x|y \in y\}$ is not an A.T.

Note that (in T) a non-admissible term can be a set (but it cannot be used to build up more complex A.T.); f.ex.: $\{x|y \in y\}$ is a set (for any y), because our comprehension schema guarantees (for the A.T. $\{x|y \in z\}$):

$$\forall y\forall z\forall t(t \in \{x|y \in z\} \leftrightarrow y \in z)$$

and so:
$$\forall y\forall t(t \in \{x|y \in y\} \leftrightarrow y \in y).$$

Remark 6.3 We could find no "trick" to get the "Russell set" back into T. The shape of this Russell set (f.ex. in [6]) could be:

$$\{y|\varnothing = \{x|y \in y\}\},$$

but it does not seem possible to show that this is a set in T (see Remark 6.2)). We hope that further investigations will bring a proof of the (relative) consistency of T.

Theorem 6.4 *There exists an interpretation of F2 for Pt-Logic in T.*

Proof: Let us first give the intuition behind the construction. Define the relations $\in^+, \in^-, D$ on V by:

$$x \in^+ y \stackrel{def}{\leftrightarrow} x \in \pi_1(y)$$

$$x \in^- y \stackrel{def}{\leftrightarrow} x \in \pi_2(y)$$

$$x\ D\ y \stackrel{def}{\leftrightarrow} ((\exists t \in^+ x \quad t \in^- y) \vee (\exists z \in^- x \quad z \in^+ y)).$$

Further, start with

$$H \stackrel{def}{=} \{(a, b)|a \cap b = \varnothing\},$$

and define the operation $\star$:

$$h^\star \stackrel{def}{=} \{x \in h | (\forall y \in^+ x \quad y \in h) \wedge (\forall z \in^- x \quad z \in h)\}.$$

Take $\tilde{H} = \cap\{H, H^\star, H^{\star\star}, \ldots\}$. The desired interpretation M will take $\tilde{H}$ as its universe, with the restrictions of $\in^+$, $\in^-$, $=$ and D to $\tilde{H}$.

As we work in T, we have to show that we can reproduce this construction in a satisfying way.

The definitions of $\in^+$, $\in^-$, D on V can stay as they are. For H, take the A.T.:

$$V^2 \cap \{t | \pi_1(t) \cap \pi_2(t) = \varnothing\}$$

(by rules 3, 5, 11). Further, $h^\star$ is the A.T. :

$$V^2 \cap \{x | x \in h \wedge \pi_1(x) \subset h\} \cap \{x | x \in h \wedge \pi_2(x) \subset h\}$$

(by rules 3, 5, 10). Using rules 4, 5, 6, 7, we get the A.T. :

$$\tilde{X} \stackrel{def}{=} \cap (\boldsymbol{P}^2 H \cap \{x | H \in x\} \cap \{x | \forall h \in x \quad h^\star \in x\}).$$

Intuitively $\tilde{X}$ is the set $\{H, H^\star, H^{\star\star}, \ldots\}$. Note that this is a descending chain:

$$H \supset H^\star \supset H^{\star\star} \ldots$$

Finally take the A.T. (by rule 6):

$$\tilde{H} \stackrel{def}{=} \cap \tilde{X}.$$

Intuitively, $\tilde{H}$ is $H \cap H^\star \cap H^{\star\star} \cap \ldots$

We can prove now the following "Transitivity Lemma".

Lemma 6.5 $\quad (a \in^+ b \in \tilde{H} \quad \vee \quad a \in^- b \in \tilde{H}) \to a \in \tilde{H}$

Proof: Suppose f.ex.: $a \in^+ b \in \tilde{H}$; $b \in \tilde{H}$ means that $\forall h \in \tilde{H} \quad b \in h$. As $\tilde{X}$ is obviously closed under the operation $h \mapsto h^\star$, we get $\forall h \in \tilde{H} \quad b \in h^\star$. But then, by the definition of $\star$; $a \in^+ b \in h^\star$ implies $a \in h$. So $\forall h \in \tilde{X}$ $a \in h$, and we conclude $a \in \tilde{H}$.

Modulo this Transitivity Lemma, it is easy to check that $M = (\tilde{H}, \in^+_M, \in^-_M, =_M, \neq_M)$ is actually a *Pt*-model of EXT2 (with $\in^+_M$ the restriction of $\in^+$ to $\tilde{H}$, $\in^-_M$ the restriction of $\in^-$ to $\tilde{H}$, $=_M$ the restriction of $=$ to $\tilde{H}$, and $\neq_M$ the restriction of D to $\tilde{H}$). Note that, for $x, y \in \tilde{H}$,

$$x \; D \; y \leftrightarrow \exists t \in \tilde{H}((t \in^+ x \wedge t \in^- y) \vee (t \in^- x \wedge t \in^+ y))$$

(by the "Transitivity Lemma").

The next step is the following "Coding Lemma."

Lemma 6.6 $\quad \forall a \forall b((a \subset \tilde{H} \wedge b \subset \tilde{H} \wedge a \cap b = \varnothing) \to (a, b) \in \tilde{H}).$

Proof: One can easily check that, for $a \subset \tilde{H}$, $b \subset \tilde{H}$ and $a \cap b = \varnothing$, we have $\forall h \in \tilde{X}$ $(a, b) \in h$, and so $(a, b) \in \tilde{H}$.

The last step in our proof of Theorem 6.4 consists in proving that, for any formula $\varphi(t, \vec{y})$ in $\mathcal{L}$, where "$\rightarrow$" does not occur, the terms $\{x | x \in \tilde{H} \wedge \varphi^+_M(x, \vec{y})\}$ and $\{x | x \in \tilde{H} \wedge \varphi^-_M(x, \vec{y})\}$ are equivalent (i.e. equal modulo T) to admissible terms (naturally, the notations φ^+_M, φ^-_M refer to the obvious interpretations of φ^+, φ^- in M).

Modulo our rules for A.T., the proof of this is just a routine verification. The atomic cases follow from rules 3, and 9, except for the case "$x_j \neq x_k$". However

$$\{x_i \in \tilde{H} | (x_j D x_k)_M\} =$$

$$\{x_i \in \tilde{H} | \exists t \in \tilde{H} ((t \in \pi_1(x_j) \wedge t \in \pi_2(x_k)) \vee (t \in \pi_2(x_j) \wedge t \in \pi_1(x_k)))\} =$$

$$\tilde{H} \cap \left[\bigcup_{t \in \tilde{H}} ((\{x_i | t \in \pi_1(x_j)\} \cap \{x_i | t \in \pi_2(x_k)\}) \cup (\{x_i | t \in \pi_2(x_j)\} \bigcap \{x_i | t \in \pi_1(x_k)\})) \right]$$

which is also an A.T. by rules 5, 6, and 9 (note that $\tilde{H}$ is a closed A.T.).

The connective and quantifier cases are completely obvious by rules 5, and 6.

So we conclude that M is a *Pt*-model for the comprehension scheme:

$$\exists z \forall x (x \in z \overset{s}{\leftrightarrow} \varphi(x, \vec{y}))$$

(for φ in $\mathcal{L}$, without "$\rightarrow$"). Indeed, by the last step,

$$a = \{x \in \tilde{H} | \varphi^+_M(x, \vec{y})\}$$

$$\text{and} \qquad b = \{x \in \tilde{H} | \varphi^-_M(x, \vec{y}))$$

are sets in T, and so, by the Coding Lemma, $(a, b) \in \tilde{H}$.
This $z \overset{def}{=} (a, b)$ exactly realizes

$$(\forall x (x \in z \overset{s}{\leftrightarrow} \varphi(x, \vec{y})))^+_M.$$

6.2 One can get variants of Theorem 6.4 by modifying the rules for the admissible terms (and so the corresponding theory T) and some details in the construction.

Let us give some examples here.

Definition 6.7 We can get an interpretation of *F1* for $Pt^{\neq}$-Logic in the theory T', obtained by strenghening rule 2: "$\{x_i | x_j \neq x_k\}$ is an A.T. for "x_i", "x_j", "x_k" (not necessarily distinct) variables". Naturally, the interpretation of $\neq$ in M should be:

$$\text{"}x \neq_M y \quad \text{iff} \quad \neg x = y\text{"}$$

(instead of "$x \neq_M y$ iff $x \; D \; y$").

Definition 6.8 We already have topological models for *F1* in *Pd*-Logic (see Section 5.2). However, let us mention that one gets an interpretation of *F1* for *Pd*-Logic by the construction in 6.2, modulo the following modifications. Take $H \overset{def}{=} V^2 \cap \{t | \pi_1(t) \cup \pi_2(t) = V\}$; replace rule 11 by the "dual" rule: "$\{x | \pi_1(x) \cup \pi_2(x) = V\}$ is an A.T.".

Definition 6.9 Call T'' the theory obtained by strengthening rule 2 as in Definition 6.7 and replacing rule 11 as in Definition 6.8. Further, define $\tilde{H}$ as in 6.8 and $\neq_M$ as in 6.7.

This produces an interpretation of *F1* for $Pd^{\neq}$-logic, in T''. Note that, as *F1* is $Pd^{\neq}$- and $Pt^{\neq}$-classical, the consistency of T' or T'' suffices to get the $Pt^{\neq}$-consistency and the $Pd^{\neq}$-consistency of *F1*.

We have here discussed the Frege versions, but one can easily check that our theories T, T', T'' also interpret suitable versions of Super Frege.

6.3 We called our theories T, T', T'' "positive". Actually they are not really "positive theories" but rather "generalized positive". As a "generalized positive set theory" already exists in the literature (see [4], [11], and [8]), we will briefly compare it with T and its variants. Let us recall here that the "generalized positive formulas" (in the language $\mathcal{L}$) are defined inductively by:

(1) Atomic formulas are G.P.F. ($\equiv$ "generalized positive formulas"),

(2) If φ, ψ are G.P.F. and "x", "y" are distinct variables, then $\varphi \wedge \psi$, $\varphi \vee \psi$, $\exists x \varphi$, $\forall x \varphi$, $\forall x \in y\, \varphi$ are G.P.F.,

(3) If $\theta(x)$ is an arbitrary formula in $\mathcal{L}$, with at most 1 free variable "x", and φ is a G.P.F., then $\forall x(\theta(x) \rightarrow \varphi)$ is a G.P.F.

The "generalized positive set theory" (GPST) has as axioms; extensionality + comprehension for the G.P.F. This theory has topological models in ZF (once more see [4], [11], and [8]). Unhappily we were unable to find a suitable adaptation of our construction of 6.2 in GPST, so we had to "create" T and its variants. T also uses comprehension for "generalized positive" formulas, but these are not GPF, even when translated in the obvious way as formulas in $\mathcal{L}$. We can even show that T is incompatible with GPST (i.e. T + GPST is inconsistent), so that neither of these two theories is a fragment of the other one, if (as we conjecture) T is consistent. Note that T could be translated in $\mathcal{L}$; it is only for clarity that we prefered to express it in $\mathcal{L}_\tau$.

To prove this inconsistency of $T+$ GPST take the admissible term

$$A \overset{def}{=} \{x | x \in V^2 \wedge \pi_1(x) \cap \pi_2(x) = \varnothing\}.$$

Then the formula "$\neg t \in t$" is equivalent to a GPF (in T + GPST):

$$\neg t \in t \leftrightarrow \exists z \in A\, \exists y(t \in y \wedge z = (t, y)).$$

Indeed if $\neg t \in t$, take $y = \{t\}$ and $z = (t, y) \in A$. The other direction is trivial.

So $\{t | \neg t \in t\}$ is a set in $T+$ GPST and we get Russell's paradox. As T' is a strengthening of T, T' + GPST is also inconsistent. We don't know whether $T''+$ GPST is inconsistent or not. We conclude by conjecturing that T, T', and T'' are consistent.

REFERENCES

[1] Arruda, A.I., "A Survey of paraconsistent logic," pp. 1–41 in *Mathematical logic in Latin America*, North-Holland, Amsterdam,1980.

[2] Boffa, M., "Décoration ensembliste de graphes par approximations," *Cahiers du Centre de Logique*, vol. 7 (1992), pp. 45–50.

[3] Crabbé, M., "Soyons positifs: la complétude de la théorie naive des ensembles," *Cahiers du Centre de Logique*, vol. 7 (1992), pp. 51–68.

[4] Forti, M. and R. Hinnion, "The consistency problem for positive comprehension principles," *The Journal of Symbolic Logic*, vol. 54 (1989), pp. 1401–1418.

[5] Gilmore, P.C., "The consistency of partial set theory without extensionality, Axiomatic set theory," *Proceedings of Symposia in Pure Mathematics: American Math. Society*, vol. 13 (1974), pp. 147–153.

[6] Hinnion, R., "Le paradoxe de Russell dans des versions positives de la théorie naive des ensembles," *Comptes Rendus des Séances de l'Académie des Sciences*, Série I , vol. 304 (1987), pp. 307–310.

[7] Hinnion, R., "The problem of the consistency of Frege's naive set theory, with extensionality, for a 3-valued logic," *Mengenlehre Kolloquium Oberwolfach*, Tagungsbericht 4 (1989).

[8] Hinnion, R., "Stratified and positive comprehension seen as superclass rules over ordinary set theory," *Zeitschrift fur Math. und Grundlagen d. Math.*, vol. 36 (1990), pp. 519-534.

[9] Kuratowski, C., "Topologie", Académie Polonaise, Tome XX, vol. 2 (1952), pp. 20–23.

[10] Lenzi, G., "Weydert's SF_3 has no (recursive) term model", *Bulletin de la Soc. Math. de Belgique*, vol. 3 (1992), pp. 311–327.

[11] Weydert, E., "How to approximate the naive comprehension scheme inside of classical logic", *Bonner Math. Schriften*, Nr. 194 (1989), pp. 1–40.

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