Notre Dame Journal of Formal Logic
Volume 34, Number 2, Spring 1993

On Closed Elementary Cuts in Recursively Saturated Models of Peano Arithmetic

BOŻENA PIEKART

Abstract We strengthen some results of Kotlarski [5] by showing that there exist infinitely many essentially different closed elementary cuts in each countable and recursively saturated model for PA.

1 Introduction and notation Let PA denote Peano Arithmetic in any of its usual formalizations. For $M \models \mathrm{PA}$ we set

$$Y^M = \{N \subseteq_e M : N \prec M\};$$

when no confusion arises we omit the superscript M. We shall study the family Y^M under the assumption that M is countable and recursively saturated. We use standard terminology and notation, assuming that the reader knows the notion of recursive saturation of models and has got some knowledge of initial segments in models of PA. See Kaye [2] for all the necessary background.

The present paper was written under Professor H. Kotlarski's direction and has grown out from his earlier papers [4] and [5].

We have organized the paper as follows. In this and the next two sections, we review earlier results on elementary cuts in countable and recursively saturated models of arithmetic. In Section 4 we prove our main result; i.e., we construct infinitely many $a_k : k \in \omega$ in M so that every $M[a_k]$ is closed and gaps $[a_k]$, $k \in \omega$ are essentially different.

Before we state results of [4] and [5], we introduce some more notation necessary for their formulation.

Let $Y_1 = \{N \in Y : N$ is not recursively saturated$\}$.

For $a \in M$ we denote

$$M(a) = \{x \in M\text{: for some parameter-free term } t(\vartheta),\ M \models x < t(a)\}$$

$$M[a] = \{x \in M\text{: for each parameter-free term } t(\vartheta),\ M \models t(x) < a\}.$$

Received July 23, 1992; revised March 27 1993

For convenience we shall use the second symbol only when $M[a]$ is non-empty (i.e., no definable element of M is greater than a); otherwise the symbol $M[a]$ will be treated as undefined.

Let us notice that $M(a)$ is the least elementary cut containing a, and $M[a]$ is the largest elementary cut not containing a (provided $M[a]$ is defined).

Theorem 1.1 *Let $M \models \mathrm{PA}$ be countable and recursively saturated. Then*
(i) *if $A \subseteq Y$ has no greatest element with respect to the inclusion, then $\bigcup A \in Y \setminus Y_1$,*
(ii) *for $N \in Y$ we have $N \in Y_1$ iff there exists an $a \in M$ such that $N = M(a)$,*
(iii) *Y_1 is of the order type of $1 +$ rationals,*
(iv) *Y is of the order type of Cantor set 2^ω, with its usual ordering,*
(v) *for each $a \in M$ greater than any definable element of M we have $M[a] \in Y \setminus Y_1$,*
(vi) *$Y \setminus Y_1$ is of the order type reals $+1$.*

Proof: See [4].

2 *Isomorphisms of elementary cuts*

The following fact is known.

Theorem 2.1 *Let $M \models \mathrm{PA}$, be countable and recursively saturated, and let $N_1, N_2 \in Y \setminus Y_1$. Then N_1 is isomorphic to N_2.*

Proof: See, e.g., Smorynski [6].

The question if all cuts $N \in Y_1$ are isomorphic has been posed by Roman Kossak. The answer is negative.

Corollary 2.2 (This result was obtained by H. Kotlarski and by C. Smorynski.) *If $M \models \mathrm{PA}$ is countable and recursively saturated then there exists an infinite family $A \subseteq Y_1$ such that if $N_1, N_2 \in A$ then N_1 is not isomorphic with N_2.*

Proof: See [5] or [7].

Let us recall that for $n \in \omega$, Tr_n denotes the natural truth definition for Σ_n-formulas. Kotlarski [5] defines the following functions F_n in PA:

$$F_n(0) = \text{The Gödel number of the formula } \vartheta_2 = \vartheta_1 + 1.$$

$$F_n(x+1) = \min y : \forall \varphi \leq F_n(x) \forall u \leq F_n(x)\ \varphi \in \Sigma_n \Rightarrow (\exists w \, \mathrm{Tr}_n(\varphi, u^\frown w) \Rightarrow \exists w \leq y \, \mathrm{Tr}_n(\varphi, u^\frown w)).$$

Thus $F_n(x+1)$ is the maximum of all examples for all Σ_n-formulas $\varphi \leq F_n(x)$ with all parameters $u \leq F_n(x)$.

The simplest properties of the funtions F_n are

Lemma 2.3
(i) $\mathrm{PA} \vdash \forall a \, F_n(a) < F_n(a+1)$,
(ii) *the formula $y = F_n(x)$ is Σ_{n+1},*
(iii) *if t is a Σ_n-term then for some a* $\mathrm{PA} \vdash \forall b > a \ t(b) < F_n(b)$.

Proof: Obvious.

Let $C_n(x)$ be formula $\exists y\, x = F_n(y)$ of PA.
Let $l_n(x) = \max z < x : C_n(z)$ and

$$p_n(x) = \min z > x : C_n(z).$$

The following lemma is obvious.

Lemma 2.4 *The following sentences are provable in* PA:

$$\forall x \exists y [C_n(x) \Rightarrow (l_n(x) = F_n(y) \;\&\; p_n(x) = F_n(y+2))$$
$$\& \neg C_n(x) \Rightarrow (l_n(x) = F_n(y) \;\&\; p_n(x) = F_n(y+1))].$$

The main lemmas about the funtions F_n are the following:

Lemma 2.5 *Let $M \models$ PA, A any infinite subset of $\omega \setminus \{0\}$ and let $a \in M$ be greater than any definable element of M and such that, for $n \in A$,*

$$M \models F_{n-1}(l_n(a)) < a \;\&\; F_{n-1}(a) < p_n(a). \tag{1}$$

Then

$$M(a) \setminus M[a] = \bigcup_{n \in A} (l_n(a), p_n(a)). \tag{2}$$

Proof: Let $x \in (l_n(a), p_n(a))$ for some $n \in A$. Then $l_n(a)$ is definable from x, indeed $l_n(a) = \max y < x : C_n(y)$ and $p_n(a)$ is definable from x as $p_n(a) = \min y > x : C_n(y)$ (cf. Lemma 2.4). Thus $x \in M(a)$, indeed $x < t(a)$ where $t = p_n$. Moreover, if $x \in M[a]$ then $l_n(a) \in M[a]$ because $M[a]$ is an elementary submodel of M. But then $p_n(a) = \min y > l_n(a) : C_n(y)$ and $p_n(a) \in M[a] < a$. Contradiction and $x \notin M[a]$.

Let us take $x \in M(a) \setminus M[a]$. There exist terms t,s such that $x < t(a)$, $s(x) \geq a$ and t,s are Σ_{n-1} for some $n \in A$. We show that $l_n(a) < x < p_n(a)$.

We claim that $x > l_n(a)$. Indeed, otherwise $x \leq l_n(a)$ and so $F_{n-1}(x) \leq F_{n-1}(l_n(a)) < a$. But we have $F_{n-1}(x) > s(x)$ by 2.3 (iii). This implies $s(x) < a$, hence we obtain a contradiction with the choice of s.

We also have $x < t(a) < F_{n-1}(a) < p_n(a)$, so $l_n(a) < x < p_n(a)$.

Lemma 2.6 *Let r be a natural number. Then for $n > 2$ there exists a natural number a such that* PA $\vdash \forall b > a$"$Card[C_{n-1} \cap (F_{n-1}(l_n(b)), \max\{e : F_{n-1}(e) < p_n(b)\})]$ *is greater than* $2^{r \cdot F_{n-1}(l_n(b))}$".

Intuitively speaking Lemma 2.6 states that between $l_n(b)$ and $p_n(b)$ (in fact between $F_{n-1}(l_n(b))$ and $\max\{e : F_{n-1}(e) < p_n(b)\}$) there are very many values of the function F_{n-1}.

Proof: See [5].

3 Closed elementary cuts For every model M we denote by $Aut(M)$ the group of all automorphisms of M. $X \subseteq M$ is *closed* iff for each $b \in M \setminus X$ there exists a $g \in Aut(M)$ such that $g(b) \neq b$ and, for all $x \in X$, $g(x) = x$.

Observe that if $M \models$ PA and $X \subseteq M$ is closed then X is the universe of an elementary submodel of M.

Kotlarski [5] proved three non-coarse theorems about such cuts.

Theorem 3.1 *If M is countable and recursively saturated and $N \in Y$ is not closed then there exists $b \in M$ such that $N = M[b]$.*

From this theorem follows that all $N \in Y$ except countably many are closed; the question as to whether models of the form $M[a]$ are closed was settled by Kotlarski in [5].

Theorem 3.2 *There exists a recursive consistent parameter-free type p in one free variable such that, for every $M \models$ PA and every $b \in M$ which realizes p, $M[b]$ is not closed.*

In order to state our results in a convenient form, let us introduce the following notions.

If $M \models$ PA and $a \in M$, the set $[a) = M(a) \setminus M[a]$ will be called the *gap around a*; once again, we define this notion only if $M[a]$ is defined, i.e., if $a > M(0)$.

We say that two gaps $[a), [b)$ in M are *essentially different* if $M(a)$ is not isomorphic to $M(b)$. It is easy to see that $[a), [b)$ are essentially different iff no $c \in [b)$ realizes $tp(a)$, equivalently, no $c \in [a)$ realizes $tp(b)$.

An analysis of the proof of 3.2 (see [5]) immediately gives the existence of infinitely many types p_k so that if a_k realizes p_k, for all k, in M then $[a_k)$ and $[a_r)$ are essentially different for $k \neq r$ and all $M[a_k]$ are not closed. (In personal communication, Kotlarski pointed out that this result may also be obtained by using minimal types in the sense of Gaifman [1]. Namely the proof of Theorem 3.9 in Gaifman [1] yields continuum many independent minimal types. It is not difficult to verify that infinitely many of them are coded in M, so they are realized in M because M is recursively saturated. Moreover if $a, b \in M$ realize two independent minimal types then $M(a)$ is not isomorphic with $M(b)$.)

Theorem 3.3 *There exists a recursive and consistent (with every completion of* PA*) parameter-free type q in one free variable such that for every countable and recursively saturated model M for* PA *and every b realizing q in M, $M[b]$ is closed.*

We strengthen this result by constructing infinitely many recursive, consistent (with each completion of PA) types q_k, $k \in \omega$ such that if, for all k, a_k realizes q_k in M then gaps $[a_k)$ and $[a_r)$ are essentially different for $k \neq r$ and all $M[a_k]$ are closed.

4 Non-isomorphic closed cuts

Let Od = set of natural odd numbers and Ev = set of natural even numbers.

Below we will define countable infinite family $\{q_r : r \in \omega\}$ of recursive types of PA such that the lemma mentioned below is true.

Lemma 4.1

(i) *For any $r \in \omega$ q_r is consistent (Exactly: if $\psi_0, \dots, \psi_{p-1} \in q_r$ then* PA $\vdash$ $\forall u \exists e > u \bigwedge_{j<p} \psi_j(e)$).

(ii) *If a realizes q_r for any r then:*

(a) $M(a) \setminus M[a] = \bigcup_{n>2, n \in Ev} (l_n(a), p_n(a))$,

(b) *for all natural even* $n > 2$ $M \models \neg C_n(a)$ *and there exists an automorphism* g *of* M *such that* $g(l_n(a)) \neq l_n(a)$ *and* $\forall x < l_{n+1}(a)\ g(x) = x$.

(iii) *If* a_l *realizes* q_l *and* a_k *realizes* q_k *(for* $k \neq l$*) then* $M(a_l) \not\equiv M(a_k)$.

We show that any such sequence meets our demands.

Theorem 4.2 *If* a_i *realizes type* q_i *and* a_j *realizes type* q_j *for* $i \neq j$ *then*:

(i) $M(a_i) \not\equiv M(a_j)$,

(ii) *for every* $i \in \omega$ $M[a_i]$ *is closed, i.e.,*

$$\forall b \notin M[a_i]\, \exists g \in Aut(M)\ g(b) = b \text{ and } \forall x \in M[a_i]\ g(x) = x.$$

Proof:

(i) follows directly from (iii) Lemma 4.1.

(ii) Let a_i realize q_i. Let us take any $b \notin M[a_i]$. Then either $b \notin M(a_i)$ or $b \in M(a_i) \backslash M[a_i]$.

Case 1. If $b \notin M(a_i)$ then there exists automorphism g such that $g(b) \neq b$ and $g \upharpoonright M(a_i) = id$, because otherwise $M(a_i)$ would not be closed and by Theorem 3.1 there would exist $c \in M$ such that $M[c] = M(a_i)$, which is impossible by Theorem 1.1. For such g $g(b) \neq b$ and $g \upharpoonright M[a_i] = id$.

Case 2. For any $b \in M(a_i) \backslash M[a_i]$ there exists a natural even $n > 2$ such that $l_n(a_i) < b < p_n(a_i)$ (by (ii) Lemma 4.1).

Let us take g such that $\forall x < l_{n+1}(a_i)\ g(x) = x$ (in particular $g \upharpoonright M[a_i] = id$) and $g(l_n(a_i)) \neq l_n(a_i)$.

Since $p_n(a_i) = \min z > a_i : C_n(z)$, either $g(l_n(a_i)) \geq p_n(a_i)$ or $g(p_n(a_i)) \leq l_n(a_i)$. If $g(l_n(a_i)) \geq p_n(a_i)$ then we have $b < p_n(a_i) \leq g(l_n(a_i)) < g(b)$; otherwise $g(b) < g(p_n(a_i)) \leq l_n(a_i) < b$. Consequently in both cases $g(b) \neq b$.

Therefore it is sufficient to find the family of types for which Lemma 4.1 is true.

Let $\mathbf{2}_0(x) = x$, $\mathbf{2}_{m+1}(x) = \mathbf{2}^{\mathbf{2}_m(x)}$, and let $\{\varphi_i : i \in \omega\}$ be some recursive enumeration of the formulas of PA. For $r \in \omega$ we put

$$\begin{aligned} q_r = {} & \{F_{n-1}(r + l_n(a)) = l_{n-1}(a) : n \in Od,\ n > 3\} \\ & \cup \{F_{n-1}(l_n(a)) < a \ \&\ F_{n-1}(a) < p_n(a) : n \in Ev,\ n > 2\} \\ & \cup \{\neg C_n(a) \ \&\ \exists d \exists w \neq l_n(a)\ (\mathbf{2}_m(l_{n+1}(a)) < d \\ & \&\ \forall x < d \bigwedge_{i<m} [\varphi_i(x, l_n(a)) \Leftrightarrow \varphi_i(x, w)]) : n > 2,\ n \in Ev,\ m \in \omega\}. \end{aligned}$$

We show that, for the family of types defiend in this way, Lemma 4.1 is true.

Proof: We first prove (iib). Let us fix $n (n \in Ev,\ n > 2)$.

Suppose a realizes q_r.

Let us consider an auxiliary type

$$\begin{aligned} \Gamma(d, w) = {} & \{\mathbf{2}_k(l_{n+1}(a)) < d : k \in \omega\} \\ & \cup \left\{\forall x < d \bigwedge_{i<m} [\varphi_i(x, l_n(a)) \Leftrightarrow \varphi_i(x, w)] : m \in \omega\right\} \cup \{w \neq l_n(a)\}. \end{aligned}$$

Γ is consistent because a realizes q_r (for any r). Let us pick d, w realizing Γ.

The following lemma is known:

Lemma 4.3 (Kotlarski, Smorynski, Vencovska) *Let $M \models \mathrm{PA}$ be countable and recursively saturated. Let $a, b, c, d \in M$ be such that:*
(i) *$M \models \mathbf{2}_n(c) < d$ for all n,*
(ii) *$M \models \forall x < d\ [\varphi(x,a) \Leftrightarrow \varphi(x,b)]$ for all formulas φ. Then there exists an automorphism g of M such that $g(a) = b$ and, for all $x < c$ $g(x) = x$.*

Proof: See, e.g., Kotlarski [5] or Kaye, Kossak, Kotlarski [3].

By the above, there exists $g \in Aut(M)$ s.t. $g(l_n(a)) = w \neq l_n(a)$ and $\forall x < l_{n+1}(a)\ g(x) = x$.

(ii.a) follows directly from Lemma 2.4 for $A = Ev \setminus \{2\}$.

Now we verify (iii) of Lemma 4.1. We only need to prove that: if $l \neq k$, a_k realizes q_k and a_l realizes q_l then no $u \in [a_k)$ realizes the type q_l; because if $M(a_k) \cong M(a_l)$, then there exists automorphism f such that $f(a_l) \in M(a_k) \setminus M[a_k]$ and $f(a_l)$ realizes q_l.

Let us assume that a_k realizes q_k and some $u \in M(a_k) \setminus M[a_k]$ realizes q_l for $l \neq k$. Then we have:

$$\bigvee_{\substack{n>3 \\ n \in Od}} F_{n-1}(k + l_n(a_k)) = l_{n-1}(a_k) \tag{1}$$

and

$$\bigvee_{\substack{n>3 \\ n \in Od}} F_{n-1}(l + l_n(u)) = l_{n-1}(u). \tag{2}$$

Since $u \in M(a_k) \setminus M[a_k)$ then by (ii) there exists natural odd $n > 3$ such that $l_{n-1}(a_k) < u < p_{n-1}(a_k)$ and $M \models \neg C_{n-1}(a_k)$. As a consequence we obtain $F_{n-1}(l + l_n(u)) = l_{n-1}(u) = l_{n-1}(a_k) = F_{n-1}(k + l_n(a_k))$. Functions F_n are $(1-1)$, and so $l + l_n(u) = k + l_n(a_k)$. By (1) and (2) we have $l_n(u) < l_{n-1}(u) < u$ and $l_n(a_k) < l_{n-1}(a_k) = l_{n-1}(u) < a_k$; hence $l_n(u) = l_n(a_k)$ and $k = l$.

(i) We prove that q_r is consistent for any fixed r.

For convenience we introduce the following abbreviations:

$$A_n^r(a) : F_{n-1}(r + l_n(a)) = l_{n-1}(a)$$

$$E_n(a) : F_{n-1}(a) < p_n(a)\ \&\ F_{n-1}(l_n(a)) < a$$

and

$$B_{m,n}(a) : \exists d \exists w \neq l_n(a) (\mathbf{2}_m(l_{n+1}(a)) < d$$
$$\&\ \forall x < d \bigwedge_{i<m} [\varphi_i(x, l_n(a)) \Leftrightarrow \varphi_i(x,w)])\ \&\ \neg C_n(a).$$

Thus $q_r = \{A_n^r(a) : n \in Od,\ n > 3\} \cup \{B_{m,n}(a) : m \in \omega,\ n \in Ev,\ n > 2\} \cup \{E_n(a) : n \in Ev,\ n > 2\}$.

Now we observe that for all n, m

$$\mathrm{PA} \vdash B_{m+1,n}(\vartheta) \Rightarrow B_{m,n}(\vartheta). \tag{3}$$

Let us take any finite subset Δ_r of q_r and the greatest n such that the formula $A_n^r(a)$, $E_n(a)$ or some formula of the form $B_{m,n}(a)$ is in Δ_r. For convenience we assume that, for this choice n, n is even and Δ_r contains both the formula $A_n^r(a)$ and the formula of the form $B_{m,n}(a)$. By (3) we may assume that $B_{m,n}(a)$ is the only formula of this form, with index n, which occurs in Δ_r.

Let us denote $E = (F_n(F_{n+1}(x)), \max\{e : F_n(e) < F_{n+1}(x+1)\})$.

Fix any non-standard x. By Lemma 2.6 if x is sufficiently big, there are more than $2^{m \cdot F_n(F_{n+1}(x))}$ elements of $C_n \cap E$.

There exists only $2^{m \cdot F_n(F_{n+1}(x))}$ sets of pairs of the form ⟨formula, parameter⟩ where formula is one of the $\varphi_0, \ldots, \varphi_{m-1}$ and parameter is smaller than $F_n(F_{n+1}(x))$, thus at least two elements of the set $E \cap C_n$ must satisfy the same set of pairs. Let one of them be z_n and the second w_n. Both of them are values of the function F_n. Let $z_n = F_n(z_n')$ and $w_n = F_n(w_n')$. Let us notice that for any value a_1 such that $z_n < a_1 < F_n(z_n' + 1)$ we have $l_n(a_1) = z_n, p_n(a_1) = F_n(z_n' + 1)$; moreover if $d = F_n(F_{n+1}(x))$, $w = w_n$ then $2_m(l_{n+1}(a_1)) < F_1(l_{n+1}(a_1)) < F_n(l_{n+1}(a_1)) = F_n(F_{n+1}(x)) = d$ and $\neg C_n(a_1)$. As a consequence we obtain $M \models B_{m,n}(a_1)$.

Moreover, if $F_{n-1}(z_n) < a < F_{n-1}(z_n + 1)$ then $F_{n-1}(a) < F_n(z_n' + 1) = p_n(a)$ (this inequality is true by Lemma 2.6 because there are more than $2^{F_{n-1}(z_n)}$ elements of the set $C_{n-1} \cap (F_{n-1}(z_n), \max\{f : F_{n-1}(f) < F_n(z_n' + 1)\}))$ and $F_{n-1}(l_n(a)) = F_{n-1}(z_n) < a$, and so we have $M \models E_n(a)$.

Therefore if $n = 4$ then we take any such a; otherwise any value a such that

$$F_{n-2}(F_{n-1}(z_n) + r) < a < F_{n-2}(F_{n-1}(z_n) + r + 1)$$

(for this choice a $F_{n-2}(l_{n-1}(a) + r) = l_{n-2}(a)$ and so $M \models A_{n-1}^r(a)$).

Now we iterate this procedure, i.e., apply it to $n-2$, $n-4$ and so on. This shows $\exists y > r \bigwedge \Delta_r(y)$; in fact, we have shown a non-empty interval of such elements y. Hence q_r is consistent. In this way the proof of Lemma 4.1 is completed.

Acknowledgments The author is grateful to Professor Henryk Kotlarski and Dr. Teresa Bigorajska for valuable suggestions and remarks.

REFERENCES

[1] Gaifman, H., "Models and types of Peano's Arithmetic," *Annals of Mathematical Logic*, vol. 9 (1976), pp. 223-306.

[2] Kaye, R., *Models of Peano Arithmetic*, Oxford Logic Guides, Oxford University Press, Oxford, 1991.

[3] Kaye, R., R. Kossak, and H. Kotlarski, "Automorphisms of recursively saturated models of arithmetic," *Annals of Pure and Applied Logic*, vol. 55 (1991), pp. 67-99.

[4] Kotlarski, H., "On elementary cuts in models of arithmetic," *Fundamenta Mathematicae*, vol. 115 (1983), pp. 27-31.

[5] Kotlarski, H., "On elementary cuts in recursively saturated models of Peano Arithmetic," *Fundamenta Mathematicae*, vol. 120 (1984), pp. 205-222.

[6] Smoryński, C., "Recursively saturated non-standard models of arithmetic," *Journal of Symbolic Logic*, vol. 46 (1981), pp. 259–286.

[7] Smoryński, C., "Elementary extensions of recursively saturated models of arithmetic," *Notre Dame Journal of Formal Logic*, vol. 22 (1981), pp. 193–203.

Zaklad Matematyki WSRP w Siedlcach
ul. Orlicz-Dreszera 19.21
08-110 Siedlce
Poland