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On Interpreting Truth Tables and Relevant Truth Table Logic

RICHARD SYLVAN

Abstract Contrary to common mythology, the two-valued truth tables do not yield classical logic. *Many* contestable assumptions are required to reach classical logic. Indeed *some* assumptions are required to get anywhere logically. In between, and in other directions, lie several other logics. For, even logically, there are many ways in which the truth tables can themselves be interpreted. In particular, they can be variously read inferentially, in one direction or two, or they may be variously read semantically. Along inferential lines, Tennant's one-way reading is reconsidered. It is argued that the tables do not lead to the logics Tennant claims to reach but can lead to various other decidedly weak logics. Along more orthodox semantical lines, it is shown how the truth tables themselves do not exclude nonclassical situations but can allow for incomplete and inconsistent set-ups. So considered, they provide the framework for a four-valued relevant logic. A four-valued implication is grafted onto this framework, simply by generalising upon two-valued material implication artifice, to deliver the familiar system *FDE* of tautological entailment. Finally, for comparison, a less contrived semantics than pure truth tabular, a semantics due to Dunn, which now admits of ready higher degree extension, is supplied for *FDE*.

It is commonplace nowadays to begin logic with the truth tables, the two-valued truth tables. These have two values, symbolized (say) as 1 and 0 – value 1 for "true", "on", "holds", "yes", "yin", etc., and value 0 for "false", "off", "fails", "no", "yang", etc. It is commonplace also to assume that these tables lead immediately to classical logic. But they do not: not without the "right" interpretation; not without both a considerable background and requisite assumptions. Consider the usual tables for "connectives" &, $\vee$, and $\sim$ in isolation (an early choice is as to basic connectives), written one of the usual ways (already condensed a little from column forms):

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&	1	0
1	1	0
0	0	0

v	1	0
1	1	1
0	1	0

~	
1	0
0	1

Outside a broad mathematical content, these "tables" could represent a wide range of things such as stylized scribbles, wallpaper designs, games of noughts and ones, and so on (as Wittgenstein emphasized in [6]). Even within a mathematical setting they could, if not idling, be doing a variety of jobs; for instance, they could be (the way Boole would perhaps have seen them) strange arithmetical tables (for an arithmetic mod 2, with & for ×, v for +, and maybe ~ for −). Even when a strictly logical setting is entered, and 'truth value' gets some stuffing, the interpretational options are considerable. Consider the table for & redepicted in column tabular form with schematic letters introduced (definitely a different pattern and game, or very odd numbers):

A	*B*	*A* & *B*
1	1	1
1	0	0
0	1	0
0	0	0

Such a column can be "read" from left to right ("constructively" building up *A* & *B* from components *A* and *B*), or from right to left ("deconstructively" in one sense), or, more commonly, *both* ways. Such a column can, moreover, be read syntactically or "proof theoretically", implicationally or inferentially, or more commonly it can be read semantically. As such columns require a good deal of interpretation, a "right" interpretation, the commonplace theme that the truth tables themselves provide an interpretation is widely astray.

To appreciate further *how much* interpretation goes into the truth tables, especially in getting a calculus from them, and what can be read out of them, it is revealing to consider in passing Tennant's provocative "Truth table logic" [5]. In this exercise Tennant manages to extract an exotic nonclassical logic from *an* (unorthodox) inferential construal of "the" truth tables. In the revised version of his article Tennant claims to be "taking a fell swoop view of what the truth tables say". Thus he reads, he says, the truth table for conjunction 'as saying':

> if the truth value of *A* and of *B* are both 1 then the truth value of *A* & *B* is 1;
>
> if the truth value of *A* is 0 then that of *A* & *B* is 0:
>
> if the truth value of *B* is 0 then that of *A* & *B* is 0.

But in fact he switches immediately, without notice, from these *conditional* forms to certain *argumentative* construals, which he calls 'truth table inferences'. Plainly, the various forms are not the same *without* substantive assumptions which can impact on the logic that results. For there are various conditional relations, and (even setting aside the significant differences between conditionality and argument) these may not match.

The first two entries, those for connectives ~ and &, in Tennant's tabulation look like this (but using 1 and 0 in place of his T and F):

Exhibit I: Truth Table Inferences (after [5])

A	$\sim A$		Sentential version	Nonredundant *demonstranda*
1	0		$A \therefore \sim\sim A$	$A \therefore \sim\sim A$
0	1		$\sim A \therefore \sim A$	
A	B	$A \& B$		
1	1	1	$A, B \therefore A \& B$	$A, B \therefore A \& B$
1	0	0	$A, \sim B \therefore \sim(A \& B)$	$\sim B \therefore \sim(A \& B)$
0	1	0	$\sim A, B \therefore \sim(A \& B)$	$\sim A \therefore \sim(A \& B)$
0	0	0	$\sim A, \sim B \therefore \sim(A \& B)$	

To arrive at the column headed 'Sentential version', moving from left to right across the column of the exhibit, it becomes evident that we are expected to apply the following rules (see [5]): Where a value 1 appears in the truth table for a wff, A e.g., write that wff, viz. A; where a value 0 appears in the truth table for a wff, B e.g., write down the negation of that wff, viz. $\sim B$. Further, for binary connectives insert commas between wff of the component wff; and insert the "inferential" symbol $\therefore$ between component (input) wff and operated-upon (output) wff of the table. Finally, transfer only the body of the table right. (Were the heading of the negation table, for instance, transposed we should obtain the powerful nonclassical inference $A \therefore \sim A$!) The procedure provides a certain one-way inferential 'reading of the truth tables' concerned.[1] It is a sort of subintuitionistic reading; it is taken to deliver $A \therefore \sim\sim A$, but not (what a reverse direction would yield) $\sim\sim A \therefore A$.

But even within one-way inferential limitations, other outcomes are easily reached. Even if similar table positions are replaced in "the same way" (i.e. location in a table makes no difference to transcription), who says that a 0 in B column is to be replaced by $\sim B$? The tables do not *say* that (nor do they show it). It could be replaced instead by $\sim\sim\sim B$, or $\sim\sim B$, or $B \& \sim B$, or. . . . Similarly for replacing 1 by wff. Lots of different one-way $\therefore$ inferences could thus emerge, some of them no doubt quite bizarre by usual straight-laced standards. Not all these procedures are perverse. If, for instance, 1 is taken to represent necessary truth, and 0 necessary falsehood, then the negation table would not unreasonably yield the (nonredundant, but one-way) inferences: $A \vee \sim A \therefore \sim(\sim A \& \sim\sim A)$, and $\sim(A \& \sim A) \therefore \sim A \vee \sim\sim A$. Differently, it could be argued that getting down from truth values for wff to wff themselves involves double negation buffering (the sort of considerations that go into Tennant's main theorem suggest as much). With double negation buffering, we can obtain, for example, the following sentential versions, illustrated for the negation table:

A	$\sim A$	DN1 sentential version[2]	DN2 sentential version[2]
1	0	$A \therefore \sim\sim\sim\sim A$	$\sim\sim A \therefore \sim\sim\sim\sim A$
0	1	$\sim\sim\sim A \therefore \sim A$	$\sim\sim\sim A \therefore \sim\sim\sim A$

Thus even within the confined space of one-way inferential elaborations, *there are many interpretations* leading with further input to a variety of weak inferential logics.

Further assumptions are imported in moving left to right from *the* acclaimed sentential version to the so-called nonredundant *demonstranda*. Certain features of the linkage ∴ [3] are taken for granted. Unless we have a basic (meta-)logic of ∴ which supplies both an identity scheme A ∴ A as well as principles of schematic replacement, we do not know that $\sim A$ ∴ $\sim A$ is redundant. After all it not derivable from the other ∴ principles given. Likewise we could hardly establish that $\sim A$, $\sim B$ ∴ $\sim(A \,\&\, B)$ is redundant unless we had been supplied with an appropriate principle of weakening for ∴ logic, for instance of the form A ∴ $B/A, C$ ∴ B. What is redundant and what is not will depend on the background (but *unexposed*) ∴ logic.

Suppose then we are given such a background ∴ logic which will be like the structural rules of some Gentzen (meta-)logic. Then the Tennant $T(\sim,\&)$ logic—to persist with just the two connectives $\sim$ and &—will consist of that background together with the nonredundant *demonstranda* of Exhibit 1. That is a truth table inferential logic of a sort. But $T(\sim,\&)$ is *considerably* weaker than either of the two systems ***T*** and ***T****** that Tennant proposes as truth table logics. So also is the full system $T(\sim,\&,\vee,\supset)$ in connective set $\{\sim,\&,\vee,\supset\}$, system TT for short. TT does not even contain such principles as those of simplification, $A \,\&\, B$ ∴ A, $A \,\&\, B \supset A$, etc. Indeed, by inspection, TT supplies *no* connective *elimination principles* for ∴ inferences.

Reaching Tennant's "truth table" logics from TT requires a macro-leap. That leap comprises several large component steps beginning with critical infiltration of a constant Λ and several schemata concerning Λ, proceeding through a substantial diversion on requirements for Kalmar's Theorem, and ending with the assumption of natural deduction techniques, above all subproof procedures which TT does not license.[4]

The schemata including Λ much exceed what the truth table for Λ (*had* it been introduced) would sustain. The modest result, Tennant style, is merely as follows:

A	Λ	
1	0	A ∴ $\sim\Lambda$
0	0	$\sim A$ ∴ $\sim\Lambda$

None of the Λ schemes (one for each connective) which Tennant lists for proof of an inferential version of Kalmar's Theorem are thereby supplied. But ***T*** and ***T****** are set up to deliver just such a theorem. It is already evident then, without entering into any further detail, that Tennant's advertised claim is mistaken. That systems ***T*** and ***T****** are systems of "*truth table logic*—a logic justified on the basis of what the truth tables *say*, rather than on the extra that they might arguably *show*"—is palpably false. Both his systems proceed far beyond what truth tables *say* on his own impoverished left-to-right inferential reading.

An obvious way to obtain a working inferential motor from the truth tables themselves, from what they appear to "say" inferentially, is to adopt a two-way reading. The resulting inferential logic, $2T(\sim,\&)$ in the $\{\sim,\&\}$ case, will presumably then include appropriate elimination principles, namely $\sim\sim A$ ∴ A; $A \,\&\, B$ ∴ A; $A \,\&\, B$ ∴ B.[5] The logic, while still exceedingly weak by all orthodox standards, is in the vicinity of nonreplacement relevant logics (investigated, e.g. by Routley and Loparic [4]). But it is weaker even than the main logics so far investigated in lacking &-∨ distribution principles (e.g. $A \,\&\, (B \vee C)$ ∴ $A \,\&\, B \vee$

A & C), and composition principles for connectives & and ∨ (e.g. $A \therefore C, B \therefore C/A \vee B \therefore C$). Nor would such a two-way inferential logic, though truth tabularly authentic enough, appeal to Tennant; for it reinstates the nonintuitionistic form of double negation, $\sim\sim A \therefore A$, the removal of which was no doubt a main virtue of the highly commended one-way reading.

It is decidedly unorthodox to construe that truth tables inferentially. The truth tables are typically taken to encapsulate *semantical* conditions. They are integrally tied, after all, with *truth* values 1 and 0. Of course the conditions may *yield* inferences, but that is only part, a by-product even, of what the conditions amount to and do. To illustrate encapsulation, one example will suffice. For instance, the rows of the table for negation ~ given below correspond to the biconditionals written to the right:

A	$\sim A$	
1	0	$\vee(A) = 1$ if and only if $\vee(\sim A) = 0$;
0	1	$\vee(A) = 0$ if and only if $\vee(\sim A) = 1$.

Here the expression $\vee(A) = 1$ is read along these lines: the value of (assigned to) A is (=) 1, i.e. "true", where 1 is the truth value *true*; similarly, for $\vee(A) = 0$, with 0 the "truth" value 0. Thus ∨ amounts to valuational function, sending expressions such as A into one or other of the truth value pair {1,0}.

So far everything looks pretty standard; we have proceeded to spell out, in entirely orthodox semantical fashion, the valuational principles the tables encapsulate. But there is a crucial difference, which makes all the difference. Ordinarily it is further assumed that values 1 and 0 are exclusive and exhaustive. That is, for any wff A, not both $\vee(A) = 1$ and $\vee(A) = 0$, and either $\vee(A) = 1$ or $\vee(A) = 0$, or, in material terms, $\vee(A) = 0$ iff $\vee(A) \neq 1$. Such assumptions hold good only for what is convenient to call *classical* situations. There are, however, two types of *nonclassical* situations where the assumptions fail, namely

- *undercomplete* situations, for A, where neither $\vee(A) = 1$ nor $\vee(A) = 0$, and
- *overcomplete* situations, for A, where both $\vee(A) = 1$ and $\vee(A) = 0$.

Examples of the first include cases of indeterminacy and vagueness; examples of the second paradoxical outcomes. While both are controversial, it is nonetheless, indeed all the more, important to investigate the logic of these situations (to determine which principles hold valid and which do not, what can be inferred and what cannot, and so on).

Admitting nonclassical situations, there are *four* cases to consider for each wff A:

T, or 1 only:	$\vee(A) = 1$ only, i.e. $\vee(A) = 1$ and $\vee(A) \neq 0$:
B, or {1,0}:	$\vee(A) = 1$ and $\vee(A) = 0$ also, i.e. A takes both values 1 and 0;
N, or { }:	$\vee(A) \neq 1$ and $\vee(A) \neq 0$ also, i.e. A takes neither value 1 nor 0;
F, or 0 only:	$\vee(A) = 0$ only, i.e. $\vee(A) = 0$ and $\vee(A) \neq 1$.

Those four cases lead us, ineluctably, to *four-valued* truth tables for each connective considered, here the set {&,∨,~}.[6] Those four-valued tables can be seen as *making more explicit* what is already contained in the two-valued tables, at least as construed semantically. Let us illustrate the derivation of some elements of the four-valued tables before setting them down in full detail. Suppose for A the situation is overcomplete, that is A gets assigned {1,0}; what happens to $\sim A$? When $v(A) = 1$, $v(\sim A) = 0$, and when $v(A) = 0$, $v(\sim A) = 1$. So when A is assigned {1,0}, $\sim A$ is assigned {0,1}, i.e. {1,0}; that is both. Or suppose under four-valued valuation I, A is assigned neither, { }, i.e. $v(A) \neq 1$ and $v(A) \neq 0$, and B is assigned both {1,0} i.e. $v(B) = 1$ and $v(B) = 0$; in short, $I(A) =$ { } and $I(B) =$ {1,0}. What is $I(A \,\&\, B)$? Well, $v(A \,\&\, B) \neq 1$, because $v(A) \neq 1$, and $v(A \,\&\, B) = 0$, because $v(B) = 0$, whence $I(A \,\&\, B) = 0$ only. Proceeding in this way, we arrive at the following tables (where 1 only and 0 only are represented by italicized symbols):

(A)

&	*1*	{1,0}	{ }	*0*
1	*1*	{1,0}	{ }	*0*
{1,0}	{1,0}	{1,0}	*0*	*0*
{ }	{ }	*0*	{ }	*0*
0	*0*	*0*	*0*	*0*

v	*1*	{1,0}	{ }	*0*
1	*1*	*1*	*1*	*1*
{1,0}	*1*	{1,0}	*1*	{1,0}
{ }	*1*	*1*	{ }	{ }
0	*1*	{1,0}	{ }	*0*

~	
1	*0*
{1,0}	{1,0}
{ }	{ }
0	*1*

As relabelled, in what will be called *S-notation*, according to the following translation scheme

$$\begin{pmatrix} 1 & \{1,0\} & \{\ \} & 0 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

these tables deliver the following tables:

(S)

&	1	2	3	4
1	1	2	3	4
2	2	2	4	4
3	3	4	3	4
4	4	4	4	4

v	1	2	3	4
1	1	1	1	1
2	1	2	1	2
3	1	1	3	3
4	1	2	3	4

~	
1	4
2	2
3	3
4	1

The S-notation simply reexpresses the four-valued matrices in a standard many-valued form, a readily recognizable form given previous logical use of these matrices.[7]

Splendid as those matrices are there is not a great deal of relevant logical work that we can accomplish with them on their own. For they yield no logical truths. Even the principle of noncontradiction, $\sim(A \,\&\, \sim A)$, takes a nondesignable value { } when A is assigned value { }. (It is nondesignable because it does not include a truth element 1.) Put differently, *the* {&,v,~} *logic delivered* where nonclassical situations are not excluded *has no theorems*. Of course the {&,v,~} logic for classical situations is the standard two-valued logic, S.

There are two evident ways out of the no-theorem predicament, ways we shall before long want to put together:

- Introduce an implicational or inferential connective of some sort, to make explicit which truth-functional wff are derivable (inferable or deducible) from which others.
- Combine classical with nonclassical situations.

Implication requires artifice in a finite-valued setting. We will generalize in a four-valued way on classical two-valued artifice, a generalization which interestingly suffices to remove basic paradox. A suitably generalized Philonian conditional cannot quite set $\vee(A \to B) = 1$ iff $\vee(A) = 0$ or $\vee(B) = 1$, in the usual disconnected fashion. For that makes no due allowance for four values; it would not even ensure $A \to A$. But an appropriate rule is not far to seek; namely

($\to_1$) $\vee(A \to B) = 1$ if $\vee(A) = 0$ only or $\vee(B) = 1$ only or $\vee(A) = \vee(B)$; and $\vee(A \to B) = 0$ otherwise.

To reach this two-valued assignment on four values, a little experimentation is required; for it is but one of many evaluation rules that might be tried. And of course it helps if experimenters know where they are headed, what results they aim to churn out. (In applying the $\to_1$-rule observe that $\vee(A) = 0$ only iff $\vee(A) = 0$ & $\vee(A) \neq 1$ and $\vee(B) = 1$ only iff $\vee(B) = 1$ & $\vee(B) \neq 0$.)

The rule delivers the following (patterned) matrix, as may be straightforwardly verified:

(*B*)

→	1	{1,0}	{ }	0
1	1	0	0	0
{1,0}	1	1	0	0
{ }	1	0	1	0
0	1	1	1	1

i.e. in *S*-notation

→	1	2	3	4
1	1	4	4	4
2	1	1	4	4
3	1	4	1	4
4	1	1	1	1

Now more of the beauty of the enterprise emerges: with matrices (*A*) and (*B*) we have arrived at precisely the matrices characteristic for first degree entailment of system *FDE*. These are the "Smiley matrices" (given in *S*-notation in [1], pp. 161–162), characteristic for "tautological entailments". The logic of *FDE* in fact matches the algebraic structure underlying the matrices, namely De Morgan lattice structure; it adds to distributive lattice logic (*DLL* of Routley [3], pp. 104ff) a De Morgan negation. In one axiomatic presentation it runs as follows (for *A* and *B* truth-functional in $\{\&, \vee, \sim\}$):

$A \to A$	$A \to B,\ B \to C / A \to C$
$A \,\&\, B \to A$	$A \to B,\ A \to C / A \to B \,\&\, C$
$A \,\&\, B \to B$	
$A \to A \vee B$	$A \to C,\ B \to C / A \vee B \to C$
$B \to A \vee B$	
$A \,\&\, (B \vee C) \to (A \,\&\, B) \vee C$	
$\sim\sim A \to A$	$A \to \sim B / B \to \sim A$

FDE is a well investigated system, with many other formulations, and with several modellings (for which see [1], chapter 3, and [3], pp. 122ff and s. 3.2).

The artifice of the finite-valued implication of the matrix formulation can be avoided very simply by switching to alternative modellings for *FDE*, in particular to a valuational semantics. Then an implication $A \to B$ is *FDE-valid* (under a valuational semantics) iff for every valuation $\vee$, when $\vee(A) = 1$ then $\vee(B) = 1$ and when $\vee(B) = 0$ then $\vee(A) = 0$. The rule is, so to say, the entirely natural requirement for implication, recalling that $\vee(A) = 1$ leaves open whether $\vee(A) = 0$ or $\vee(A) \neq 0$, so that both forward and backward truth value preservation should be considered. Then the theorems of *FDE* coincide exactly with the *FDE*-valid wff.[8]

NOTES

1. In the initial printing of his article, Tennant simply assumed that the truth tables were read, from the given column of forms, left to right; in revision he makes that assumption explicit. But many other assumptions are not explicit, beginning with the idea that the truth tables are primarily inferential devices, construed through a coupling ∴, which gives way to "natural deduction" linear inference.

2. The DN1 version can be reached by replacing 1 by A_1 and 0 by $\sim A_{1^\circ}$, then eliminating sub 1° through double negation and sub 1 vacuously. The procedure was suggested by relevant truth table games. The DN2 version eliminates sub 1 also through double negation. A DN3 would remove 1 only, and not 1°, through double negation. Other versions could further multiply up negation or other connectives.

3. The symbol is nowhere explained by Tennant but presumably is transcribable as "therefore". The *non*hypothetical "therefore" is *much* more restrictive than usual inferential schemes, including those of natural deduction, subsequently substituted for them, which are suppositional. Nor are truth tables so narrowly restricted, admitting of hypothetical construal (e.g. were A true, $\sim A$ would be false). Accordingly "therefore", ∴, is too narrow both for proper construal of the truth tables and for Tennant's subsequent natural-deduction enterprise.

4. How much scene setting *goes into* a natural deduction formulation is evident from better investigations of the topic.

5. There is further ado here, which we have glossed over, since two-way inferential readings are merely a sidestream in the present setting. For example, right to left readings strictly involve inferential forms which are multiple on the right; then there are further slides in getting down to singular forms.

6. We are here retracing a beautiful route mapped out by Dunn [2]. We should observe that further assumptions are entering as we proceed (there is no pure assumptionless way), e.g. syntactical assumptions as to the availability of symbols, and semantical assumptions as to the role of 1 in distinguishing designated values.

7. Observe that the truth-tabular analysis of these (S)-matrices provides but one analysis explaining the matrices; we shall arrive at, or refer to, other analyses. Some matrices will be familiar from other settings. The matrices for & and ∨ are those both distributive lattice theory and Boolean algebra supply and which were adopted by Parry for Lewis modal logics. The negation matrix is that given by De Morgan lattice negation. These (S)-matrices were assembled by Smiley; see Anderson and Belnap [1].

8. See e.g. [3], p. 198. From there, moreover, it is an easy step to the next stage, the full first degree logic, *FD*. For details see [3].

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The Australian National University
P.O. Box 4
Canberra, ACT
2600 Australia