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# The Fraenkel–Mostowski Method, Revisited

NORBERT BRUNNER*

**Abstract** Permutation models generated by isomorphic topological groups satisfy the same choice principles (Boolean combinations of injectively bounded statements). As an application the group $J_p$ of $p$-adic integers is characterized: A monothetic linear group $G$ generates a model that satisfies the same choice principles that hold in the model corresponding to $J_p$ iff the $G$-model satisfies the well-orderable selection principle, and $AC_q$ holds, $q$ prime, iff $q \neq p$. The main result is a strengthening of a previous theorem of Pincus: All Fraenkel–Mostowski–Specker independence proofs concerning choice principles can be proved in finite support models.

***1 Introduction*** In this note we comment on some aspects of the structure theory of permutation models which are related to their historical development fifty years ago. Following some ideas of Fraenkel, permutation models were invented by Lindenbaum and Mostowski [16], [17] as a device for proving independence results on the axiom of choice in ZFA set theory (a weakening of ZF which permits a set $A$ of atoms). In [17] Mostowski used models with only finite supports. Later he extended this method to models with infinite supports [18], and finally Specker [24] presented the most general construction of a permutation model from a group-generated Hausdorff topological group. This line of research seems to have been based on the conviction that more general construc-

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tions are needed for more sophisticated independence proofs. We will show that all Fraenkel-Mostowski independence proofs for ZFA which concern the choice principles in Jech [12] can be performed in finite-support models. In fact, one can stipulate the existence of a support function, though not for least supports. Hence, the flexibility gained from Specker's construction does not materialize in provability strength. As was first announced by Pincus ([23], p. 137), for each $\Pi_2$ sentence in a Specker model there is an infinite-support Fraenkel-Mostowski model that satisfies it.[1]

*1.1 Notation* Our surrounding set theory is ZFC, $(\mathrm{V},\in,=)$ denoting the real world. Following an idea of Truss, we define a ZFA-universe as follows: For $X \in \mathrm{V}$ we set $\mathrm{V}(X) = \bigcup\{\mathrm{V}_\alpha : \alpha \in \mathrm{On}\}$, where $\mathrm{V}_0 = X \times \{0\}$ and $\mathrm{V}_\alpha = \{(A,\alpha) : \alpha$ minimal, such that $A \subseteq \bigcup\{\mathrm{V}_\beta : \beta \in \alpha\}\}$. We define $x \in_x y$, if $x \in A$ and $y = (A,\alpha)$ for some $\alpha > 0$ and $\alpha > A$, and we set $x =_x y$ iff $x = y$ in V. $(\mathrm{V}(X), \in_x, =_x)$ satisfies ZFA + AC, and $\varnothing_x = (\varnothing,1)$, $A_x = (\mathrm{V}_0,1)$ is the set of atoms. If there is no danger of confusion, we shall omit the subscript. A faithful representation $d: G \to S(X)$ ($G$ a group, $S(X)$ the full symmetry group, and $d$ an injective homomorphism) is extended to all of $\mathrm{V}(X)$ as in Specker [24]: $(\hat{d}g)(x,0) = ((dg)x,0)$ for $x \in X$, and we recursively set $(\hat{d}g)x = \{(\hat{d}g)y : y \in_x x\}$ (within $\mathrm{V}(X)$) for $x \in \mathrm{V}(X)\setminus\mathrm{V}_0$, defining an $\in$-automorphism. We write $\mathrm{sym}_{dG}x = \mathrm{sym}_d x = \{g \in G: (\hat{d}g)x = x\}$ for the stabilizer of $x \in \mathrm{V}(X)$ and $\mathrm{fix}_d x = \bigcap\{\mathrm{sym}_d y : y \in_x x\}$ for the pointwise stabilizer. A topological group $(G,\cdot,\mathbf{G})$ ($\mathbf{G}$ the topology) generates the permutation model $\mathrm{PM} = \mathrm{PM}(d,G,\cdot,\mathbf{G},X) = \{x \in \mathrm{V}(X) : \forall y \in_x \mathrm{TC}(\{x\}) : \mathrm{sym}\, y \in \mathbf{G}\}$, where $\mathrm{TC}(\cdot)$ is the transitive closure in V(X); a proper V-class $C \subseteq \mathrm{PM}$ defines a class $(C,\mathrm{On})$ of PM iff $\mathrm{sym}\, C \in \mathbf{G}$. Then $(\mathrm{PM},\in_x,=_x)$ satisfies ZFA. We shall *assume* that $A_x \subseteq \mathrm{PM}$. If possible, we shall not mention $d$.

This approach to permutation models is more flexible than that in Brunner and Rubin [7], insofar as it allows us to compare models on different sets of atoms. Also, we prefer to work with atoms instead of with irreflexive sets, since the latter can be used as in Felgner [8] to introduce additional information on the $\in$-structure which cannot be reconstructed from the group action.

$\mathbf{G}$ is linear (group-generated), if $\mathbf{1}$ (i.e., the identity) has a neighborhood base consisting of open groups. For example, $\mathbf{G}^d_{\mathrm{fin}}$, which is generated by $\{\mathrm{fix}\, e: e \in [A_x]^{<\omega}\}$, and $\mathbf{G}^d_{\mathrm{nat}}$, which is generated by $\{\mathrm{sym}\, x: x \in \mathrm{PM}\}$, are group-generated Hausdorff topological groups. If $\mathbf{G}$ generates PM, so does $\mathbf{G}_{\mathrm{nat}} \subseteq \mathbf{G}$, whence it suffices to restrict one's attention to linear Hausdorff groups. We shall always *assume*, then, that $\mathbf{G} = \mathbf{G}_{\mathrm{nat}}$.

*1.2* $\mathbf{G} \neq \mathbf{G}_{\mathrm{nat}}$ is possible: the "second Fraenkel model" [12] is generated by $G = \mathbb{Z}_2^\omega$ with the product topology $\mathbf{P}$ of $2^\omega$. $H < G$ is the direct sum of countably many $\mathbb{Z}_2$'s. We let $\mathbf{G}$ be the group topology generated by $\mathbf{P} \cup \{H\}$. Then $H$ is dense in $G$ with respect to $\mathbf{P}$, whence $H$ and the product topology $\mathbf{P} \upharpoonright H$ generates the Fraenkel model (see [7]). Since $H$ is an open subgroup of $(G,\cdot,\mathbf{G})$ with the relative topology $\mathbf{G} \upharpoonright H = \mathbf{P} \upharpoonright H$, $(G,\cdot,\mathbf{G})$ generates the Fraenkel model too. But $\mathbf{G}_{\mathrm{nat}} = \mathbf{P} \neq \mathbf{G}$. Moreover, the assumption $A_x \subseteq \mathrm{PM}$ is not automatically true.

*2 Persistency* As was shown by Ahlbrandt and Ziegler [1], under some additional conditions countably categorical structures with isomorphic topological automorphism groups are each interpretable in the other. For permutation models the topology is more closely related to the structure: permutation models with isomorphic topological automorphism groups satisfy the same choice principles. To assign a meaning to this statement, we observe that the group of $\in$-automorphisms of PM $\subseteq$ V($X$) naturally corresponds to a subgroup Aut PM of $S(X)$ and that $\mathbf{Aut}_{\text{nat}}$ generates PM. Moreover, as motivated by Pincus (5.1 of [21]) we specify that a "choice principle" is a negation of a Jech–Sochor bounded statement. In [6] such a result was announced for some examples of choice principles, but the proof could be completed for the axiom of choice, AC, only.

*2.1* A sentence $\phi$ of the ZF-language is *persistent* if the following holds: If $(G,\cdot,\mathbf{G}) \in$ V is a topological group, $d_i: G \to S(X_i)$, $i \in 2$, are injective representations, and $\text{PM}_i = \text{PM}(d_i, G, \cdot, \mathbf{G}, X_i)$ are the corresponding permutation models with $\in$-relations $\in_i$, such that $A_{X_i} \subseteq \text{PM}_i$ and $\mathbf{G}_{\text{nat}}^{d_i} = \mathbf{G}$, then $\text{PM}_0$ satisfies $\phi$ iff $\text{PM}_1$ does (with $\in$ replaced by the respective $\in_i$'s).

If $\phi$ is persistent, then its validity depends only on the group, whence we say that $G$ satisfies $\phi$ iff some/all permutation models generated by $G$ satisfy $\phi$. In view of 4.3 this makes sense for all group-generated Hausdorff topological groups.

**Theorem** *Boolean combinations of Jech-Sochor bounded statements are persistent.*

As an application, it follows that in the "second Fraenkel model" there is a complex vector space which has Hamel bases of different cardinalities. For as was shown by Läuchli [14], the group $(\mathbb{Z}_2 \oplus \mathbb{Z}_2)^\omega$ satisfies this statement. The theorem follows from the following lemma and the proof of the Jech–Sochor Theorem (see [13]).

**2.2 Lemma** *For each $\sigma \in$ On there is an $\in_0 - \in_1$-isomorphism $F: P_0^{\bar\sigma}(A_0) \to P_1^{\bar\sigma}(F''A_0)$ in* V. *Here $\bar\sigma$ is the $\sigma$-th ordinal in* V($\emptyset$) *(which is the same as the $\sigma$-th ordinal in* $\text{PM}_i$*), $A_0 = A_{X_0}$ and $P_i^{\bar\sigma}(\cdot)$ is the $\sigma$-fold iteration of the power set operation in* $\text{PM}_i$.

*Proof:* We first define $F$ on $A_0$. We let $\{\text{orb}_0\, a_\alpha : \alpha \in \lambda\}$ be the partition of $A_0$; here $\text{orb}_i x = \{(\hat{d}_i g)x : g \in G\}$, $\text{sym}_i = \text{sym}_{d_i}$, $\alpha \to a_\alpha$ in V. As was shown by Truss [25], for each open $H < G$ there is an $x \in \text{PM}_1$ such that $\text{sym}_1 x = H$. For since $\mathbf{G} = \mathbf{G}_{\text{nat}}^{d_1}$ there is a $y \in \text{PM}_1$ such that $H \supseteq \text{sym}_1 y$, and in $\text{PM}_1$ we may set $x = \{(\hat{d}_1 g)y : g \in H\}$. Since $H \in$ V, $x \in$ V$(X_1)$ is a set and since $x \subseteq \text{PM}_1$ and $\text{sym}_1 x = H$, $x \in \text{PM}_1$. With $H = \text{sym}_0 a_\alpha$, in V we get a function $\alpha \to x_\alpha \in \text{PM}_1$ such that $\text{sym}_1 x_\alpha = \text{sym}_0 a_\alpha$. Modulo some obvious manipulations, we may assume that $x_\alpha$ is of the form $x_\alpha = \{S_\alpha \times \overline{(\kappa + \alpha)}\}$, where $S_\alpha \neq \phi$ and $\kappa$ exceeds $\lambda$ and the V-cardinality of $P^\sigma(X_0)$. We now define $F \upharpoonright A_0$ as the orbit $F = \{\langle(\hat{d}_0 g)a_\alpha, (\hat{d}_1 g)x_\alpha\rangle : g \in G \text{ and } \alpha \in \lambda\}$. One easily verifies that $F$ is a one-to-one function $F: A_0 \to \text{PM}_1$ in V such that $\text{sym}_1 Fa = \text{sym}_0 a$ and $(\hat{d}_1 g)F(a) = F((\hat{d}_0 g)a)$.

We recursively extend $F$ to $P^{\bar\sigma}(A_0)$ by $Fx = F''x$; $F: P^{\bar\sigma}(A_0) \to$ V$(X_1)$. $F(x) \notin F(a)$ for $a \in A_0$, for if $x \in A_0$ this follows from the definition of $F(x)$

as a singleton, and if $x \notin A_0$ then $F(x) \in F(a)$ implies that $F''x = Fx = S_\alpha \times \overline{(\kappa + \alpha)}$ has more elements (with respect to cardinality in V) than in $V(X_1)$ $P^{\bar{\sigma}}(F''A_0) \supseteq F''x \geq \kappa$. From this one proves by induction on $\max\{rkx, rky\}$ that $x = y$ iff $Fx = Fy$; $x \in y$ iff $Fx \in Fy$; and the identity $(\hat{d}_1 g)F(x) = F((\hat{d}_0 g)x)$. This implies that the image of $F$ is contained in $PM_1$; for if $g \in sym_0 x$ and $x \in A_0$, then $g \in sym_1 Fx$ from the definition of $F \upharpoonright A_0$, and if $x \notin A_0$ then $(\forall y \in x)(\exists z \in x): (\hat{d}_0 g)y = z$, whence $(\hat{d}_1 g)Fy = Fz \in Fx$ and $(\hat{d}_1 g)Fx \subseteq Fx$, proving that $sym_0 x \subseteq sym_1 Fx$. So we have that $F$ is an $\in$-isomorphism from $P_0^{\bar{\sigma}}(A_0)$ into $P_1^{\bar{\sigma}}(F''A_0)$. That it is onto follows by induction from the following claim: If $X = \{Fy : y \in Y\}$, $Y \subseteq PM_0$ in $V(X_0)$, then $sym_0 Y \supseteq sym_1 X$ (by a previous remark this gives $sym_0 x = sym_1 Fx$), whence preimages of $PM_1$-subsets of im $F$ are in $PM_0$. For if $g \in sym_1 X$, then $(\forall y \in Y)(\exists z \in Y)$: $(\hat{d}_1 g)F(y) = F(z)$, whence $F((\hat{d}_0 g)y) = F(z)$, and by injectivity $(\hat{d}_0 g)y = z$, proving that $(\hat{d}_0 g)Y \subseteq Y$.

*2.3* 2.1 can be improved to show that injectively bounded statements $\exists x \phi(x)$ are persistent, where $\phi(x) \equiv \forall y((y^* \leq b(x) \wedge TC(x) \cap y = \emptyset) \rightarrow c(x,y))$, $b, c$ are bounded, i.e., for some absolute term $\alpha$, $b(x) \Leftrightarrow b^{P^\alpha(x)}(x)$ and $c(x,y) \leftrightarrow c^{P^\alpha(x \cup y)}(x,y)$, and $y^*$ is the Hartogs number. We sketch a proof.

We assume that in $PM_0$ $\phi(x)$ holds. Then with $\sigma = \sigma(x)$ and $F: P_0^{\bar{\sigma}}(A_0) \rightarrow PM_1$ of 2.2 we have $\phi(Fx)$ in $PM_1$; $\sigma(x)$ is an ordinal such that for all $u$ and $y$ with $y^* \leq b(x)$, $P^\alpha(u)$ $\in$-isomorphic to $P^\alpha(x)$, and $TC(u) \cap y = \emptyset$, there is a $v$ such that $v^* \leq b(x)$, $TC(v) \cap x = \emptyset$, and $P^\alpha(x \cup v)$ is $\in$-isomorphic to $P^\alpha(u \cup y)$ with the isomorphism being an extension of the previous one. The proof resembles that of C15 in [21]. It follows that $b(x) = b(Fx) = \bar{\beta}$. We let $\tau = \sigma(Fx)$ and $G: P_1^{\bar{\tau}}(A_1) \rightarrow PM_0$ be the mapping of 2.2. Then the following sets are $\in$-isomorphic: $P_1^{\bar{\alpha}}(Fx \cup y)$, where $y$ is any set in $PM_1$ such that $y^* \leq \bar{\beta}$ and $y \cap TC(x) = \emptyset$; $P_0^{\bar{\alpha}}(GFx \cup Gy)$, where $(Gy)^* \leq \bar{\beta}$; $P_0^{\bar{\alpha}}(GFx \cup v)$, where $v$ is defined from C14 in [21] as at p. 730 in [21] to satisfy $TC(x) \cap v = \emptyset$ in addition to $v^* \leq \bar{\beta}$; $P_0^{\bar{\alpha}}(x \cup w)$, some $w$ such that $w \cap TC(x) = \emptyset$ and $w^* \leq \bar{\beta}$. Since $c(x,w)$ holds by $\phi(x)$, $c(Fx,y)$ follows, proving $\phi(Fx)$.

Examples for statements covered by 2.3 but not by 2.1 are "every field has an algebraic closure" and "there is a set of $2^{\aleph_0}$ representatives for the Dedekind-finite cardinals".

*2.4* Choice principles seem to behave poorly under products. If $G = \Pi\langle G_i : i \in I\rangle$ is an infinite product ($|G_i| \geq 2$ and $I$ infinite), then $G$ does not satisfy $AC^\omega$ (i.e., PM does not satisfy $AC^{\bar{\omega}}$). For if $G_i$ generates $PM_i$ in $V(X_i)$ and $d_i: G_i \rightarrow S(X_i)$, then we let $d: G \rightarrow X = \bigcup\{X_i \times \{i\} : i \in I\}$ be the sum representation and PM the corresponding model which meets our assumptions $A_x \subseteq PM$ and $\mathbf{G}_{nat}$ = product topology. If we set $P_i = ((X_i \times \{i\}) \times \{0\}, 1)$ (in $V(X)$ the set of atoms that comes from $X_i$), then in PM $< P_i : i \in I\rangle$ is a well-ordered family of nonempty sets which has no infinite subfamily with a choice function. Also, in V there is an $\in$-isomorphism $F: PM_i \rightarrow \{x \in PM : A_x \cap TC(x) \subseteq P_i\}$, whence G satisfies all Jech–Sochor bounded statements holding in $G_i$.

If $G$ is a topological ultraproduct, modulo a countably incomplete ultrafilter, then $G$ is a P-space, i.e., **G** is closed under countable intersections, whence the principle of dependent choices DC holds. Hence the bounded statement $AC^{\omega}$ is not inherited by continuous open homomorphisms.[2] However, the axiom of choice for families of finite sets, $AC_{fin}$, fails.

**Theorem** *If $G$ is abelian or a* P*-space, then $G$ does not satisfy* $AC_{fin}$*, unless $G$ is discrete.*

*Proof:* The common property of P-groups and abelian groups which we use is: For $x \in$ PM and $g \in G$, $\text{fix}\{(\hat{dg}^k)x : k \in \mathbb{Z}\} \in \mathbf{G}$. We show by an application of Howard's argument [11], that if $G$ satisfies $AC_{fin}$, it satisfies AC, whence $G$ is discrete. If $X \in$ PM, we let $f$ be a choice function on $[X]^{wo} \backslash \{\varnothing\}$ (which exists by $AC_{wo}$). Then fix $X \geq$ sym $(X,f)$ and $X$ is well-orderable. We assume, on the contrary, that $x \in X$ and $g \in$ sym $(X,f) \backslash$sym $x$. Then $Y = \{(\hat{dg}^k)x : k \in \mathbb{Z}\} \in$ PM. Since $Y$ is wo (well-orderable), we can form $y = f(Y) \in Y$. Since $g \notin$ sym $x$, $(\hat{dg})y \neq y$, contradicting $(\hat{dg})y = ((\hat{dg})f)((\hat{dg})Y) = y$. So in particular the set $A$ of atoms is well-orderable, whence $\{\text{id}\} = \text{fix } A \in \mathbf{G}$.

*3 **The translation problem*** The problem of finding explicit topological characterizations is open for most persistent statements. In 1967 Mathias observed that $G$ satisfies AC iff $G$ is discrete, and recently Blass has given a combinatorial (in terms of Ramsey groups) characterization of the Boolean prime ideal theorem. In [3] he discovered an axiom SVC (small violations of choice) which holds in all permutation models, and also gave translations of choice principles in terms of SVC-witnesses. We show that such translations automatically establish the corresponding global (class-) forms of these choice principles in the model, and also give equivalent properties of the group, though they are clumsy to state.

A $\chi$-set $S$ is an SVC-witness, i.e., for each set $x$ there are $\alpha \in$ On and a surjective $f : \alpha \times S \to x$. Dually, $S$ is a $\gamma$-set, if for each $x$ there exist $\alpha \in$ On and an injective $f : x \to \alpha \times S$. Each $\gamma$-set is $\chi$, and if $S$ is $\chi$ then $P(S)$ is $\gamma$. SVC states that there is a $\chi$-set. For example, in ZFA, AC is equivalent to the existence of a $\chi$-set $S$ with a Dedekind-finite $P(S)$. It is an open problem to determine the status of the assertion that every $\chi$-set is $\gamma$ in the hierarchy of choice principles in ZFA + SVC. The partition principle (if $P$ is a family of disjoint nonempty sets, then $P \leq \cup P$) implies this assertion.

**3.1 Theorem** *Let* PM *be a permutation model generated in* V$(X)$ *by* $G$, $\mathbf{G} = \mathbf{G}^d_{nat}$, *and* $S \in$ PM.

(1) *The following statements are equivalent:*
 (i) *$S$ is $\chi$*
 (ii) *There is a surjective function* $F : S \times \text{On} \to \text{PM}$ *in* PM
 (iii) *There is an open $H < G$, such that* $\{\text{sym}_H x : x \in S\}$ *is a neighborhood base of* id *in $H$ (this is the reason for coining the term $\chi$-set), where* $\text{sym}_H x = H \cap \text{sym}_{dG} x$.

(2) *The following statements are also equivalent.*

(i) *$S$ is a $\gamma$-set*

(ii) *There is an injective function $F$*: $\mathrm{PM} \to S \times \mathrm{On}$ *in* PM

(iii) *There is an open $H < G$ such that* $\{\mathrm{sym}_H x : x \in S\}$ *is the set of all open subgroups of $H$.*

*Proof:* We avoid mentioning $d$. The first observation is abstracted from [3]:

*There is a set* $B \in \mathrm{PM}$, *such that* $\{\mathrm{sym}\, x : x \in B\} = \{H < G : H \textit{ open}\}$.

As was observed in the proof of 2.2, for each $H < G$ there is an $x_H \in \mathrm{PM}$, such that $\mathrm{sym}\, x_H = H$, $H \to x_H$ in V. We set $B = \bigcup\{\mathrm{orb}\, x_H : H < G\}$, which is a set in $\mathrm{V}(X)$ since $G \in V$. As $B \subseteq \mathrm{PM}$ and $\mathrm{sym}\, B = G$, $B \in \mathrm{PM}$.

We now prove (1). (ii) ⇒ (i) follows from replacement. For (i) ⇒ (iii) we set $H = \mathrm{sym}_G f$, where $f : \alpha \times S \to B$ is onto. For (iii) ⇒ (ii), we first observe that $\mathrm{orb}_H\langle s,x\rangle$ is a surjective mapping from $\mathrm{orb}_H s$ onto $\mathrm{orb}_H x$, if $\mathrm{sym}_H s < \mathrm{sym}_H x$. Hence if $\mathrm{orb}_H x_\alpha$, $\alpha \in \mathrm{On}$, is an enumeration of the $H$-orbits of PM and $\mathrm{sym}_H s_\alpha < \mathrm{sym}_H x_\alpha$, where $\alpha \to (s_\alpha, x_\alpha)$ is in $\mathrm{V}(X)$, then $F = \bigcup\{\mathrm{orb}_H\langle\langle s_\alpha, \alpha\rangle, x_\alpha\rangle : \alpha \in \mathrm{On}\}$ is a mapping from $\mathrm{dom}\, F \subseteq S \times \mathrm{On}$ onto PM and $\mathrm{sym}\, F = H$.

The proof of (2) proceeds similarly. For (i) ⇒ (iii) one sets $H = \mathrm{sym}_G f$ for some one-to-one $f : B \to \alpha \times A$; $\mathrm{sym}_H f(x) \supseteq \mathrm{sym}_H x$ is trivial and "$\subseteq$" follows from $f(x) = f(\hat{d}hx)$ for $h \in \mathrm{sym}_H f(x)$ and injectivity. For (iii) ⇒ (ii) we take $\mathrm{sym}_H s_\alpha = \mathrm{sym}_H x_\alpha$ and observe that $\mathrm{orb}_H\langle x_\alpha, s_\alpha\rangle$ is a bijective mapping between the orbits, whence $F = \bigcup\{\mathrm{orb}_H\langle x_\alpha, \langle s_\alpha, \alpha\rangle\rangle : \alpha \in \mathrm{On}\} : \mathrm{PM} \to S \times \mathrm{On}$ is a symmetric one-to-one mapping.

*3.2* It follows from (2) that PM is covered by a well-orderable class of finite sets (the class form CMC of Levy's axiom of multiple choice MC), iff PM satisfies MC. Consequently, as a corollary to [7], we obtain: *G satisfies* MC, *iff G is locally bounded.* In [7] it was shown that PM satisfies CMC iff the automorphism group Aut is locally compact.

If Aut is locally compact, it is locally bounded and hence satisfies CMC. And if Aut satisfies MC, it is locally bounded and therefore its Weil-completion $G$ is a locally compact subgroup of $S(A_X)$ (working with the natural representation of Aut). Since each $g \in G$ corresponds to a Cauchy net in $\mathbf{Aut}_{\mathrm{nat}}$, for $x \in \mathrm{PM}$ a value $g(x) \in \mathrm{PM}$ can be defined, thus identifying $g$ with an automorphism, i.e., $G = \mathrm{Aut}$.

*3.3* The first independence theorem which used an uncountable set of atoms (V-cardinality) appeared in Mostowski [20], whose proof showed that $\mathrm{AC}^\kappa$ (AC for families of infinite cardinality at most $\kappa \in \mathrm{On}$) does not imply well-orderable choice $\mathrm{AC}^{\mathrm{wo}}$. Moreover, as follows from the following cardinal inequality, countably many atoms do not suffice. Concerning a possible converse of this inequality, it was observed by Levy [15] that the character $\chi(\mathbf{G}_{\mathrm{nat}})$ can be arbitrarily high, and G would still satisfy the same persistent statements as $S(\omega)$ with the product topology.

**Theorem** *If G satisfies* $\mathrm{AC}^\kappa$ *and* $\chi(\mathbf{G}_{\mathrm{nat}}) \le \kappa$, *then G satisfies* $\mathrm{AC}^{\mathrm{wo}}$.

*Proof:* Since there will be no confusion, we write $\alpha$ instead of $\bar{\alpha}$. $S = \langle S_\alpha : \alpha \in \lambda\rangle$ is a well-orderable sequence of nonempty sets, and $H = \text{sym } S \in \mathbf{G}_{\text{nat}}$. We construct a choice function $\sigma \in$ PM. There is a neighborhood base of id of the form $\langle \text{sym}_G x_\alpha : \alpha \in \kappa\rangle$, $\alpha \to x_\alpha \in$ PM in $V(X)$. From $AC^\kappa$ we obtain a choice function $f \in$ PM for the transfinite sequence $\langle \text{orb}_H x_\alpha : \alpha \in \kappa\rangle \in$ PM, such that $f(\alpha) \in \text{orb}_H x_\alpha$. In $V(X)$, where AC holds, we let $s$ be a choice function for $S$, $s(\alpha) \in S_\alpha$, and we define $\phi \in \kappa^\lambda \subseteq$ PM such that sym $x_{\phi(\alpha)} <$ sym $s(\alpha)$, $\alpha \in \lambda$. Then $F = \bigcup\{\text{orb}_H\langle\langle x_{\phi(\alpha)}, \alpha\rangle, s(\alpha)\rangle : \alpha \in \lambda\}$ is a function, $F \in$ PM, and we define $\sigma(\alpha) = F\langle f(\phi(\alpha)), \alpha\rangle \in \text{orb}_H s(\alpha) \subseteq S_\alpha$, a choice function of $S$ in PM.

***4 Support functions*** We now investigate structural properties of permutation models that are not persistent. PM $\subseteq V(X)$ is a finite support model, if there exist a group $G$ and a representation $d: G \to S(X)$, such that $\mathbf{G}^d_{\text{fin}}$ generates PM (i.e., PM = PM$(d, G, \cdot, \mathbf{G}^d_{\text{fin}}, X)$); $e \in [A_x]^{<\omega}$ is a support of $x \in$ PM, if sym $x \supseteq$ fix $e$. In $V(X)$ there is a function $S$: PM $\to [A_x]^{<\omega}$ such that sym $x \supseteq$ fix $S(x)$. If, in addition, $S$ is a class of PM, it is called a *support function* for $G$. PM has a support function iff there is a group $G$ with a support function, such that $\mathbf{G}_{\text{fin}}$ generates PM. PM is an *M-model*, if every $x$ has a least support supp$(x)$. This notion was introduced in Mostowski [18]. We start with a first-order characterization of the finite-support-model property.

**4.1 Theorem**
(1) PM *is a finite-support model iff* $A^{<\omega} = \bigcup\{A^n : n \in \omega\}$ *is a* $\chi$*-set*
(2) PM *has a support function iff* $[A^{<\omega}]^{<\omega}$ *is a* $\gamma$*-set.*

Moreover, if PM is generated by $G$, then it is a finite-support model iff for some open $H < G$ $\mathbf{H}_{\text{fin}} = \mathbf{H}_{\text{nat}}$, and it has a support function iff there is a support function for some open $H < G$. From this one easily deduces that the usual permutation models constructed from normal ideals of infinite sets cannot be obtained from a different group and the ideal of finite sets of atoms.

*Proof:* (1): If $\mathbf{G}_{\text{fin}}$ generates PM for some group $G$, then $\{\text{sym}_G f : f \in A^{<\omega}\}$ is a local base of id for $\mathbf{G}_{\text{fin}}$, whence $A^{<\omega}$ is $\chi$. If conversely $\{\text{sym}_H f : f \in A^{<\omega}\}$ is a local base of id for some $H < G$ in $\mathbf{G}_{\text{nat}}$, where $\mathbf{G}_{\text{nat}}$ generates PM, then $\{\text{fix}_H(\text{im } f) : f \in A^{<\omega}\} \subseteq \mathbf{H}_{\text{fin}}$ is an open base of id for $\mathbf{H}_{\text{nat}}$. Hence $\mathbf{H}_{\text{nat}} = \mathbf{H}_{\text{fin}}$, and since $H$ is open, it generates PM too.

We now prove (2): If $[A^{<\omega}]^{<\omega}$ is a $\gamma$-set, and $F(x) = \langle T(x), \alpha_x\rangle$, $x \in$ PM, is the mapping from 3.1.(2)(ii), then $\text{sym}_H T(x) = \text{sym}_H x$ for $H = \text{sym}_G T$ and $S(x) = \bigcup\{\text{im } f : f \in T(x)\}$ is a support function for $\mathbf{H}_{\text{nat}}$ : $\text{sym}_H x \supseteq \text{fix}_H S(x)$. Hence $\mathbf{H}_{\text{nat}} = \mathbf{H}_{\text{fin}}$, and since $H$ is open $\mathbf{H}_{\text{nat}}$ generates PM, which therefore has a support function. If conversely $S$: PM $\to [A]^{<\omega}$ is a support function, then we set $H = \text{sym}_G S$. By the proof of 2.2, every open subgroup $K$ of $H$ is of the form $\text{sym}_H x$, for some $x \in$ PM. We let $f: n \to S(x)$ be any bijection and set $F = \text{orb}_K f$, $K = \text{sym}_H x$. Then $K = \text{sym}_H F$, and since $F \subseteq S(x)^n$ is finite, $F \in [A^{<\omega}]^{<\omega}$.

***4.2*** If $G$ is totally bounded, then every permutation model PM generated by $G$ has a support function $S$ for $G$. For by 3.4 of [7] $\mathbf{G}_{\text{nat}} = \mathbf{G}_{\text{fin}}$. We set $O =$

$\{\text{orb } a : a \in A_x\} \subseteq [A_x]^{<\omega}$, and observe that in PM $[O]^{<\omega}$ can be well-ordered by some relation $<$, sym $< = G$. Hence $S(x) = \cup e$, $e \in [O]^{<\omega}$ the $<$-least element such that sym $x \supseteq \text{fix}(\cup e)$, defines a support function for $G$, sym $S = G$. But in general a group has a representation without a support function.

**Example** *There exists a finite support model without a support function.*

*Proof:* We work in $V(\mathbb{Z} \times \omega)$, $\mathbb{Z}$ the integers, and let $G = \mathbb{Z}^\omega$ operate on the set $A$ of atoms through $dg(z,n) = (g(n) + z,n)$. PM is the model generated from $\mathbf{G}_{\text{fin}}$. If PM has a support function, then there is an open $H < G$ such that there is a support function $S$ for $\mathbf{H}_{\text{fin}} = \mathbf{H}_{\text{nat}}$. Since $S$ is a support function for each open subgroup of $H$, we may assume that $H = \text{fix}_G e = \text{sym}_H S$ for some $e \in [A]^{<\omega}$. We define the following sets of atoms of PM: $P_n = \{(z,n,o) : z \in \mathbb{Z}\}$, $E_n = \{(z,n,O) : z \text{ even}\}$, $O_n = P_n \setminus E_n$, and we observe that $\text{sym}_G P_n = \text{sym}_G\{E_n, O_n\} = G$, whence $f(n) = S(E_n) \cup S(O_n) \in [A]^{<\omega}$ defines a function in PM, $\text{sym}_H f = H$. If $e \cap P_n = \emptyset$, then $f(n) \cap P_n \neq \emptyset$; for otherwise $g \in \text{fix}_H\ S(E_n)$, where $g(n) = 1$ and $g(i) = 0$ for $i \neq n$, despite the fact that $\hat{d}gE_n = O_n$. We now obtain a contradiction: A standard permutation argument shows that there does not exist a function $f : \omega \to [A]^{<\omega}$ such that $f(n) \cap P_n \neq \emptyset$ for infinitely many $n$.

*4.3* To complete the picture, we show that every linear group generates some finite-support model. This together with 2.1 proves that, for the purpose of independence proofs, the class of finite-support models with arbitrarily large sets of atoms (in view of 3.3) is sufficient. With a proper class of atoms the situation might be different.

**Theorem** *For every group-generated Hausdorff topological group* $(G,\cdot,\mathbf{G})$ *there exists a permutation model* PM in some $V(X)$ *and a representation* $d$ *of* $G$ *as a subgroup of* $S(X)$, *such that* $\mathbf{G}^d_{\text{fin}} = \mathbf{G}$ *generates* PM, $A_x \subseteq$ PM, *and* PM *has a support function* $S$ *for* $\mathbf{G}^d_{\text{fin}}$ *such that* $S(x)$ *is a minimal support of* $x$.

*Proof:* We begin with an adaptation of a standard result from permutation group theory (see, e.g., [10]).

If $(G,\cdot,\mathbf{G})$ is a group-generated Hausdorff topological group, then there is a permutation model QM in some $V(X)$ and a representation $t$ of $G$ as a subgroup of $S(X)$, such that $\mathbf{G}^t_{\text{fin}} = \mathbf{G}^t_{\text{nat}} = \mathbf{G}$ generates PM.

We set $X = \{h \cdot H : h \in G \text{ and } H < G \text{ open}\}$ and let $G$ operate on $X$ by $(tg)(hH) = ghH$. In $V(X)$ we form the permutation model QM generated by $\mathbf{G}^t_{\text{fin}}$. Then $A_x \subseteq$ QM, $\mathbf{G}^t_{\text{fin}} = \mathbf{G}^t_{\text{nat}}$, and since $\text{sym}_{tG}(hH,0) = hHh^{-1}$, $\mathbf{G}_{\text{fin}} = \mathbf{G}$. Since $\mathbf{G}$ is Hausdorff, $t$ is an injective representation of $G$.

We let $B$ be the set found in the proof of 3.1 for QM. In V we form $V(B)$ and define the operation $d$ of $G$ on $B$ by $(dg)(b) = (\hat{t}g)b$, $\hat{t}g$ the $\in$-automorphism of QM. PM is generated from $\mathbf{G}^d_{\text{fin}}$ in $V(B)$. Since $\{\text{sym}_{dG}(b,0) : b \in B\} = \{\text{sym}_{tG} b : b \in B\} = \{H < G : H \text{ open}\}$, $\mathbf{G}^d_{\text{fin}} = \mathbf{G}$ and $A_B$ is a $\gamma$-set in PM; hence there exists a support function $S : \text{PM} \to [A_B]^{\leq 1}$ in PM (as in the proof of 4.1). Obviously, $S(x)$ is a minimal support of $x$.

M-models cannot be obtained in this way. As follows from [4], if PM is an M-model then $\text{AC}^{\text{wo}}_{\text{wo}}$ holds, the axiom of choice for well-ordered families of well-orderable sets, as does $\mathrm{PAC_{wo}}$, which says that every family of well-orderable sets has an infinite subfamily with a choice function. Therefore, a compact group generates an M-model iff it is discrete.[3]

*4.4* The groups that appear in the literature are usually the automorphism groups of some first-order structure on the set of atoms which contains all the information about the permutation model, as in Pincus [22]. To give another example of the above construction, we consider the group $J_p$ of $p$-adic integers, $p$ a prime number. Each $g \in J_p$ is thought of as a formal series $\sum_{n=0}^{\infty} g_n \cdot p^n$, $g_n \in p$, and addition is defined accordingly (cf. 10.2 of Hewitt and Ross [9]). The topology is defined by a nonarchimedean metric, the groups $L_n = \{g \in J_p : g_k = 0, \text{ for } k < n\}$ forming an open neighborhood base of the identity. In fact, each open subgroup of $J_p$ is an $L_n$. $J_p$ is a compact linear monothetic Hausdorff group. For $a \in J_p$, $n \geq 0$, we set $x(a,n) = \{x \in J_p : x_k = a_k \text{ for } k \in n\}$ and $X = \{x(a,n) : a \in J_p, n \geq 0\}$. An operation $d$ of $J_p$ is defined as $(dg)x(a,n) = x(a + g,n)$, giving $\mathrm{sym}(x(a,n),0) = L_n$, whence $\mathrm{PM}(d, J_p, +, \mathbf{J_p}, X)$ is the finite-support model that results from 4.3 when applied to $J_p$. As an application, we add some weak forms of AC to an independence result of Zuckerman [26].

**Lemma** *If $E$ is a finite set of primes, then $G = \Pi \langle J_p : p \in E \rangle$ satisfies* MC, $\mathrm{KW^{wo}}$ *(the well-orderable selection principle, that for each well-ordered family* $\mathbf{F}$ *of sets $F$ with at least two elements there is a selection $S : \varnothing \neq S(F) \subseteq F$ and $F \backslash S(F) \neq \varnothing$), and* $\mathrm{AC}_n$ *(choice for families of n-element sets) iff n is not an integer combination $n = \sum_{p \in E} n(p) \cdot p$, $n(p) \in \omega$, of E.*

*Proof:* PM is a model generated by $G$. If $H < G$ is open, because of the special structure of $G$ it is a product of the $L_{n(p)}$'s, whence the index $[H:K]$ is a product of primes in $E$, $K < H < G$ open. In particular, for $K = H \cap \mathrm{sym}\, x$, the cardinality $|\mathrm{orb}_H x| = [H:K]$ has all its prime factors in $E$. If $|S| = n$ and $n$ is not an integer combination of $E$, there is some $x \in S$ such that $|\mathrm{orb}_H x| = 1$, $H = \mathrm{sym}\, S$, because $S = \bigcup\{\mathrm{orb}_H x : x \in S\}$, i.e., $\mathrm{sym}\, x \supseteq \mathrm{sym}\, S$. It follows from [11] that every family of $n$-element sets has a choice function. If, on the other hand, $n = \sum_{p \in F} n(p) \cdot p$, $\varnothing \neq F \subseteq E$, then $\mathbf{F} = \bigcup\{\mathrm{orb}\, F_n : n \in \omega\}$, $F_n = \bigcup\{n(p) \times \mathrm{orb}_{H(p,n)} x_n^p : p \in F\}$, $H(p,n) = L_n \times \prod_{q \neq p} J_q$, $L_n < J_p$, and $\mathrm{sym}\, x_n^p = H(p, n+1)$, is a counterexample to $\mathrm{PAC}_n$ (every family of $n$-element sets has an infinite subfamily with a choice function). We now prove $\mathrm{KW^{wo}}$. In view of MC it suffices to prove $\mathrm{KW^{wo}_{fin}}$, and we may assume that the family has the form $\mathbf{F} = \langle F_\alpha : \alpha \in \kappa \rangle$, $F_\alpha = \mathrm{orb}_H\, x_\alpha$, $H = \mathrm{sym}\, \mathbf{F}$, where $H = \prod_{p \in E} L_{n(p)} \left[ \left( \prod_p p^{n(p)} \right) \mathbb{Z} \right]$ and $\mathrm{sym}\, x_\alpha = \prod_{p \in E} L_{m(p,\alpha)}$. If $m(p,\alpha) \geq n(p)$ for all $p \in E$, $F_\alpha$ is a singleton, a contradiction. Hence $\kappa = \bigcup\{K_p : p \in E\}$, $K_p = \{\alpha \in \kappa : m(p,\alpha) < n(p)\}$, and we define the selection function on $K_p$ as follows: $S_p(\alpha) = \mathrm{orb}_{H(p)} x_\alpha$, where $H(p) = L_{n(p)-1} \times \prod_{q \neq p} L_{n(q)}$. Since $G$ is abelian, sym $(\text{orb}_H\, x_\alpha) = \langle H, \text{sym } x_\alpha\rangle$, the group generated by $H$ and sym $x_\alpha$, and as $\langle H, \text{sym } x_\alpha\rangle \neq \langle H(p), \text{sym } x_\alpha\rangle$ in the $p$th coordinate, $S_p(\alpha) \neq F_\alpha$.

Despite their importance in topological algebra, the $p$-adic groups seem to have escaped attention by set theorists. They prove the following new independence results: (i) Mostowski's independence results [19] can be augmented by $\text{KW}^{\text{wo}} + \text{AC}_{\leq n} \nrightarrow \text{AC}_m$, if $m, n$ do not satisfy the condition $(S)$; (ii) $G = \Pi\ \langle J_p : p \text{ prime}\rangle$ is a monothetic group, whence it satisfies $\text{AC}_n^{\text{wo}}$ for each $n \geq 1$ (cf. [7], p. 158), but by 2.4 and the lemma no instance of $\text{PAC}_n$, $n \geq 2$, holds.

With respect to persistency, finite products of $J_p$ are the only linear monothetic groups which satisfy $\text{KW}^{\text{wo}}$. For if $G$ is linear and monothetic, it is discrete or totally bounded, since all nontrivial subgroups of $\mathbb{Z}$ have a finite index. Hence by 25.16 of [9] $G$ satisfies the same persistent statements as $\Pi\langle G_p : p \text{ prime}\rangle$, where $G_p$ is $J_p$ or $\mathbb{Z}\ (p^{n(p)})$. If there are infinitely many factors $G_p \neq 1$, $\text{PKW}^{\omega}$ fails (counterexample: $\text{orb}_G x_i$, $x_i \in P_i$ from 2.4). Otherwise, if $G$ is not discrete, it contains finitely many factors $J_p$ and their product is an open subgroup of $G$, generating the same model. If $G$ is discrete, it satisfies the same persistent statements as $1 =$ the empty product. We may conclude:

**Corollary** *A monothetic linear group G satisfies the same persistent statements as some finite product* $P = \Pi\langle J_p : p \in E\rangle$ *iff G satisfies* $\text{KW}^{\text{wo}}$.

Moreover, $p \in E$ iff $\text{AC}_p$ fails and, as follows from the proof of the lemma and 2.4, $\text{PKW}_{\overline{p}}$ is false for $J_p$ and hence for $P$, if $p \in E$.

Another application of the $p$-adic groups is motivated by quantum mechanics. Benioff [2] has considered several extension mappings $\text{T} \to \tilde{\text{T}}$ of quantum mechanical operators T on a Hilbert-space $H$ of some model $\text{PM} \subseteq \text{V}(X)$ to $\tilde{\text{T}}$ on a corresponding Hilbert-space $\tilde{H}$ in the "real world" $\text{V}(X)$; e.g., if $\tilde{H}$ is the completion of $H$ in $\text{V}(X)$ then $\tilde{\text{T}}$ is the unique extension of T to $\tilde{H}$. Benioff has applied his results to solve ontological questions.[4] Here we investigate the spectral behavior under this extension for the following spaces.

**Example** If $G$ is a nondiscrete linear monothetic group, it satisfies the following statement: "There is a Hilbert space $H$ which is not finite dimensional while each orthonormal system is finite".

*Proof:* As follows from the previous remarks, $\text{PKW}_{\text{fin}}$ is false; let $\langle F_n : n \in \text{M}\rangle$ be a counterexample.

We set $H = \bigcap\{\text{Ker } \phi_n : n \in \text{M}\}$, where $\phi_n : \text{l}_2(\bigcup\{F_n : n \in \text{M}\}) \to \mathbb{C}$ is the continuous linear functional $\phi_n(x) = \sum_{a \in F_n} x(a)$. If $\langle x_n : n \in \omega\rangle$ is a sequence in $H$, then $s = \bigcup\{s(x_n) : n \in \omega\}$ is finite, where $s(x_n) = \{a : x(a) \neq 0\}$, whence $H$ is locally sequentially compact. For $\phi_n(x) = 0$ implies $F_n \backslash s_+(x) \neq \emptyset$, $s_+(x) = \{a : \text{Re } x(a) > 0 \text{ or } (\text{Re } x(a) = 0 \text{ and } \text{Im } x(a) > 0)\}$, and $s(x) \cap F_n \neq \emptyset$ implies $s_+(x) \cap F_n \neq \emptyset$, whence an infinite $s$ defines a selection function $S(n) = F_n \cap s_+(x_m)$, $m$ minimal with $s(x_m) \cap F_n \neq \emptyset$, with an infinite domain dom $S = \{n \in \text{M} : s \cap F_n \neq \emptyset\}$. If $D$ is an orthonormal subset of $H$, then $[D]^{<\omega}$ is Dedekind-finite (cf. [5]), and the axiom of multiple choice implies that $D$ is finite.

Since $G$ satisfies MC, $D = \bigcup\{F_n : n \in M\}$ can be represented as a well-orderable union of finite sets, whence there is a diagonal operator $S$ on $l_2(D)$ for which the topological approximate point spectrum $\Pi(S) \neq \sigma_p(S)$, the point spectrum (in the model PM generated by $G$). If $T$ is the restriction of $S$ to $H$, then the sequential approximate point spectrum $\sigma_a(T) = \sigma_p(T) \neq \Pi(\mathrm{T})$ (see [5] for the background). For $\tilde{H}$ we have $\sigma_a(\tilde{T}) = \Pi(\tilde{T}) = \Pi(T)$; hence $\sigma_a(T) \neq \sigma_a(\tilde{T})$ changes. If $\hat{H}$ is the nonstandard hull of $H$ in $\mathrm{V}(X)$ and $\bar{H} = H$, the nonstandard hull in PM (with respect to a countable ultrapower), then for the extension $\hat{T}$ to $\hat{H}$ we have $\sigma_p(T) \neq \sigma_p(\hat{T}) \supseteq \Pi(T)$.

We finally note that the results of this paper do not apply if the permutation model is formed in a subclass of $\mathrm{V}(X)$, the most interesting case being $\mathrm{V}_1(X) = \{x \in \mathrm{V}(X) : \mathrm{TC}(x) \cap A_x \text{ is finite}\}$, which is due to Levy.

## NOTES

1. In [23] the result is stated for $\Sigma_2$. In a letter sent on May 5, 1988, Pincus communicated to me a proof that works for sentences $\Pi_2$ in the power set operation.
2. This is to be expected in view of the following fact: If $x \in$ PM and PM$(x)$ is the $\mathrm{V}(x)$ of Section 1.1 relativized to $\mathrm{PM}_1$ then PM$(x)$ is a permutation model, generated by the quotient group sym $(x)/$fix $(x)$. Conversely, each $G/H$, $H$ a closed normal subgroup of $G$, generates some PM$(x)$. Hence quotients preserve countable choice.
3. The Pincus construction may be strengthened as follows. There is some ordinal $\beta$ such that each open subgroup $H$ of $G$ is $H =$ sym $(x)$, for some $x \in$ PM of rank rk $(x \leq \beta)$. The Pincus model QM is constructed in $\mathrm{V}(\{x \in \mathrm{PM} : \mathrm{rk}\ (x) \leq \beta\})$ with the group action induced by the action of $G$ in PM.
4. For instance, it was observed by W. Boos (*The Journal of Symbolic Logic*, vol. 53 (1988), p. 1289) that in appropriate generic extensions the extended model contains hidden parameters for the ground model.

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*Institut für Mathematik*
*und Angewandte Statistik*
*Universität für Bodenkultur*
*Gregor Mendel-Strasse 33*
*A-1180 Wien*
*Österreich*