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Constructive Predicate Logic with Strong Negation and Model Theory

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In this paper, we attempt to investigate the so-called constructive predicate logic with strong negation from a model-theoretical point of view.

Strong negation was first introduced by Nelson [8] in connection with Kleene's *recursive realizability*. At the same time, Markov [7] showed independently that intuitionistic (Heyting) negation can be defined by strong negation and implication. Then Vorob'ev [14] formulated constructive propositional logic with strong negation. Polish logicians such as Rasiowa [9],[10] studied in terms of lattice theory. Additionally, we can find a Gentzen-type formulation by Almukdad and Nelson [1] and Ishimoto [5],[6], and a model-theoretic study by Thomason [13] and Routley [11], to cite only a few.

Strong negation is a constructive negation different from *Heyting negation*. For example, in intuitionistic logic, $\neg(A \wedge B)$ is not equivalent to the derivability of at least one formula of $\neg A$ or $\neg B$. And we cannot prove the equivalence between $\neg\forall x A(x)$ and $\neg A(t)$. But these are equivalent in constructive logic with strong negation.

In this paper, we research this system on the basis of Kripke's *many worlds semantics* (the so-called *Kripke model*), and try to provide a *Henkin-type proof* of the completeness theorem for the system. We will also inquire into the relationships among constructive predicate logic with strong negation, classical logic, and intuitionistic logic.

1 Constructive predicate logic with strong negation Constructive predicate logic with strong negation, instead of Heyting negation, is designated as S. As stated above, Heyting negation can be defined in S by way of strong negation and implication, as S does not have it as one of its primitives. We call the sys-

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tem with Heyting negation S^+. The well-formed formulas (w.f.f.) in S and S^+ are defined in terms of parameters (free variables), bound variables, predicate variables including equality (=) as a binary predicate and function symbols with any finite number of arguments as well as seven logical symbols, namely, $\wedge$ (conjunction), $\vee$ (disjunction), $\supset$ (implication), $\neg$ (Heyting negation), $\sim$ (strong negation), $\forall$ (universal quantifier), and $\exists$ (existential quantifier). Here we ought to pay attention to the fact that implication is a logical symbol in the sense of intuitionistic logic. In fact Rasiowa [10] introduced a symbol $\Rightarrow$ defined in terms of $\rightarrow(\supset)$ as follows:

$$A \Rightarrow B \equiv (A \rightarrow B) \wedge (\sim B \rightarrow \sim A). \tag{1.1}$$

Markov [7], on the other hand, used *complete equality* $\doteqdot$;

$$A \doteqdot B \equiv (A \equiv B) \wedge (\sim A \equiv \sim B). \tag{1.2}$$

Then, S is the system in which strong negation is embedded in intuitionistic logic.

We can define the Hilbert-type version of S as follows:

$$\vdash A \supset . B \supset A \tag{1.3}$$
$$\vdash (A \supset . B \supset C) \supset . (A \supset B) \supset (A \supset C) \tag{1.4}$$
$$\vdash A \supset (B \supset . A \wedge B) \tag{1.5}$$
$$\vdash A \wedge B . \supset . A \tag{1.6}$$
$$\vdash A \wedge B . \supset . B \tag{1.7}$$
$$\vdash (A \supset C) \supset . (B \supset C) \supset (A \vee B . \supset C) \tag{1.8}$$
$$\vdash A \supset . A \vee B \tag{1.9}$$
$$\vdash B \supset . A \vee B \tag{1.10}$$
$$\vdash \sim A \supset . A \supset B \tag{1.11}$$
$$\vdash \sim (A \supset B) \equiv . A \wedge \sim B \tag{1.12}$$
$$\vdash \sim (A \wedge B) \equiv . \sim A \vee \sim B \tag{1.13}$$
$$\vdash \sim (A \vee B) \equiv . \sim A \wedge \sim B \tag{1.14}$$
$$\vdash A \equiv \sim\sim A \tag{1.15}$$
$$\vdash \forall x A(x) \supset A(t) \tag{1.16}$$
$$\vdash A(t) \supset \exists x A(x) \tag{1.17}$$
$$\vdash \sim \forall x A(x) \equiv \exists x \sim A(x) \tag{1.18}$$
$$\vdash \sim \exists x A(x) \equiv \forall x \sim A(x). \tag{1.19}$$

This version of S is closed under *detachment* and the rules of restriction on variables in quantification.

$$\vdash A \supset B(t) \Rightarrow \vdash A \supset \forall x B(x) \tag{1.20}$$
$$\vdash A(t) \supset B \Rightarrow \vdash \exists x A(x) \supset B. \tag{1.21}$$

$A \equiv B$ is an abbreviation for $(A \supset B) \wedge (B \supset A)$. Deleting Axiom (1.12) from the list of axioms, Fitch's system [3] is obtained. It should be noted here that the formula $\forall x(A \vee B(x)) \supset (A \vee \forall x B(x))$ (x not free in A) is not provable in S. But it is provable in Fitch's system (see Fitch [3] and Thomason [13] for details). Fitch's system has a Kripke semantics with constant domains, whereas S requires variable domains, as discussed later.

As stated above, Heyting negation can be defined in terms of strong negation;

$$\neg A \equiv A \supset \sim A. \tag{1.22}$$

In S^+ involving Heyting negation, the following formulas are provable:

$$\vdash (A \supset B) \supset . (A \supset \neg B) \supset \neg A \tag{1.23}$$
$$\vdash \neg A \supset . A \supset B \tag{1.24}$$
$$\vdash A \equiv \sim\neg A. \tag{1.25}$$

Naturally we can find that S^+ is equal to Vorob'ev's system. Moreover, the following two axioms for equality can be added to S:

$$\vdash a = a \tag{1.26}$$
$$\vdash a = b \supset . A(a) \supset A(b). \tag{1.27}$$

2 Kripke's many-worlds semantics The Kripke model for S is a quintuple $\langle G, R, V_P, V_N, P\rangle$ where:

(2.1) G is a nonempty set (of possible worlds).
(2.2) R is a relation, reflexive and transitive, defined on two elements belonging to $G \times G$.
(2.3) V_P and V_N are functions, each of which maps every propositional variable belonging to a subset of G satisfying the following:
 a. $V_P(A) \cap V_N(A) = \varnothing$
 b. $\forall\Gamma^*(\Gamma \in V_P(A) \Rightarrow \Gamma^* \in V_P(A))$
 c. $\forall\Gamma^*(\Gamma \in V_N(A) \Rightarrow \Gamma^* \in V_N(A))$.
(2.4) P is a function which maps from G to a nonempty set of parameters which holds $P(\Gamma) \subseteq P(\Gamma^*)$ for any Γ satisfying $\Gamma R \Gamma^*$.

Here A denotes a propositional variable, and Greek capitals such as Γ, Δ, etc. denote meta-variables ranging over elements of G. And $\forall\Gamma^*$ and $\exists\Gamma^*$ are quantifications over the elements $\Gamma^* \in G$ such that $\Gamma R \Gamma^*$ (cf. Fitting [4]). Namely, the relation $\Gamma R \Gamma^*$ says that Γ^* is better than Γ because of partial order relation of R.

We can define two forcings, $\models_P$ and $\models_N$, with respect to any element of G and propositional variable in the Kripke model $\langle G, R, V_P, V_N, P\rangle$ as follows;

$$\Gamma \models_P A \Leftrightarrow \Gamma \in V_P(A) \tag{2.5a}$$
$$\Gamma \models_N A \Leftrightarrow \Gamma \in V_N(A). \tag{2.5b}$$

Thus, the following relations as regards any formulas in S or S^+ are proved by induction on the length of the formulas;

$$\Gamma \models_P A \wedge B \Leftrightarrow \Gamma \models_P A \text{ and } \Gamma \models_P B \tag{2.6}$$
$$\Gamma \models_P A \vee B \Leftrightarrow \Gamma \models_P A \text{ or } \Gamma \models_P B \tag{2.7}$$
$$\Gamma \models_P A \supset B \Leftrightarrow \forall\Gamma^*(\Gamma^* \models_P A \Rightarrow \Gamma^* \models_P B) \tag{2.8}$$
$$\Gamma \models_P \neg B \Leftrightarrow \forall\Gamma^*\Gamma^* \not\models_P B \text{ (not } \Gamma^* \models_P B) \tag{2.9}$$
$$\Gamma \models_P \sim B \Leftrightarrow \Gamma \models_N B \tag{2.10}$$
$$\Gamma \models_P A \Leftrightarrow \forall\Gamma^*\Gamma^* \models_P A \text{ for any atomic } A \tag{2.11}$$
$$\Gamma \models_P \forall x A(x) \Leftrightarrow \forall\Gamma^*\Gamma^* \models_P A(t) \text{ for all } t \in P(\Gamma^*) \tag{2.12}$$
$$\Gamma \models_P \exists x A(x) \Leftrightarrow \Gamma \models_P A(t) \text{ for some } t \in P(\Gamma) \tag{2.13}$$
$$\Gamma \models_N A \wedge B \Leftrightarrow \Gamma \models_N A \text{ or } \Gamma \models_N B \tag{2.14}$$
$$\Gamma \models_N A \vee B \Leftrightarrow \Gamma \models_N A \text{ and } \Gamma \models_N B \tag{2.15}$$
$$\Gamma \models_N A \supset B \Leftrightarrow \Gamma \models_P A \text{ and } \Gamma \models_N B \tag{2.16}$$

$$\Gamma \models_N \neg B \Leftrightarrow \Gamma \models_P B \tag{2.17}$$
$$\Gamma \models_N \sim B \Leftrightarrow \Gamma \models_P B \tag{2.18}$$
$$\Gamma \models_N A \Leftrightarrow \forall\Gamma^*\Gamma^* \models_N A \text{ for any atomic } A \tag{2.19}$$
$$\Gamma \models_N \forall x A(x) \Leftrightarrow \Gamma \models_N A(t) \text{ for some } t \in P(\Gamma) \tag{2.20}$$
$$\Gamma \models_N \exists x A(x) \Leftrightarrow \forall\Gamma^*\Gamma^* \models_N A(t) \text{ for all } t \in P(\Gamma^*). \tag{2.21}$$

Given a model $\langle G, R, V_P, V_N, P\rangle$, if $\Gamma \models_P A$ for any $\Gamma \in G$, then A is valid in the model. If a formula is valid in any model, it is valid. We obtain the model for S by deleting relations (2.9) and (2.17) involving Heyting negation. Moreover, only either of the universal and existential quantifiers is needed since one of them can be defined in terms of the other due to (1.18) and (1.19).

Lemma 1 *Given a model* $\langle G, R, V_P, V_N, P\rangle$,

$$\forall\Gamma^*(\Gamma \models_P A \Rightarrow \Gamma^* \models_P A),$$
$$\forall\Gamma^*(\Gamma \models_N A \Rightarrow \Gamma^* \models_N A)$$

for any $\Gamma \in G$ *and* A.

Proof: We can prove this lemma by induction on the length of the formulas. The induction steps are as follows:

a. $\Gamma \models_P A \wedge B \Leftrightarrow \Gamma \models_P A$ and $\Gamma \models_P B$
$\quad\Rightarrow \Gamma^* \models_P A$ and $\Gamma^* \models_P B \Leftrightarrow \Gamma^* \models_P A \wedge B.$

b. $\Gamma \models_P A \supset B \Leftrightarrow \forall\Gamma^*(\Gamma^* \models_P A \Rightarrow \Gamma^* \models_P B)$
$\quad\Rightarrow \forall\Gamma^{**}(\Gamma^{**} \models_P A \Rightarrow \Gamma^{**} \models_P B) \Leftrightarrow \Gamma^* \models_P A \supset B.$

c. $\Gamma \models_P \forall x A(x) \Leftrightarrow \forall\Gamma^*\Gamma^* \models_P A(t)$ for all $t \in P(\Gamma^*)$
$\quad\Rightarrow \forall\Gamma^{**}\Gamma^{**} \models_P A(t)$ for all $t \in P(\Gamma^{**})$
$\quad\Leftrightarrow \Gamma^* \models_P \forall x A(x).$

d. $\Gamma \models_P \sim A \Leftrightarrow \Gamma \models_N A \Rightarrow \Gamma^* \models_N A \Leftrightarrow \Gamma^* \models_P \sim A.$

e. $\Gamma \models_P \neg A \Leftrightarrow \forall\Gamma^*\Gamma^* \not\models_P A \Rightarrow \forall\Gamma^{**}\Gamma^{**} \not\models_P A \Leftrightarrow \Gamma^* \models_P \neg A.$

In the case of $\models_N$, the induction steps are similar to those in the case of $\models_P$. Lemma 1 tells us that $\models_P$ and $\models_N$ have a monotonicity property.

Lemma 2 *Given a model* $\langle G, R, V_P, V_N, P\rangle$ $\Gamma \models_P A$ *and* $\Gamma \models_N A$ *never come out simultaneously for any* $\Gamma \in G$ *and* A.

The proof is carried out by induction on the length of the formulas on the basis of (2.3a).

Corollary 1 *Given a model* $\langle G, R, V_P, V_N, P\rangle$ $\Gamma \models_P A$ *and* $\Gamma \models_P \sim A$ *never come out simultaneously for any* $\Gamma \in G$ *and* A.

3 The properties of constructive predicate logic with strong negation In this section we are first going to prove the soundness and completeness for S. The soundness can be derived from the fact that the axioms for S, namely (1.3)–(1.19), are valid in any model. Thus we can obtain the consistency theorem for S.

Theorem 1 (Consistency Theorem) *If* A *is provable in* S *then it is valid.*

Proof: We can prove the theorem by induction on the length of the proof. The proof of axioms without strong negation is similar to that of classical logic. The induction steps for other axioms are exemplified as follows:

$$\begin{aligned}
&\Gamma \models_P (\sim A \vee \sim B) \supset \sim(A \wedge B)\\
&\Leftrightarrow \forall\Gamma^*(\Gamma^* \models_P \sim A \vee \sim B \Rightarrow \Gamma^* \models_P \sim(A \wedge B))\\
&\Leftrightarrow \forall\Gamma^*(\Gamma^* \models_P \sim A \text{ or} \Gamma^* \models_P \sim B \Rightarrow \Gamma^* \models_N A \wedge B)\\
&\Leftrightarrow \forall\Gamma^*(\Gamma^* \models_N A \text{ or } \Gamma^* \models_N B \Rightarrow \Gamma^* \models_N A \text{ or } \Gamma^* \models_N B).
\end{aligned}$$

$$\begin{aligned}
&\Gamma \models_P A \wedge \sim B \supset \sim(A \supset B)\\
&\Leftrightarrow \forall\Gamma^*(\Gamma^* \models_P A \wedge \sim B \Rightarrow \Gamma^* \models_P \sim(A \supset B))\\
&\Leftrightarrow \forall\Gamma^*(\Gamma^* \models_P A \text{ and } \Gamma^* \models_P \sim B \Rightarrow \Gamma^* \models_N A \supset B)\\
&\Leftrightarrow \forall\Gamma^*(\Gamma^* \models_P A \text{ and } \Gamma^* \models_N B \Rightarrow \Gamma^* \models_P A \text{ and } \Gamma^* \models_N B).
\end{aligned}$$

$$\begin{aligned}
&\Gamma \models_P \sim\forall x A(x) \supset \exists x \sim A(x)\\
&\Leftrightarrow \forall\Gamma^*(\Gamma^* \models_P \sim\forall x A(x) \Rightarrow \Gamma^* \models_P \exists x \sim A(x))\\
&\Leftrightarrow \forall\Gamma^*(\Gamma^* \models_N \forall x A(x) \Rightarrow \Gamma^* \models_P \sim A(t) \text{ for some } t \in P(\Gamma^*))\\
&\Leftrightarrow \forall\Gamma^*(\Gamma^* \models_N A(t) \Rightarrow \Gamma^* \models_N A(t)) \text{ for some } t \in P(\Gamma^*).
\end{aligned}$$

Likewise we can prove the consistency theorem for S^+:

$$\begin{aligned}
\Gamma \models_P \sim\neg A \supset A &\Leftrightarrow \forall\Gamma^*(\Gamma^* \models_P \sim\neg A \Rightarrow \Gamma^* \models_P A)\\
&\Leftrightarrow \forall\Gamma^*(\Gamma^* \models_N \neg A \Rightarrow \Gamma^* \models_P A)\\
&\Leftrightarrow \forall\Gamma^*(\Gamma^* \models_P A \Rightarrow \Gamma^* \models_P A).
\end{aligned}$$

$$\begin{aligned}
\Gamma \models_P \neg A \supset .A \supset \sim A. &\Leftrightarrow \forall\Gamma^*(\Gamma^* \models_P \neg A \Rightarrow \Gamma^* \models_P A \supset \sim A)\\
&\Leftrightarrow \forall\Gamma^{**}(\Gamma^{**} \models_N A \Rightarrow. \Gamma^{**} \models_P A \Rightarrow \Gamma^{**} \models_N A).
\end{aligned}$$

$$\begin{aligned}
\Gamma \models_P \neg A \supset .A \supset B. &\Leftrightarrow \forall\Gamma^*(\Gamma^* \models_P \neg A \Rightarrow \Gamma^* \models_P A \supset B)\\
&\Leftrightarrow \forall\Gamma^{**}(\Gamma^{**} \not\models_P A \Rightarrow. \Gamma^{**} \models_P A \Rightarrow \Gamma^{**} \models_P B)\\
&\Leftrightarrow \forall\Gamma^{**}(\Gamma^{**} \models_N A \Rightarrow. \Gamma^{**} \models_N B \Rightarrow \Gamma^{**} \models_N A).
\end{aligned}$$

Next we show that the converse of Theorem 1, namely, the completeness theorem, is provable. The proof is carried out by means of the Henkin-type proof construction. Thus we need to introduce several concepts for the proof. According to Shoenfield [12] and van Dalen [2], the intuitionistic counterparts of the Henkin theory can be used in our proof.

Definition A set of sentences Γ satisfying the following conditions is a *prime theory* with respect to L:

(3.1) Γ is closed under $\vdash$
(3.2) $A \vee B \in \Gamma \Rightarrow A \in \Gamma$ or $B \in \Gamma$
(3.3) $\exists x A(x) \in \Gamma \Rightarrow A(c) \in \Gamma$ for some constant c in L.

The constant c in (3.3) is an analog of the *witness constant* in Henkin theory. Then we extend language L to L' by adding a suitable set of witness constants. Thus the following lemma is provable.

Lemma 3 *Let Γ and ϕ be closed. If $\Gamma \not\vdash \phi$, then there exists a prime theory Γ' such that $\Gamma' \not\vdash \phi$.*

Proof: The prime theory of Γ can be defined as:

$$L_n = L \cup c_n$$
$$C_{n+1} = C_n \cup \{c \mid c \text{ is a witness constant}\}.$$

Then Γ' is obtained by series of extensions $\Gamma_0 \subseteq \Gamma_1 \subseteq \ldots$. That is, Γ' is a union of Γ and the set of witness constants satisfying (3.3). Now put Γ_k to be given such that $\Gamma_k \nvdash \phi$ and contains a lot of witnesses. Then two cases can be considered. First, suppose that k is even. Take the first existential sentence $\exists x A(x)$ in L' that has not been treated, such that $\Gamma_k \vdash \exists x A(x)$, and assume that d is a witness constant not appearing in Γ_k. Then $\Gamma_{k+1} = \Gamma_k \cup \{A(d)\}$ holds.

Suppose, on the other hand, that k is odd. Take the first disjunctive sentence $A \vee B$ such that $\Gamma_k \vdash A \vee B$ was not treated before:

$$\Gamma_{k+1} = \Gamma_k \cup \{A\} \text{ if } \Gamma_k, A \nvdash \phi$$
$$\Gamma_k \cup \{B\} \text{ if } \Gamma_k, A \vdash \phi.$$

From (3.2), it is not needed that both $\Gamma_1, A \vdash \phi$ and $\Gamma_2, B \vdash \phi$ hold simultaneously. Γ' consists of all Γ_k's satisfying the above-mentioned properties, $\Gamma' = \bigcup_{k \geq 0} \Gamma_k$. As a result, it suffices to show the existence of the prime theory in which $\Gamma' \nvdash \phi$ holds.

Case 1. $\Gamma' \nvdash \phi$. First, we prove $\Gamma_i \nvdash \phi$ by induction on i. If i is even, then assuming $\Gamma_{i+1} \vdash \phi$, $\Gamma_i, A(d) \vdash \phi$ holds. Though we obtain $\Gamma_i \vdash \phi$ since $\Gamma_i \vdash \exists x A(x)$, this contradicts the assumption. Thus $\Gamma_{i+1} \nvdash \phi$ holds. For any i, $\Gamma_i \nvdash \phi$ holds by induction on i. If $\Gamma' \vdash \phi$, $\Gamma_i \vdash \phi$ for some i. But by contraposition and $\Gamma_i \nvdash \phi$, $\Gamma' \nvdash \phi$ holds.

Case 2. Γ' is a prime theory. We may check three conditions on a prime theory:

(3.2): Let $A \vee B \in \Gamma'$ and k is the smallest number such that $\Gamma_k \vdash A \vee B$. By assumption, for any h such that $k \leq h$, $\Gamma_h \vdash A \vee B$. Then $A \in \Gamma_{h+1}$ or $B \in \Gamma_{h+1}$. Hence $A \in \Gamma'$ or $B \in \Gamma'$.

(3.3): Let $\exists x A(x) \in \Gamma'$ and k, h are described as before. By assumption, since $\Gamma_h \vdash \exists x A(x)$, $A(d) \in \Gamma_{h+1} \subseteq \Gamma'$ holds. Also it is trivial that (3.1) holds. Then, we get Lemma 3.

Next, we have to prove that there exists a model for unprovable formulas:

Theorem 2 (Model Existence Lemma) *If* $\Gamma \nvdash \phi$ *then there is a Kripke model such that* $\Gamma_0 R^n \Gamma$ *and* $\Gamma_0 \nvdash \phi$. [Here R^n is an n-time application of R defined in (2.2).]

Proof: Let $C = \{c_i \mid i \geq 0\}$ be the set of constants not appearing in L. In the first place, we shall extend Γ to the prime theory Γ_i such that $\Gamma_i \nvdash \phi$. Consider the language L' such that for any k belonging to the set of natural numbers $\mathbb{N}$ it is a union of L and the set of constants $C' = \bigcup_{i=0}^{k-1} c_i$. Then we can define a Kripke model satisfying Theorem 2. According to the definition in Section 2, if G is $\mathbb{N}$, (2.3) is sufficient since $\mathbb{N}$ is a partially ordered set. Also V_P and V_N are defined as follows:

$$V_P\colon \mathbb{N} \to \{n \mid n \in \mathbb{N}, n \text{ is even}\}$$
$$V_N\colon \mathbb{N} \to \{n \mid n \in \mathbb{N}, n \text{ is odd}\}.$$

Moreover we shall consider a mapping P from $\mathbb{N}$ to the set of constants and mapping Σ from $\mathbb{N}$ to the set of atomic formulas containing a constant over $P(\mathbb{N})$. As $0R^k k$, it is clear that $P(0)$ is a set of constants in Γ_0 and $\Sigma(0)$ is a set of atomic formulas in Γ_0.

Now, let N correspond to ordered pairs (α_0, β_0), $(\alpha_1, \beta_1), \ldots$ in L' satisfying $\Sigma(n)$, $\alpha_i \nvdash \beta_i$. And for each i, apply Lemma 3 to $\Sigma(n) \cup \{\alpha_i\}$ and β_i. As a result the prime theory Γ_i is obtained. Now $\Sigma(n)$ is the constants and atomic formulas belonging to Γ_i. This model must be satisfied by the following statements since it is a prime theory:

$$n \models_P \psi \Leftrightarrow \Sigma(n) \vdash \psi.$$

Proof: The proof is carried out by induction on the formula ψ. The inductive step is exemplified as below:

$$\begin{aligned}
A \vee B: \Rightarrow) \quad & n \models_P A \vee B \Leftrightarrow n \models_P A \text{ or } n \models_P B \\
& \Rightarrow \Sigma(n) \vdash A \text{ or } \Sigma(n) \vdash B \Leftrightarrow \Sigma(n) \vdash A \vee B. \\
\Leftarrow) \quad & \Sigma(n) \vdash A \vee B \Leftrightarrow \Sigma(n) \vdash A \text{ or } \Sigma(n) \vdash B \\
& \Rightarrow n \models_P A \text{ or } n \models_P B \Leftrightarrow n \models_P A \vee B.
\end{aligned}$$

$A \supset B$: $\Rightarrow)$ Assume $\Sigma(n) \nvdash A \supset B$. By definition $\Sigma(n)$, $A \nvdash B$ holds. And there exists some $n' \geq n$ such that $\Sigma(n) \cup \{A\} \subseteq \Sigma(n')$ and $\Sigma(n') \nvdash B$. From the inductive basis, $n' \models_P A$ and from $n' \geq n$ and $n \models_P A \supset B$, $n' \models_P B$ but it is a contradiction since $\Sigma(n') \vdash B$ by hypothesis. Therefore $\Sigma(n) \vdash A \supset B$.

$$\begin{aligned}
\Leftarrow) \quad & \Sigma(n) \vdash A \supset B \Leftrightarrow \Sigma(n) \vdash A \Rightarrow \Sigma(n) \vdash B \\
& \forall m(m \models_P A \Rightarrow m \models_P B) \text{ for } m \geq n \Leftrightarrow n \models_P A \supset B. \\
\exists x A(x): \Rightarrow) \quad & n \models_P \exists x A(x) \Leftrightarrow \exists m(m \models_P A(t)) \text{ for } m \geq n, \text{ some } t \in P(m) \\
& \Rightarrow \Sigma(m) \vdash A(t) \Leftrightarrow \Sigma(n) \vdash \exists x A(x). \\
\Leftarrow) \quad & \text{it is trivial since } \Sigma(n) \text{ is a prime theory.}
\end{aligned}$$

N and G are in one-to-one correspondence and N is a partially ordered set, thus $\Gamma_0 \nvdash \phi$ holds. By the model existence lemma, the completeness theorem for S is obtained:

Theorem 3 (Completeness Theorem) *A is provable in S iff A is valid in S.*

Proof: Necessity is presented in the proof of Theorem 1. Sufficiency, that is, completeness, is clear since by the model existence lemma there exists a model for it assuming $\Gamma_i \nvdash \phi$.

Theorem 4 (Compactness Theorem) *There exists a model for Γ iff there exists a model for finite subset Δ of Γ.*

Proof: The proof is carried out by contraposition. Namely, there is no model for Γ iff there is no model for $\Delta \subseteq \Gamma$. Sufficiency is trivial since Δ is a finite subset of Γ. Necessity is proved as follows: By Theorem 2, if there exists no model for Γ, then Γ is a set of inconsistent formulas. Since $\Gamma \vdash \perp$, there is a set of formulas $\Delta_1, \ldots, \Delta_n \in \Gamma$ satisfying $\Delta_1, \ldots, \Delta_n \vdash \perp$. That is, there is no model for $\Delta = \{\Delta_1, \ldots, \Delta_n\}$. Now consider the domain $\bigcup_{\Gamma \in G} P(\Gamma)$ for a Kripke model $\langle G, R, V_P, V_N, P\rangle$. Then the following theorem holds:

Theorem 5 *A is valid in all models iff A is valid in all models with the set of denumerable parameters.*

Proof: Necessity is trivial. Sufficiency can be proved as follows: Consider the model $\langle G, R, V_P, V_N, P\rangle$ with nondenumerable parameters in which A is not valid. By Theorem 3, A is also unprovable even if parameters are restricted to denumerable numbers since A is not provable. Then there exists a model with denumerable parameters corresponding to the primary model.

Next we shall investigate the relationships among S, intuitionistic logic H, and classical logic C. To do so, we first define the mapping T from all formulas in H to formulas in S recursively.

$$TA = A \text{ for any atomic } A \tag{3.4}$$
$$T(A \wedge B) = TA \wedge TB \tag{3.5}$$
$$T(A \vee B) = TA \vee TB \tag{3.6}$$
$$T(A \supset B) = TA \supset TB \tag{3.7}$$
$$T(\neg A) = TA \supset {\sim}TA \tag{3.8}$$
$$T(\forall x A(x)) = \forall x T(A(x)) \tag{3.9}$$
$$T(\exists x A(x)) = \exists x T(A(x)). \tag{3.10}$$

We provide a Kripke model for H before we consider the relationships between S and H. The Kripke model for H is a quadruple $\langle G, R, V_H, P\rangle$. G, R, and P are described in (2.1), (2.2), and (2.4), V_H is a mapping from every propositional variable to a subset of G satisfying the property:

$$\forall \Gamma^*(\Gamma \in V_H(A) \Rightarrow \Gamma^* \in V_H(A)).$$

The forcing $\models_H$ is defined in terms of the forcing $\models_P$ except in the case of strong negation.

Lemma 4 *Given models $\langle G, R, V_H, P\rangle$ for H and $\langle G, R, V_P, V_N, P\rangle$ for S*

$$\Gamma \models_H A \text{ iff } \Gamma \models_P TA \qquad \textit{for any } \Gamma\,(\in G).$$

Proof: The proof is carried out by induction on the length of formulas. Examples of inductive steps are given below:

$$\begin{aligned}\Gamma \models_H A \wedge B &\Leftrightarrow. \Gamma \models_H A \text{ and } \Gamma \models_H B \Rightarrow \Gamma \models_P TA \text{ and } \Gamma \models_P TB. \Leftrightarrow \Gamma \models_P TA \wedge TB.\\ &\Leftrightarrow \Gamma \models_P T(A \wedge B).\end{aligned}$$

$$\begin{aligned}\Gamma \models_H A \supset B &\Leftrightarrow \forall \Gamma^*(\Gamma^* \models_H A \Rightarrow \Gamma^* \models_H B) \Rightarrow \forall \Gamma^*(\Gamma^* \models_P TA \Rightarrow \Gamma^* \models_P TB)\\ &\Leftrightarrow \Gamma \models_P TA \supset TB \Leftrightarrow \Gamma \models_P T(A \supset B).\end{aligned}$$

$$\begin{aligned}\Gamma \models_H \neg A &\Leftrightarrow \forall \Gamma^* \Gamma^* \not\models_H A \Rightarrow \forall \Gamma^* \Gamma^* \not\models_P TA \Leftrightarrow \forall \Gamma^*(\Gamma^* \models_P TA \Rightarrow \Gamma^* \models_P {\sim}TA)\\ &\Leftrightarrow \Gamma \models_P TA \supset {\sim}TA \Leftrightarrow \Gamma \models_P T(\neg A).\end{aligned}$$

$$\begin{aligned}\Gamma \models_H \forall x A(x) &\Leftrightarrow \forall \Gamma^* \Gamma^* \models_H A(t) \text{ for all } t \in P(\Gamma^*)\\ &\Rightarrow \forall \Gamma^* \Gamma^* \models_P TA(t) \text{ for all } t \in P(\Gamma^*)\\ &\Leftrightarrow \Gamma \models_P \forall x TA(x) \Leftrightarrow \Gamma \models_P T(\forall x A(x)).\end{aligned}$$

Theorem 6 (Embedding Theorem) *A is valid in H iff TA is valid in S.*

Proof: First we should prove the necessity condition. If we assume that A is not valid in H then there exists a model $\langle G, R, V_H, P\rangle$ for H such that $\Gamma \nvDash_H A$. For the counter-model for S, V_P and V_N are defined as follows:

$$V_P(A) = V_H(A),\ V_N(A) = \{\Gamma \mid \forall\Gamma^*\Gamma^* \notin V_H(A)\}.$$

This model is a model for S when $\Gamma \vDash_H \neg A \Leftrightarrow \Gamma \vDash_P {\sim}A$ holds. For any atomic formula A

$$\Gamma \vDash_H A \Leftrightarrow \Gamma \in V_H(A) \Leftrightarrow \Gamma \in V_P(A) \Leftrightarrow \Gamma \vDash_P A \Leftrightarrow \Gamma \vDash_P TA.$$

By Lemma 4, TA is not valid in a model $\langle G, R, V_P, V_N, P\rangle$ for S. Thus Theorem 6 is proved. Now assume the equivalence between provability of TA in S and that of A in H. Then we get the completeness theorem for H by way of the completeness theorem and the embedding theorem for S.

Corollary 2 *A is valid in H iff A is provable in H.*

Similarly, the compactness theorem for H is obtained. The proof on the basis of Kripke semantics can also be found in Fitting [4].

Next consider the relationships between classical logic C and S. The following theorem is first proved by Ishimoto [5],[6] in the Gentzen-Schütte type formulation:

Theorem 7 (Embedding Theorem) *A is provable in classical predicate logic C iff ${\sim}A \supset A$ is valid in S, where A is not containing $\supset$.*

Proof: For proving necessity, assume ${\sim}A \supset A$ is not valid in S. Thus there exists a model for S such that $\Gamma \vDash_P {\sim}A$ and $\Gamma \nvDash_P A$. Now let us define the function V_C in C corresponding to V_P and V_N in S, namely,

$$\Gamma \vDash_P A \Rightarrow V_C(A) = T \tag{3.11}$$
$$\Gamma \vDash_N A \Rightarrow V_C(A) = F \tag{3.12}$$

where A is a formula not containing $\supset$. Proving necessity is carried out by induction on the length of formulas. Several examples of inductive steps are given below:

$$\begin{aligned}\Gamma \vDash_P A \wedge B \Leftrightarrow .\ \Gamma \vDash_P A \text{ and } \Gamma \vDash_P B \Rightarrow .\ V_C(A) = T \text{ and } V_C(B) = T\\ \Leftrightarrow V_C(A \wedge B) = T.\end{aligned}$$

$$\Gamma \vDash_P {\sim}A \Leftrightarrow \Gamma \vDash_N A \Leftrightarrow V_C(A) = F \Leftrightarrow V_C({\sim}A) = T.$$

$$\begin{aligned}\Gamma \vDash_P \forall x A(x) &\Leftrightarrow .\ \forall\Gamma^*\Gamma^* \vDash_P A(t) \text{ for all } t \in P(\Gamma^*).\\ &\Rightarrow V_C(A(t)) = T \text{ for all } t.\\ &\Leftrightarrow V_C(\forall x A(x)) = T.\end{aligned}$$

$$\begin{aligned}\Gamma \vDash_N A \vee B \Leftrightarrow .\ \Gamma \vDash_N A \text{ and } \Gamma \vDash_N B \Rightarrow V_C(A) = F \text{ and } V_C(B) = F.\\ \Leftrightarrow V_C(A \vee B) = F.\end{aligned}$$

$$\begin{aligned}\Gamma \vDash_N \forall x A(x) &\Leftrightarrow \Gamma \vDash_N A(t) \text{ for some } t \in P(\Gamma).\\ &\Rightarrow V_C(A(t)) = F \text{ for some } t.\\ &\Leftrightarrow V_C(\forall x A(x)) = F.\end{aligned}$$

By (3.12) $\Gamma \vDash_P {\sim}A$ holds since $V_C(A) = F$. That is, A is not valid in C. Sufficiency is easily proved by induction on the length of formulas. As a corollary of Theorem 7, the completeness theorem for C is obtained. Thus if A is valid in C then $\sim A \supset A$ is a theorem of S. Though $\sim A \supset A$ is a theorem of C since S is a subsystem of C, this is equivalent to A. Then A is a theorem of C. The following corollary holds by means of $\neg A \equiv A \supset \sim A$.

Corollary 3 (Embedding Theorem) *A is valid in C iff $\neg\sim A$ is valid in S^+ where A is a formula not containing $\supset$.*

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