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Semantical Antinomies in the Logic of Sense and Denotation

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0 Informal introduction Frege's semantical ideas about sense and denotation have very often been discussed as if the main problem were to work out a satisfactory *intensional semantics* for the natural languages. Frege himself was always content to discuss matters of intensionality in connection with unformalized natural languages.[1] But given his very Platonistic views, it seems clear that he did not regard this as getting to the heart of the matter. From that perspective it is arguable that the main thing that is needed is an adequate *intensional logic* or (what may not be quite the same thing) a *theory of intensional entities.* Once this is in hand, it would seem to be a relatively simple matter to use the theory to construct an intensional semantics as desired. It was presumably with some such idea in mind that Alonzo Church constructed his Logic of Sense and Denotation ([5], [7], [8]). The name is actually somewhat misleading (as Wells notes in [14]): the Logic of Sense and Denotation is a theory of the entities *suitable* to be senses of this or that expression in this or that actual or possible language—i.e., it is a theory of *concepts* (in Church's sense, not Frege's). No language nor class of languages is singled out for discussion and no *intensional semantics* is constructed with some of the concepts *as* senses and various things *as* denotations.[2] The main concern is with the intensional entities and their relations to one another and to other things (especially the *concept relation*, which holds between a concept and the entity, if any, falling under it).

Unfortunately, when the actual details are supplied, contradictions threaten to appear. Some of these are analogous to the Epimenides Paradox in that they require additional, empirical but clearly possible, premises. Others are analogous to the Zermelo-König Antinomy and produce contradictions only in the presence of the Axiom of Choice.[3] Still others are intrinsic to the Logic of Sense and Denotation (without the Axiom of Choice) but turn on the adoption of a "criterion of identity" for concepts of a stringent, but quite plausible, sort.

A satisfactory completion of the Frege–Church project requires some reso-

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lution of these difficulties.[4] In attempting to deal with these matters, I have become convinced that what is required is a return to language. If we think of concepts as senses of expressions in (actual and possible) languages and regard the concept relation as carrying with it a reference to the language from which it was abstracted, then the arguments of the antinomies produce *theorems* concerning the possibility of expressing this or that concept in such-and-such language.[5] A concept relation, on this way of looking at things, is *essentially* the relative product of *is expressed by in language i* and *denotes in language i*—for some particular language *i*. If this is the correct perspective, then the concept relation of Church's original formulation is ill-defined (or, better, ill-conceived); it would correspond to existentially generalizing *i* in the explanation just given. Indeed, in [8] Church supplies superscripts on the symbols for the concept relations and explains that these represent a sort of Tarskian hierarchy, but combined into a single language. I propose to further identify the two approaches by construing the superscripts as referring to particular possible languages. In addition, I introduce variables over a domain of possible languages and allow such variables to serve as superscripts on the delta symbols representing concept relations. It can be shown (see Section 6) that if we further introduce quantifiers for possible languages, then the language being used cannot belong to the domain of possible languages being considered.

Frege of course never faced these matters. But given his methodology (as displayed, for example, in [10]), it is not impossible that he could have been brought to agree that for some purposes language and sense could not be separated.

1 Intuitive formulation of an Epimenides Paradox[6] Consider the sentence:

(0) Church's favorite proposition is not true.

Now if Church should happen to prefer the sense of this sentence above all others, then paradox results—a form of the Epimenides Paradox applied to *propositions* (as opposed to sentences). Let us use 'c_2' to abbreviate the phrase 'Church's favorite proposition', and adopt the convention of surrounding a sentence with '$[\ ]_2$' in order to obtain a designation of the sense of (i.e., the proposition expressed by) the sentence. According to the Frege-Church theory of meaning, a proposition is *true* if it is a concept of, well, *The True*. Roughly, this means that the proposition is expressed (in *some* possible language) by a sentence which *denotes the truth-value t* (or, if you prefer, *is true*). The logic mentioned uses the symbol Δ for the (converse of) the concept relation. The appropriate definition of truth for propositions is then:

$$\mathrm{Tr}(p_2) \rightarrow \Delta T_0 p_2$$

where p_2 is a proposition variable and 'T_0' abbreviates '$(a_0)(a_0 \supset a_0)$'—a necessary truth (0 is the type of truth-values). Thus a proposition p_2 is true if and only if p_2 is a concept of T_0.

Granted all these, our assumptions for the paradox are:

(1) $c_2 = [\sim\mathrm{Tr}(c_2)]_2$

"Church's favorite proposition is the proposition that Church's favorite proposition is not true";

(2) $\Delta(\sim\mathrm{Tr}(c_2))[\sim\mathrm{Tr}(c_2)]_2$

"The proposition that Church's favorite proposition is not true is a concept of the truth-value thereof that Church's favorite proposition is not true".

A contradiction follows from these two assumptions using the Logic of Sense and Denotation as formulated in [5] (see Section 3 below for the formalities). But (1) seems possible and (2) has the look of necessary truth.

It was this and other "semantical" antinomies (and the rather different Russell–Myhill antinomy) which led to the proposal of this paper. The suggestion, as explained in Section 0, is that the concept relation of the Logic of Sense and Denotation be *relativized to possible languages.* There is a certain artificiality, however, in retaining this connection in the concept relation, but I argue that it is mitigated to some extent by adjustments within the logic and hence of its consequences. The resulting resolution of the antinomies resembles Tarski's, but with some differences. The Logic of Sense and Denotation treats intensions, i.e. propositions, properties, and other concepts, directly within the object language. No single hierarchy of language, analogous to the Tarskian hierarchy of metalanguages, results. And the theory is simpler than Russell's Ramified Theory of Types in the respect of not restricting quantification over propositions. But there is of course the complexity, not present in Russell's theory, of the Fregean hierarchy of senses.

2 Alternative (0) with possible language variables Let the *simple types* be 0 (truth-values), 2 (propositions), 4 (concepts of propositions), . . . ; 1 (individuals), 3 (concepts of individuals), 5 (concepts of concepts of individuals). . . . In addition, we postulate a *special type* $\mathfrak{L}$ (possible languages). *Complex types* are obtained by beginning with the simple types and applying the rule that if a and b are (simple or complex) types, then (ab) is a type (functions defined everywhere in b and taking values in a). Note that the special type of possible languages does not appear in the system of simple and complex types. The type symbols are the Arabic numerals '0', '1', . . . , and $\ulcorner(t_1,t_2)\urcorner$ whenever t_1 and t_2 are type symbols. Small Roman letters with type symbols as subscripts are variables of the indicated types. The letters $i, j, k, \bar{i}, \bar{j}, \bar{k}$ . . . are *possible language variables.* Possible language variables are *possible language terms.* Arabic numerals for the natural numbers are *possible language terms.* If i is a possible language variable, then the expression $i + 1$ is a *possible language term.* Informally, it is to be understood that the Arabic numerals will stand for particular formalized languages, sublanguages of the Logic of Sense and Denotation, to be described more fully below. (This use of the numerals is entirely distinct from their use to refer to types. No confusion should result since possible language terms occur only as superscripts on symbols Δ.) If a particular possible language is assigned as value to a possible language variable i, then $i + 1$ will take as value a possible language which is related to i much the way a metalanguage is related to its object language. This will be clearer when we describe a particular model below (Section 7). Possible language variables are not subject to quantification and appear only as superscripts in the new symbols Δ^i or in the new symbols Δ^{i+1}, where in each case i is a possible language variable. (That is, in every case where the symbol Δ^m is a primitive symbol of [1], the corresponding expressions with i and $i+1$ in place of m are additional primitive symbols of our new logistic system.) Otherwise the notation follows [1] and [8] except for obvious changes necessitated by the present type designations.

The intuitive understanding of $\Delta^i x_a x_\alpha$ is that the concept x_α is a concept of the entity x_a *relative to the possible language i.* (This illustrates our use of metalinguistic variables: if a is any type, then α is the type of concepts of a's; similarly for b and β, c and γ, and so on.) And x_α is a concept of x_a relative to language i if some expression of i denotes x_a (in i) and expresses x_α (in i). The relation Δ^i is then the relative product of the converse of *expresses-in-i* and *denotes-in-i.* And the relativity to particular possible languages must be retained in all uses of the concept relations. A proposition, for example, is true relative to i if some sentence expresses that proposition *in i* and denotes *t in i*—i.e., if the proposition is a concept-i of t.

Adopt the following definitions (the second of which makes official our convention of abbreviation used informally in Section 1):

$$e^i_{0\alpha} \rightarrow \lambda x_\alpha (\exists x_a) \Delta^i x_a x_\alpha$$
$$[\mathbf{A}_a]_2 \rightarrow \mathbf{A}_\alpha, \text{ where } \mathbf{A}_\alpha \text{ is the first ascendant of } \mathbf{A}_a$$
$$\mathbf{F}^2_{\alpha\beta} \rightarrow [\lambda x_b (\mathbf{F}_{ab} x_b]_2.$$

In connection with the second of these, we also abbreviate $[[\mathbf{A}_a]_2]_2$ as $[\mathbf{A}_a]_4$ and so on. And we suppose that the definition of 'Tr' in Section 1 is modified by supplying a superscript, representing a possible language term, on both 'Tr' and 'Δ'. With these changes and additions we can reformulate the Logic of Sense and Denotation for Alternative (0) (see [1]) as follows. Axioms C1–C6 are unchanged in appearance (except for obvious changes resulting from our new scheme for designating types) but it is to be understood that the superscript m (in C4 and C6) is to represent any numeral or possible language variable. Further, the expression $m+1$ in those axioms is to be understood to represent the next numeral in the natural order if m is a numeral, and to represent the possible language term $i+1$ if m is a possible language variable i. We adopt this same convention in axioms to be stated below. The axioms C7 and C8 are replaced by the following:

C7^{ab}. $\Delta^m f_{ab} f^2_{\alpha\beta} \supset .\Delta^m x_b x_\beta \supset .\Delta^{m+1}(f_{ab} x_b)(f_{\alpha\beta} x_\beta).$

C8^a. $\Delta^i x_a x_\alpha \supset .\Delta^j y_a x_\alpha \supset .x_a = y_a$, *where **i** and **j** are possible language terms.*

(It is in fact the closures of these which are axioms, but we adopt here and below the convention of omitting initial universal quantifiers.)

The unpleasant and unnatural relativity of the concept relation is somewhat mitigated by axiom C8^a. According to it if a concept is nonvacuous relative to any language i, then it is a concept of the same thing relative to any language j in which it has an extension. I justify this informally by the following considerations. Let us just suppose that a language in the present sense must include within it whatever is required to fix the denotations (if any) of its expressions. In applying the logic to natural languages, we correspondingly require that whatever contextual factors are required to determine denotation are fixed throughout. Once this is done, there can be no objection to adopting the principle that, within a given language, two expression with the same meaning must have the same denotation. And for different languages it is taken to be partly definitive of correct translation that the denotations cannot differ. I regard it as a matter of convention whether or not we wish to allow that a denoting expression may have a translation into another language which fails to denote therein. On the whole it seems best to allow for this possibility.

The axioms specific to Alternative (0) require only the routine replacement of type designations except for A(0)3, which is to be replaced by:

$$\text{A(0)3}^{mab}.\quad e^m f_{\alpha\beta} \supset .e^m(f_{\alpha\beta}x_\beta) \supset .\text{Con } x_\beta \supset e^{m+1}x_\beta.$$

3 The Epimenides Paradox formalized Now let $\imath p_2 C_{02} p_2$ mean 'the proposition favored by Church above all others' and abbreviate this further by 'c_2'. We consider the formulas

(1) $c_2 = [\sim\text{Tr}^i c_2]_2$,
(2) $\Delta^j(\sim\text{Tr}^i c_2)[\sim\text{Tr}^i c_2]_2$,

and designate them (1) and (2) respectively, as indicated. The reasoning will be free-and-easy in that detailed formally correct proofs will not be given. But the indications are thought to be sufficient for the interested reader to construct rigorous proofs. In particular we proceed as if theorems might contain free variables whereas Church ([5]) does not actually allow this. (However, see [8] for metatheorems which justify our procedure.) The Deduction Theorem (DT), rules of inference for the propositional calculus (PC), and the usual laws of quantification theory (QT) hold in all cases we shall need (the last of these hold generally only in nonempty types). We shall need also the following:

Derived Rule T *From* $\mathbf{A}_0$ *to infer* $\mathbf{A}_0 = T_0$, *and vice-versa.*

Proof sketch: $\vdash T_0$ (recall that T_0 is the wff $(a_0)(a_0 \supset a_0)$), hence $\vdash \mathbf{A}_0 \supset T_0$, by PC. And $\mathbf{A}_0 \vdash T_0 \supset \mathbf{A}_0$ by PC. But we have as an axiom ([5]) a law of extensionality: $\vdash p_0 \supset q_0 \supset .q_0 \supset p_0 \supset .p_0 = q_0$. Hence, by PC, we have $\mathbf{A}_0 \vdash \mathbf{A}_0 = T_0$. Conversely, since $\vdash T_0$, we have $\mathbf{A}_0 = T_0 \vdash T_0$. So, by substitutivity of identity (subst. iden. holds quite generally), $\mathbf{A}_0 = T_0 \vdash \mathbf{A}_0$.

Derived Rule F *From* $\mathbf{A}_0$ *to infer* $\sim\mathbf{A}_0 = F_0$, *and vice-versa.*

Proof is similar using $\vdash \sim F_0$ (F_0 is $(a_0)a_0$).

An analogue of the Tarski Schema T is a theorem of the Logic of Sense and Denotation:

Theorem 1 $\Delta^j p_0 p_2 \supset .\text{Tr}^j p_2 \equiv p_0$.

Proof:	1.1 $\Delta^j p_0 p_2, \Delta^j T_0 p_2 \vdash p_0 = T_0$	C8^0 (set $\boldsymbol{i} = \boldsymbol{j} = i$), QT, PC
	.2 $\Delta^j p_0 p_2, \Delta^j T_0 p_2 \vdash p_0$	.1, Rule T
Also,	1.3 $\Delta^j p_0 p_2, p_0 \vdash p_0 = T_0$	Rule T
	.4 $\Delta^j p_0 p_2, p_0 \vdash \Delta^j T_0 p_2$	.3, subst. iden.
	.5 $\vdash \Delta^j p_0 p_2 \supset .\Delta^j T_0 p_2 \equiv p_0$	.2, .4, DT, PC.

Using this we can develop a theorem corresponding to the Epimenides Antinomy.

Theorem 2 $c_2 = [\sim\mathrm{Tr}^i c_2]_2 \supset .\Delta^j(\sim\mathrm{Tr}^i c_2)[\sim\mathrm{Tr}^i c_2]_2 \supset .\mathrm{Tr}^j c_2 \equiv \sim\mathrm{Tr}^i c_2.$

Proof: 2.1 (2) $\vdash \Delta^j T_0[\sim\mathrm{Tr}^i c_2]_2 \equiv \sim\mathrm{Tr}^i c_2$ T1, QT, PC, Def. of 'Tr^j'
.2 (1), (2) $\vdash \Delta^j T_0 c_2 \equiv \sim\mathrm{Tr}^i c_2$ .1, subst. iden.
.3 (1), (2) $\vdash \mathrm{Tr}^j c_2 \equiv \sim\mathrm{Tr}^i c_2$ .2, Def. of 'Tr^j'.
Now use DT.

"If Church's favorite proposition is that Church's favorite proposition is not true-*i* and that proposition is expressed by something in language *j*, then it is true-*j* if and only if it is not true-*i*." There is no paradox here so far. The apparent difficulty arises when we try to identify the languages *i* and *j*.

Theorem 3 $c_2 = [\sim\mathrm{Tr}^i c_2]_2 \supset \sim\Delta^i(\sim\mathrm{Tr}^i c_2)[\sim\mathrm{Tr}^i c_2]_2.$

Proof: T2 (set $i = j$), PC.

This is the desired theorem. If Church should smile with favor upon the feared proposition, then no sentence in language *i* can express that proposition and denote its proper truth-value—in the sense indicated by the consequent of T3.

The antinomy would be restored in the sense that (1) and (2) would lead to a contradiction if we had: $\Delta^j p_0 p_2 \supset .\mathrm{Tr}^i p_2 \equiv p_0$. This would in effect obliterate distinctions between possible languages. "Half of" this is a theorem:

Theorem 4 $\Delta^j p_0 p_2 \supset .\mathrm{Tr}^i p_2 \supset p_0.$

Proof: 4.1 $\Delta^j p_0 p_2, \Delta^i T_0 p_2 \vdash p_0 = T_0$ C8^0, QT, PC
.2 $\Delta^j p_0 p_2, \Delta^i T_0 p_2 \vdash p_0$ .1, Rule T.
Now use DT (twice) and the definition of 'Tr^i'.

We can now justify the remark above that the antinomy would be restored if we could strengthen the second conditional to an equivalence.

Metatheorem *If we had* $\vdash \Delta^j p_0 p_2 \supset .p_0 \supset \mathrm{Tr}^i p_2$, *then we would have that* (1), (2) $\vdash \mathrm{Tr}^i c_2 \equiv \sim\mathrm{Tr}^i c_2$ *(and hence that* $\vdash (1) \supset \sim(2)$*)*.

Proof: Suppose $\vdash \Delta^j p_0 p_2 \supset .p_0 \supset \mathrm{Tr}^i p_2$. Then by T4 and PC, $\vdash \Delta^j p_0 p_2 \supset .\mathrm{Tr}^i p_2 \equiv p_0$. And hence by QT, we would have that $\vdash \Delta^j(\sim\mathrm{Tr}^i c_2)[\sim\mathrm{Tr}^i c_2]_2 \supset .\mathrm{Tr}^i[\sim\mathrm{Tr}^i c_2]_2 \equiv \sim\mathrm{Tr}^i c_2$. Hence, (1) $\vdash \Delta^j(\sim\mathrm{Tr}^i c_2)[\sim\mathrm{Tr}^i c_2]_2 \supset .\mathrm{Tr}^i c_2 \equiv \sim\mathrm{Tr}^i c_2$ by subst. iden. Thus: (1), (2) $\vdash \mathrm{Tr}^i c_2 \equiv \sim\mathrm{Tr}^i c_2$ by PC.

In such a case our empirical assumption about Church's propositional preferences and the claim that the proposition in question is expressible in *some* language *j* would lead to a contradiction using the logic. This cannot be permitted. Fortunately, there is no reason to expect the despised formula as theorem. A little reflection shows that it does not accord with the intuitively described models. (Below, in section 7, we sketch the construction of a particular model and describe some constraints on admissible models.)

The situation is still somewhat obscure. Does the proposition c_2 *have* a truth-value in language *i*? No.

Theorem 5 $c_2 = [\sim\mathrm{Tr}^i c_2]_2 \supset .\Delta^j(\sim\mathrm{Tr}^i c_2)[\sim\mathrm{Tr}^i c_2]_2 \supset \sim\mathrm{Tr}^i c_2.$

Proof: 5.1 $\Delta^j(\sim\mathrm{Tr}^i c_2)[\sim\mathrm{Tr}^i c_2]_2 \supset .\mathrm{Tr}^i[\sim\mathrm{Tr}^i c_2]_2 \supset \sim\mathrm{Tr}^i c_2$ — T4, QT
.2 (2) $\vdash \mathrm{Tr}^i[\sim\mathrm{Tr}^i c_2]_2 \supset \sim\mathrm{Tr}^i c_2$ — .1, PC
.3 (1), (2) $\vdash \mathrm{Tr}^i c_2 \supset \sim\mathrm{Tr}^i c_2$ — .2, subst. iden.
.4 (1), (2) $\vdash \sim\mathrm{Tr}^i c_2$ — .3, PC.
Now use DT twice.

Definition $\mathrm{Fls}^i p_2 \rightarrow \Delta^i F_0 p_2.$

Theorem 6 $c_2 = [\sim\mathrm{Tr}^i c_2]_2 \supset .\Delta^j(\sim\mathrm{Tr}^i c_2)[\sim\mathrm{Tr}^i c_2]_2 \supset \sim\mathrm{Fls}^i c_2.$

Proof: 6.1 (1), (2) $\vdash \Delta^j(\sim\mathrm{Tr}^i c_2)c_2$ — subst. iden.
.2 (1), (2), $\Delta^i F_0 c_2 \vdash \sim\mathrm{Tr}^i c_2 = F_0$ — .1, $C8^0$, QT, PC
.3 (1), (2), $\Delta^i F_0 c_2 \vdash \mathrm{Tr}^i c_2$ — .2, Rule F
.4 (1), (2), $\Delta^i F_0 c_2 \vdash \Delta^i T_0 c_2$ — .3, Def. of 'Tr^i'
.5 (1), (2), $\Delta^i F_0 c_2 \vdash T_0 = F_0$ — .4, $C8^0$, QT, PC
.6 (1), (2) $\vdash \sim\Delta^i F_0 c_2$ — .5, PC (since $\vdash T_0 \neq F_0$).
Now use DT and the definition of 'Fls^i'.

Theorems 5 and 6 then guarantee that if c_2 is as described and has the "right" truth-value in some language j, then it has no truth-value in language i. Our account thus combines some features of the "truth-value gap" approach with the Tarski-type resolution.

We can settle the question of the truth-value of c_2 in any language in which it has a truth-value:

Theorem 7 $c_2 = [\sim\mathrm{Tr}^i c_2]_2 \supset .\Delta^j(\sim\mathrm{Tr}^i c_2)[\sim\mathrm{Tr}^i c_2]_2 \supset \mathrm{Tr}^j c_2.$

Proof: By T2, T5, and PC.

So the proposition c_2 has no truth-value in language i but is true-j in any language in which it has a truth-value; in particular it is true-$i+1$—where $i+1$ is a "metalanguage" for i.

It is perhaps time for a general assessment of the situation insofar as this is possible without precise models. The sentence (1) (the antecedent of T7) will in general be a contingent truth (although for certain choices of meaning for 'C_{02}' we can make it a necessary truth). In our example it is dependent upon Church's preferences about propositions. This means that the extension of the denotation relation for the language of which (1) is taken to be a sentence is determined by those preferences. And any such language will have to be distinct from the language i involved in (1). It is natural to regard this as deriving from the fact that the proposition c_2 involves the concept relation for the language i. (Notice that our standard sentence expressing the proposition c_2 contains an expression denoting that concept relation.) Suppose, contrary to what will be argued, that some language contained an expression D denoting the concept relation Delt for that language. Then the pair $([D]_2, \mathrm{Delt})$, where $[D]_2$ is the sense of the expression D, would belong to Delt. But this is contrary to the principle, embodied in the Axiom of Foundation of set theory, that a set cannot contain itself or a set which contains it, and so on. Here we have a relation which would contain a pair which contains that relation as element. In the usual construction of relations in extension, it is a consequence of the Axiom of Foundation that this cannot occur. In any event it is the violation of this principle about extensions which seems to be the basic source of the paradox.[7] By indexing the concept relation to possible languages and explaining it in terms of the denotation relation language, we trace the difficulty back to its source: no language can have such a denotation relation. What shall we say about the *expression* relation which holds between expressions of language *i* and their senses in *i*? We may say, if we like, that it is possible for some expression of language *i* to *express* the concept relation of language *i*. But that expression will then have no denotation in *i*.

The Grelling's Antinomy produces a similar theorem about the possibility of denoting in a given language certain concepts involving the concept relation for that language, but the analysis provides nothing essentially new. We go on to consider a less familiar antinomy in this context.

4 The Zermelo-König Antinomy Suppose that not all entities of some particular type are concepted. Then, assuming that the Axiom of Choice holds for the entities of this type, there will be a well-ordering of the domain in question. If there are unconcepted entities of this type, then there is a least such according to the well-ordering. But the concept "the least unconcepted element of the well-ordering" is a concept of the unconcepted entity in question—a contradiction. Well, perhaps we should conclude that all entities in all such types must be concepted. Unfortunately, the logic itself guarantees that certain entities cannot be concepted (see the Russell-Myhill Antinomy below in Section 5).

Another suggestion is that perhaps the well-ordering relation itself is not concepted. Hence the words quoted above do not express a definite concept, contrary to appearances. Early opponents of the Axiom of Choice argued that the function in question (and hence the well-ordering relation too) may not be "definable" and hence should not be postulated. If we avoid this extreme conceptualist position, we will find that we must hold that in *no possible language* is the function (or relation) in question denoted by an expression therein! But if we are not in the grips of some conceptualistic position, what possible reason could there be for such a view? We proceed to analyze a form of this antinomy in the Logic of Sense and Denotation.

For the following theorems concerning the Axiom of Choice, we adopt the following designations:

(AC^b) $(\exists x_b) f_{0b} x_b \supset f_{0b}(\iota f_{0b})$,
(3) $(\exists x_b) \sim (\exists x_\beta) \Delta^i x_b x_\beta$,
(4) $\Delta^i(\iota U^i_{0b})[\iota U^i_{0b}]_2$

where the definition of U^i_{0b} is given by:

$$U^i_{0b} \rightarrow \lambda x_b \sim (\exists x_\beta) \Delta^i x_b x_\beta .$$

AC^b is a strong form of the axiom of choice—"global choice" (in the type b). The ι function therein acts as a description function in the logic without this axiom, but now is to be construed as a choice function. We reconstruct the Zermelo-König Antinomy directly in terms of this rather than detouring through a well-ordering of the type b.

Theorem 8 $AC^b \supset .(\exists x_b)\sim(\exists x_\beta)\Delta^i x_b x_\beta \supset .\sim\Delta^i(\iota U^i_{0b})[\iota U^i_{0b}]_2$.

Proof: 8.1 $AC^b \vdash (\exists x_b)U_{0b}x_b \supset U^i_{0b}(\iota U^i_{0b})$ QT
.2 (3) $\vdash(\exists x_b)(\lambda x_b \sim(\exists x_\beta)\Delta^i x_b x_\beta)x_b$ λ-abstraction
.3 (3) $\vdash(\exists x_b)U^i_{0b}x_b$ .3, Def. of 'U^i_{0b}'
.4 AC^b, (3) $\vdash U^i_{0b}(\iota U^i_{0b})$ .1, .3, PC
.5 AC^b, (3) $\vdash\sim(\exists x_\beta)\Delta^i(\iota U^i_{0b})x_\beta$ .4, Def. of 'U^i_{0b}', λ-contraction
.6 (4) $\vdash(\exists x_\beta)\Delta^i(\iota U^i_{0b})x_\beta$ EG
.7 AC^b, (3) $\vdash\sim$(4) .5, .6, DT, PC.
Now use DT.

For certain types b the second antecedent of T8 is provable, as noted above. For such types, if we adopt the Axiom of Choice, the consequent of T8 will be a theorem. The acute reader will have noticed that we could have just as well used the existential generalization of (4) (on the second argument) to derive a contradiction from AC^b and (3).

Theorem 9 $AC^b \supset .(\exists x_b)\sim(\exists x_\beta)\Delta^i x_b x_\beta \supset .\sim (\exists x_\beta)\Delta^i(\iota U^i_{0b})x_\beta$.

Proof: Similar to T8.

Thus we conclude that if some entities in the type b are not concepted-i (i.e., concepted *in* possible language i), then there is a *particular* entity of the type which is not concepted-i. There's no contradiction in this. Indeed, everything here is compatible with:

($4j$) $\Delta^j(\iota U^i_{0b})[\iota U^i_{0b}]_2$,

that the entity in question is concepted *in language j* (distinct from i). If the choice function exists at all (and it does), it can have a name and fall under some concept in some possible language. Any restriction of the means of definition to those available in the mathematical language as of a certain date seems entirely arbitrary—and if all possible means of definition are allowed, there is no reason not to take the choice function as a primitive in a certain language.

From the present perspective the failure of (4^i) (i.e., (4) above) is to be explained by the impossibility of there being in language i an expression denoting the (appropriate) concept relation for i. It is worth noticing that i might contain a name of the choice function ι and, if it has sufficient expressive power to express the other concepts involved in U^i, then the only obstacle to (4^i) is the failure to express (or better, to denote) its own concept relation.

The last antinomy we shall analyze is of a rather different character.

5 The Russell-Myhill Antinomy This antinomy seems to arise only when one adopts a stringent criterion of identity for concepts. Informally we may express it thus: if the proposition that all propositions are F is identical with the proposition that all propositions are G, then the property F *is* the property G. Now correlate with each set of propositions F the proposition that all propositions are F. This correlation then constitutes a 1-1 correspondence between (a subset of) the set of propositions and the set of all sets of propositions—contrary to Cantor's Theorem.

The reasoning as thus explained is subject to various objections. The *prop-*

erty F (of propositions) is conflated with the *set* of propositions having F. But all the argument requires is that each set of propositions falls under (or is "determined" by) a property and that each property determines at most one set. Someone might complain that there is no such thing as the set of *all* propositions. Again, the antinomy goes through on various weaker assumptions, for example, that the class of all universal propositions (about propositions) is a set. Rather than pursue these matters informally, we proceed at once to the more controlled setting of the Logic of Sense and Denotation. The proof to follow is in all essentials due to John Myhill.

Under Alternative (0), the following theorem is required (and is provable in [1]):

Theorem 10 $\quad \mathrm{Con}\, f_{2\alpha} \supset .\mathrm{Con}\, g_{2\alpha} \supset .\pi f = \pi g \supset .f = g.$

(We omit the proof. See [1] for the definition of 'Con'.) Intuitively, and very roughly, this says that if f and g are properties (of things of type a), then if universal proposition $[(x_a)fx_a]_2$ is identical with universal proposition $[(x_a)gx_a]_2$, then the property f is identical with the property g.

Definition $\quad (p_2 \in^i q_2) \rightarrow (\exists f_{02})(\exists f_{24})(\Delta^i f_{02} f_{24} \,.\, q_2 = \pi f_{24} \,.\, f_{02} p_2)$

("The proposition p_2 'belongs to'-i the proposition q_2 if there is a set of propositions f_{02} falling-i under a property (of propositions) f_{24} such that q_2 *is* the proposition that all propositions have f_{24} and p_2 belongs to f_{02}".)

Definition $\quad R^j_{02} \rightarrow \lambda p_2 {\sim} (p_2 \in^j p_2).$

(R^j_{02} is the set of all propositions which do not 'belong to'-j themselves.) We omit the subscript on this in the proof to follow.

Adopt the following designations:

(5) $\Delta^i R^j [R^j]_2$.
(6) $\pi[R^j]_2 \in^i \pi[R^j]_2$

Theorem 11 $\quad \Delta^i R^j [R^j]_2 \supset .\pi[R^j]_2 \in^i \pi[R^j]_2 \equiv \pi[R^j]_2 \notin^j \pi[R^j]_2).$

Proof:	11.1	(6) $\vdash (\exists f_{02})(\exists f_{24})(\Delta^i f_{02} f_{24} \,.\, \pi[R^j]_2 = \pi f_{24} \,.\, f_{02}(\pi[R^j]_2))$	Def. of '$\in^i$'
Also:	.2	$\mathrm{Con}[R^j]_2, \mathrm{Con} f_{24}, \pi[R^j]_2 = \pi f_{24} \vdash [R^j]_2 = f_{24}$	T10,QT
But:	.3	$(5) \vdash \mathrm{Con}[R^j]_2$	Axiom A(0)2 (of [1]), QT, PC
And:	.4	$\Delta^i f_{02} f_{24} \vdash \mathrm{Con} f_{24}$	Axiom A(0)2, QT, PC
Hence:	.5	$(5), \Delta^i f_{02} f_{24}, \pi[R^j]_2 = \pi f_{24} \vdash [R^j]_2 = f_{24}$	.2, .3, .4, PC
	.6	$(5), \Delta^i f_{02} f_{24}, \pi[R^j]_2 = \pi f_{24} \vdash \Delta^i f_{02} [R^j]_2$	.5, subst. iden.
	.7	$(5), \pi[R^j]_2 = \pi f_{24}, \Delta^i f_{02} f_{24} \vdash f_{02} = R^j$	.6, C8^{02}, QT, PC
	.8	$(5), \pi[R^j]_2 = \pi f_{24}, \Delta^i f_{02} f_{24}, f_{02}(\pi[R^j]_2) \vdash R^j(\pi[R^j]_2)$	.7, subst. iden.
	.9	$(5), (6) \vdash R^j(\pi[R^j]_2)$	.8, PC, EI, .1, PC
	.10	$(5), (6) \vdash {\sim}(\pi[R^j]_2 \in^j \pi[R^j]_2)$	.9, Def. of 'R^j', λ-contraction
Further,	.11	$({\sim}\pi[R^j]_2 \in^j \pi[R^j]_2) \vdash R^j(\pi[R^j]_2)$	λ-abstraction, Def. of 'R^j'
	.12	$(5), \pi[R^j]_2 = \pi[R^j]_2, R^j(\pi[R^j]_2) \vdash (6)$	PC, EG (twice)
	.13	$(5), {\sim}(\pi[R^j]_2 \in^j \pi[R^j]_2) \vdash (6)$	.11, .12, QT.

Now use DT on .10 and .13, then PC and DT.

If we now identify i and j in T11, we obtain by PC:

Theorem 12 $\sim\Delta^i R^i[R^i]_2$.

Indeed, inspection of the proof of T11 will reveal that we could prove its analogue with a variable in place of '$[R^j]_2$'. Hence

Theorem 13 $\sim(\exists f_{24})\Delta^i R^i f_{24}$.

Thus we may conclude that there are entities which are not concepted-i, and this independently of whatever other assumptions we may make about language i. Though we have proved this for the particular case of R^i being of type (02), the corresponding things are theorems in all types $(\alpha\beta)$ where type α provably contains at least two entities and β is an intensional type (see [1]).

Again, the difficulty may be traced to the fact that language i cannot contain a name of its own concept relation. Indeed, we can prove in [1]:

Axiom 15^m $\sim\Delta^m\Delta^m[\Delta^m]_2$, if m is a numeral.

We omit the proof of this. The basic idea is that from the denial of A15, it follows (by a generalization of the Sense Relationship Theorem of Church [8], called "Principle (A)" in [1]) that T12 (for $i = m$) fails. And this implication can be shown by simply proceeding step by step from the m-conceptedness of the delta relation to the m-conceptedness of R^m, using especially C1, C2, CS5, and C7 of [1].

The change in C7, suggested above, thwarts the proof in the present system. Still, it is reasonable to adopt A15 and its generalization to variable i in place of m as axioms. The models sketched below validate these axioms. Basically this results from the formal counterpart of the argument presented above that no language can contain an expression denoting the concept relation for that language.

We conclude that the Russell–Myhill Antinomy, the Zermelo–König Antinomy, and the Epimenides Paradox have the same foundation: it is the truth of A15 and its generalization. In each case what is proved is that it is impossible for any language to contain an expression denoting the concept relation of that language.

6 The Super-Epimenides If we extend the language of the logic so that quantifiers containing possible language variables are permitted, then we obtain a generalization of the Epimenides Paradox. Contemplate such an extension (with obvious extension of the definition of 'well-formed formula') and adopt the definition:

$$TRp_2 \rightarrow (\exists i)\Delta^i T_0 p_2.$$

Thus a proposition is TRUE (simpliciter) if it is true-i for some possible language i. We can then prove a Tarski-like theorem:

Theorem 14 $\Delta^j p_0 p_2 \supset .TRp_2 \equiv p_0$.

Proof: 14.1 $\Delta^j p_0 p_2,\ (\exists i)\Delta^i T_0 p_2 \vdash p_0 = T_0$ — C8^0, EI, PC

.2 $\Delta^j p_0 p_2,\ (\exists i)\Delta^i T_0 p_2 \vdash p_0$ — .1, Rule T

Also .3 $\Delta^j p_0 p_2, p_0 \vdash p_0 = T_0$ Rule T
.4 $\Delta^j p_0 p_2, p_0 \vdash \Delta^j T_0\, p_2$ .3, subst. iden.
.5 $\Delta^j p_0 p_2, p_0 \vdash (\exists i)\Delta^i T_0 p_2$ .4, EG
.6 $\Delta^j p_0 p_2 \vdash ((\exists i)\Delta^i T_0 p_2) \equiv p_0$ .2, .5, DT (twice), PC.

Now use DT and definition of 'TR'.

Now adopt the designations:

(7) $c_2 = [\sim TRc_2]_2$
(8) $(\exists i)\Delta^i(\sim TRc_2)[\sim TRc_2]_2$.

Theorem 15 $c_2 = [\sim TRc_2]_2 \supset \sim(\exists i)\Delta^i(\sim TRc_2)[\sim TRc_2]_2$.

Proof: 15.1 $\vdash \Delta^j(\sim TRc_2)[\sim TRc_2]_2 \supset .TR([\sim TRc_2]_2) \equiv \sim TRc_2$ T14, QT
.2 (7) $\vdash \Delta^j(\sim TRc_2)[\sim TRc_2]_2 \supset .TRc_2 \equiv \sim TRc_2$ .1, subst. iden.
.3 (7) $\vdash (\exists i)\Delta^i(\sim TRc_2)[\sim TRc_2]_2 \supset .TRc_2 \equiv \sim TRc_2$ .2,QT
.4 (7) $\vdash \sim$(8) .3, PC
.5 $\vdash$ (7) $\supset \sim$(8). .4, DT.

Thus, given an appropriate empirical premise, there will be *no* possible language in which the appropriate proposition is expressible. That is, there is no such language *in the domain* $\mathcal{L}$. Certainly the proposition in question is expressible in the language being used to prove Theorem 15. I conclude that the domain $\mathcal{L}$ cannot include the language of the Logic of Sense and Denotation. This is to be expected. If we think of a language as including a denotation relation (or the corresponding value functions), then it is not surprising that language being used, together with its denotation (or value) function, cannot be in a domain over which its variable range. We discuss this further below.

7 ***Models*** Returning to the formulation of the Logic of Sense and Denotation which does not contain quantification on possible language variables, we can sketch a construction which yields models.

The principal change from [1] in the conception of a model is that there is to be included a type $\mathcal{L}$ of possible languages. Each element of $\mathcal{L}$ is to consist of a set of typed well-formed expressions together with value-functions associated with each form and constant of the language. (The value-functions for constants give a fixed value for every assignment of values to any sequence of variables—which value is to be the denotation of the constant.) And we place the following requirements on the domain $\mathcal{L}$: For each language i belonging to $\mathcal{L}$, there is to be a language $i + 1$ which contains every symbol of i and, in addition, contains a name of the concept relation for i. Intuitively, this is a requirement which $i + 1$ would be required to satisfy if it could serve as (intensional) metalanguage for i. Furthermore, suppose that the language $i + 1$ is "explicit" in the sense that if the language i contains a name denoting a function F_{ab} and a name denoting the value $F_{ab}A_b$ of that function for an argument A_b, then $i + 1$ contains a name of that argument A_b. This will validate A(0)3. And we demand that if F_{ab} is a closed expression of indicated type, and A_b is a closed expression of type indicated by its subscript, then the language $i + 1$ is to contain an expression which expresses the concept $[F_{ab}A_b]_2$. In the base language of the Logic of Sense and Denotation such an expression is obtained by juxtaposing F_{ab} and A_b in that order. We require that each of the languages $i+1$ contains some such device for the expressions of i of the indicated types. This serves to validate C7.

In general we suppose that there may be given a set of stipulated synonymies or, better, stipulated *sense-concurrences* for some of the constants or forms of particular possible languages of $\mathfrak{L}$, setting them identical in meaning with forms (for all possible substitutions of meaningful constants for variables) or with constants, respectively, already present in that same language. Let

$$\mathbf{A}_a[x_b, y_c, \ldots, z_d] \propto \mathbf{B}_a[x_b, y_c, \ldots, z_d]$$

mean that the form (or constant) $\mathbf{A}_a$ is *stipulated sense-concurrent* with the form (or constant) $\mathbf{B}_a$. This relation is subject to the conditions that $\mathbf{A}_a$ and $\mathbf{B}_a$ are of the same type, contain (if any) exactly the same distinct free variables x_b, $y_c, \ldots, z_d$ (not necessarily all of the same type), and $\mathbf{A}_a$ contains no well-formed proper part except the mentioned variables. Further $\mathbf{A}_a$ is not to consist of a single such variable standing alone.

We assume about sense-concurrence the following principle:

(*) If $\mathbf{A}_a$ is sense-concurrent to $\mathbf{B}_a$, then $\dot{S}^{x_b}_{\mathbf{C}_b}\mathbf{A}_a|$ is sense-concurrent to $\dot{S}^{x_b}_{\mathbf{C}_b}\mathbf{B}_a|$.

(If variables of $\mathbf{C}_b$ would be captured in $\mathbf{B}_a$ by such a substitution, we require that $\mathbf{B}_a$ be first subjected to alphabetic changes of bound variables to avoid such capture.)

We require further of the relation $\propto$ that it shall be a function and that stipulations of sense-concurrence shall be noncircular: every nonempty set of forms (or constants) shall contain a form (or constant) no well-formed part of which is stipulated sense-concurrent with any other expression in the set. (This also guarantees that stipulated sense-concurrence shall be grounded—there can be no infinite sequences of such stipulations without foundation.)

Then a well-formed expression $\mathbf{D}_a$ is *sense-isomorphic* to an expression $\mathbf{E}_a$ if and only if its being so follows from the rules:

1. If two expressions are stipulated to be sense concurrent or their sense concurrence follows from (*), then they are sense-isomorphic.
2. Expressions which differ only by alphabetic change of bound variables are sense-isomorphic.
3. If $\mathbf{E}_a$ can be obtained from $\mathbf{D}_a$ by a finite sequence of replacements of sense-isomorphic expressions, then $\mathbf{D}_a$ is sense-isomorphic to $\mathbf{E}_a$.

Sense isomorphism as thus characterized is a modified version of Church's synonymous isomorphism. The axioms of Alternative (0) are not affected by this change—it only serves to accommodate certain apparent natural language counterexamples to synonymous isomorphism as criterion of synonymy (see [2]).

There seems to be no objection to allowing stipulations of sense-concurrence *between* different languages in the domain. Because this will obviously require certain further constraints on the domains, we do not pursue the completely general conception of a model further in this place. Besides the cross language synonymies, it seems natural to allow constraints on meaning within a given language of a weaker sort—such as might correspond to Carnap's idea of meaning postulates [4].

To construct a model given such a domain of possible languages, proceed as follows. Add to the Logic of Sense and Denotation symbols to denote each of the concept relations (of all types) associated with the languages of $\mathfrak{L}$. (This might be done by simply adding names, not necessarily denumerably many, for each of the possible languages in $\mathfrak{L}$ and then substituting these for the possible language variable in the expressions Δ^i.) It follows, of course, that the concept relations of the base language will not be included among those thus denoted. The Logic of Sense and Denotation is not itself one of the languages included in the domain $\mathfrak{L}$. Now collect closed expression of a given type together into equivalence classes on the basis of sense-isomorphism. These will be used to represent concepts of entities of that type. The domain of individuals is to be the union of all the domains of individuals for the particular languages of $\mathfrak{L}$, together, if desired, with a domain of individuals for the base language. The domain of possible languages is required to include the sublanguages L_0, $L_1, \ldots$, where L_m ($m = 0, 1, 2, \ldots$) is obtained from the Logic of Sense and Denotation by deleting all symbols Δ^n, except those where n is a numeral designating a number less than m. Proceed to assign denotations (or more generally, value-functions) for these languages as outlined in [1].

Because the notion of an admissible model has not been made completely determinate, no notion of validity is yet defined. We can, however, guarantee the consistency of the logic by constructing a particular model in such a way that there are no stipulated synonymies of any kind and the possible language domain contains exactly the languages $L_0, L_1. \ldots$

8 Concluding remarks The ultimate justification of the scheme I have proposed will come, if it comes at all, from the development of its consequences as to the underlying explanations of the various paradoxes and as to the validation of various inferences concerning intensionalities. Further details of the general conception of a possible language and of synonymies across such languages are needed. But we can give some informal reasons for the suggestion that a concept relation might carry with it the language from which it was abstracted.

Let us consider the idea of a proposition and its truth or falsity as a case in point. As a matter of historical fact, the notion of a proposition has arisen by abstraction from the idea of different sentences having the same meaning in the same or in different languages (see, for example, [6]). It is by no means obvious that it makes sense to attribute such a property as truth to such abstracta independently of their connection with the languages from which they were abstracted. If we take language in a broad sense (as we do), it is clear that no example of a proposition which is true or false independently of a language has ever been or could be given. To point out, for example, that it was true before there were any people that the earth went around the sun does not guarantee that the proposition need have no connection even with *possible languages*. These are just as abstract and necessary as propositions (if either are).

Whether Frege himself would approve of this idea is not completely clear. Certainly the tendency of his thought suggests that he would not be content with a notion of truth tied so closely to systems of representation. But he had not yet had to face the semantical paradoxes. In any case the question is of minor historical interest.

NOTES

1. His only suggestion of what a formalized intensional logic might contain is in a letter to Russell [11], p. 153.
2. If we regard expressions as just anything at all, for example, numbers, then of course the Logic of Sense and Denotation contains sufficient machinery to discuss languages and assignments of meaning to them.
3. The Zermelo-König Antinomy (or paradox) was so-called by Beth ([3], p. 488). It was discovered by König.
4. Russellian intensional logic, as Russell of course knew, encounters some of these same difficulties. These are all blocked therein by the Ramified Theory of Types. This is true even of the Russell-Myhill Antinomy as it appears in a detailed formulation of Russell's intensional logic implicit in *The Principles of Mathematics*. For a detailed formulation of that logic and the resolution in Ramified Type Theory, see Church [9].
5. Really it is the impossibility of *denoting* certain things which is proved and the impossibility of expressing concepts *of them* in the language is then a consequence. We may say, if we wish, that the very same concepts are expressible in the language, but we must then allow that they have no extensions therein. Kripke [13] has especially emphasized that the paradoxes should be construed as proving theorems about the possibility of doing such-and-such in languages of a certain sort.
6. David Kaplan observed in 1962 that the intensional analogue of the Epimenides is formalizable in the Logic of Sense and Denotation.
7. The present suggestion may be viewed as accepting Gödel's claim ([12], footnote 17, p. 134) that the theory of simple types suffices for avoiding also the "epistemological paradoxes".

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