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An Intuitionistic Sheffer Function

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The purpose of this note is to present a ternary propositional function which is a Sheffer function in the Heyting propositional calculus. We shall also consider some related Sheffer functions in positive logic.

Although it is easy to guess what should be the general notion of a Sheffer function in a propositional calculus, we shall first fix our terminology. Following [2] and [1], we shall say that a set of functions F is a *Sheffer set* for a set of functions G iff every member of G can be defined by a finite number of compositions from the members of F. A set F is an *indigenous Sheffer set* for G iff F is a Sheffer set for G and G is a Sheffer set for F. A function f is an (*indigenous*) *Sheffer function* for G iff $\{f\}$ is a (indigenous) Sheffer set for G. Of course these notions will interest us here only when the functions in question are propositional functions. Unless stated otherwise, $\to$, $\wedge$, $\vee$, $\neg$, $\leftrightarrow$, $\perp$, and $\top$ will stand for the usual Heyting propositional functions.

In [1] Hendry has shown that there is no binary indigenous Sheffer function for $\{\to, \wedge, \vee, \neg\}$. In that paper it is also stated that $\{\leftrightarrow, \vee, \neg\}$ is an indigenous Sheffer set for $\{\to, \wedge, \vee, \neg\}$. More precisely, $\{\leftrightarrow, \vee\}$ is an indigenous Sheffer set for $\{\to, \wedge, \vee\}$, since in the Heyting propositional calculus we can prove

$$(A \to B) \leftrightarrow ((A \vee B) \leftrightarrow B)$$
$$(A \wedge B) \leftrightarrow ((A \vee B) \leftrightarrow (A \leftrightarrow B)) \ .$$

(A useful survey of such equivalences can be found in [3] and [4], p. 21.)

Some further economy was achieved by Schroeder-Heister in [7]. He shows that $\{s, \perp\}$ is an indigenous Sheffer set for $\{\to, \wedge, \vee, \neg\}$, where s is a ternary propositional function defined by

$$s(A_1, A_2, A_3) \leftrightarrow ((A_1 \leftrightarrow A_2) \vee A_3) \ .$$

Then in the Heyting propositional calculus we can prove

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$$\begin{aligned}\top &\leftrightarrow s(A, A, B)\\ (A \leftrightarrow B) &\leftrightarrow s(A, B, \bot)\\ (A \vee B) &\leftrightarrow s(A, \top, B)\\ \neg A &\leftrightarrow s(A, \bot, \bot)\ .\end{aligned}$$

Instead of s we could as well use the ternary propositional function s_1 for which we have

$$s_1(A_1, A_2, A_3) \leftrightarrow ((A_1 \leftrightarrow A_2) \vee (A_1 \leftrightarrow A_3))$$

$$\begin{aligned}\top &\leftrightarrow s_1(A, A, B)\\ (A \leftrightarrow B) &\leftrightarrow s_1\ (A, B, B)\\ (A \vee B) &\leftrightarrow s_1(\top, A, B)\\ \neg A &\leftrightarrow s_1(A, \bot, \bot)\ .\end{aligned}$$

The advantage of s_1 over s is that s_1 is an indigenous Sheffer function for $\{\rightarrow, \wedge, \vee\}$, as can be seen from the equivalences above. Another ternary propositional function which can replace s or s_1, and is also an indigenous Sheffer function for $\{\rightarrow, \wedge, \vee\}$, is s_2 for which we have

$$s_2(A_1, A_2, A_3) \leftrightarrow ((A_1 \vee A_2) \leftrightarrow A_3)$$

$$\begin{aligned}\top &\leftrightarrow s_2(A, A, A)\\ (A \leftrightarrow B) &\leftrightarrow s_2(A, A, B)\\ (A \vee B) &\leftrightarrow s_2(A, B, \top)\\ \neg A &\leftrightarrow s_2(A, A, \bot)\\ (A \rightarrow B) &\leftrightarrow s_2(A, B, B)\\ (A \wedge B) &\leftrightarrow s_2(A, B, A \leftrightarrow B)\ .\end{aligned}$$

There is no binary indigenous Sheffer function for $\{\rightarrow, \wedge, \vee\}$. This can be inferred from the following. If f were a binary indigenous Sheffer function for intuitionistic $\{\rightarrow, \wedge, \vee\}$, it would also be an indigenous Sheffer function for classical $\{\rightarrow, \wedge, \vee\}$, because every equivalence provable in the Heyting propositional calculus is a two-valued tautology. And that there is no binary indigenous Sheffer function for classical $\{\rightarrow, \wedge, \vee\}$ is shown by surveying all the binary two-valued propositional functions.

Let f_1 and f_2 be n-ary propositional functions in the Heyting (or classical) propositional calculus. Then f_1 and f_2 are mutually equivalent iff for some permutations P_1 and P_2 of the sequence $A_1, \dots, A_n$, in the Heyting (or classical) propositional calculus we can prove $f_1(P_1) \leftrightarrow f_2(P_2)$. It follows easily that if s_1 and s_2 are defined in terms of classical $\leftrightarrow$ and $\vee$, they are indigenous Sheffer functions for classical $\{\rightarrow, \wedge, \vee\}$. Since it is not difficult to show that in that case s_1 and s_2 are nonequivalent, the Heyting propositional functions s_1 and s_2 are also nonequivalent.

Now we introduce the ternary propositional function t which is an indigenous Sheffer function for the whole set $\{\rightarrow, \wedge, \vee, \neg\}$. This function is defined by

$$t(A_1, A_2, A_3) \leftrightarrow ((A_1 \vee A_2) \leftrightarrow (A_3 \leftrightarrow \neg A_2))\ .$$

Then in the Heyting propositional calculus we can prove

$$\begin{aligned}
\neg A &\leftrightarrow t(A, A, A) \\
(A \vee B) &\leftrightarrow t(A, B, \neg B) \\
\bot &\leftrightarrow t(\neg A, A, A) \\
(A \leftrightarrow B) &\leftrightarrow t(A, \bot, B) \\
\top &\leftrightarrow t(A, \bot, A) \\
(A \rightarrow B) &\leftrightarrow t(A \vee B, \bot, B) \\
(A \wedge B) &\leftrightarrow t(A \vee B, \bot, A \leftrightarrow B) \ .
\end{aligned}$$

To obtain an n-ary ($n > 3$) indigenous Sheffer function for $\{\rightarrow, \wedge, \vee, \neg\}$ just substitute $B_1 \wedge \ldots \wedge B_{n-2}$ or $B_1 \vee \ldots \vee B_{n-2}$ for A_i in $(A_1 \vee A_2) \leftrightarrow (A_3 \leftrightarrow \neg A_2)$. If this substitution is made for say A_1, and the resulting function is $f_n(B_1, \ldots, B_{n-2}, A_2, A_3)$, then in the Heyting propositional calculus we can prove

$$f_n(A_1, \ldots, A_1, A_2, A_3) \leftrightarrow t(A_1, A_2, A_3) \ .$$

We conclude this note with a question. What is the number of mutually nonequivalent ternary indigenous Sheffer functions for $\{\rightarrow, \wedge, \vee, \neg\}$? This number (which is, of course, finite in the classical case) is greater than one. It is easily shown that functions like the following

$$\begin{aligned}
&t(A_1, A_2, A_3) \vee (A_1 \leftrightarrow (A_3 \leftrightarrow \neg A_2)) \\
&t(A_1, A_2, A_3) \vee (\neg A_1 \wedge A_2 \wedge A_3) \ ,
\end{aligned}$$

which are indigenous Sheffer functions for $\{\rightarrow, \wedge, \vee, \neg\}$ (in the definitions of $\neg$, $\vee$, and $\leftrightarrow$ proceed as with t using $\bot \leftrightarrow \neg(A \vee \neg A)$), are nonequivalent to t.[1]

NOTE

1. After completing this paper I was informed by H. E. Hendry that a function equivalent to t, as well as its extensions to n-ary ($n > 3$) cases, were discovered independently by G. N. Haven. I have also learned that another example of a ternary indigenous Sheffer function for $\{\rightarrow, \wedge, \vee, \neg\}$ was given by A. V. Kuznetsov in [5], viz. $((A_1 \vee A_2) \wedge \neg A_3) \vee (\neg A_1 \wedge (A_2 \leftrightarrow A_3))$. In this paper Kuznetsov shows that there is no indigenous Sheffer function for $\{\rightarrow, \wedge, \vee, \neg\}$ with fewer than five occurrences of variables when written in terms of $\rightarrow$, $\wedge$, $\vee$, and $\neg$. The function t does not contradict this result, since when it is written in terms of $\rightarrow$, $\wedge$, $\vee$, and $\neg$ it has more than five occurrences of variables. Kuznetsov also anticipates [1] in demonstrating that there is no binary indigenous Sheffer function for $\{\rightarrow, \wedge, \vee, \neg\}$. Another paper relevant to our topic is [6], which treats of criteria for Sheffer sets in the Heyting propositional calculus.

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