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Incompactness in Regular Cardinals

SAHARON SHELAH*

Annotated Contents

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If $PT(\lambda, \aleph_1)$ fails, then there are countable $A_i (i < \lambda)$ such that $\{A_i: i < \lambda\}$ has no transversal but for every $\alpha < \lambda$ we can find $\beta_i (i < i(*))$ $\alpha = \{\beta_i: i < i(*)\}$ and $A_{\beta_i} - \bigcup_{j<i} A_{\beta_j}$ is infinite.

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Section 5 λ-freeness of Abelian groups: We prove that, if enough axioms are satisfied, then incompactness in λ implies $PT(\lambda, \kappa^+)$ fails. We then prove that $PT(\lambda, \aleph_1)$ is equivalent to "every λ-free Abelian group is λ^+-free" and to "every strongly λ-free Abelian group is λ^+-free".

We finish with some concluding remarks on further research.

Introduction

Context: U is a fixed set (we shall deal with subsets of it) and F a family of pairs of subsets of it. We write $A/B \in F$ or say "A/B is free" or "A is free over B" when $(A, B) \in F$, χ will be a fixed cardinal.

Axiom II *A/B is free if $A \cup B/B$ is free; A/A is free.*

Axiom III *If A/B, B/C are free then A/C is free.*

Axiom IV *If $A_i (i < \lambda)$ is increasing, for $i < \lambda$ $A_i \Big/ \bigcup_{j<i} A_j \cup B$ is free then $\bigcup_{i<\lambda} A_i/B$ is free.*

Definition 0.1 We say "for the χ-majority of $X \subseteq A$, $P(X)$" if there is an algebra $\mathfrak{A}$ with universe A and χ functions such that any $X \subseteq A$ closed under those functions satisfies P. We can replace $X \subseteq A$ by $X \in \mathcal{P}(A)$ or $X \in \mathcal{P}_{<\lambda}(A)$; alternatively we say $\{X \subseteq A: \mathcal{P}(A)\}$ is a χ-majority.

Axiom VI *If A is free over $B \cup C$ then for the χ-majority of $X \subseteq A \cup B \cup C$, $A \cap X/(B \cap X) \cup C$ is free.*

Axiom VII *If A is free over B, then for the χ-majority of $X \subseteq A \cup B$, $A/(A \cap X) \cup B$ is free.*

Definition 0.2 A/B is κ-free if for the χ-majority of $X \subset A \cup B$, if $|\bar{X}| < \kappa$ then $A \cap X/B$ is free.

Definition 0.3 $E^{\kappa}_{\kappa}(A)$ is the filter on $\mathcal{P}_{\leq\kappa}(A)$ generated by the sets $G_n(F) =_{df} \left\{ \bigcup_{i<\kappa} A_i \colon A_i \subseteq A,\ |A_i| < \kappa,\ F(\langle A_j \colon j \leq i \rangle) \subseteq A_{i+1} \right\}$ where $F_i\colon {}^{\kappa>}[\mathcal{P}_{<\kappa}(A)] \to \mathcal{P}_{<\kappa}(A)$ (we use κ regular $> \aleph_0$).

Theorem 0.4 (Shelah) *Suppose $|A| = \lambda$, λ is singular, $\lambda = \sum_{i<cf\lambda} \lambda_i$, λ_i increasing continuous. Then A/B is free iff A/B is λ-free iff for every i $\{X \in \mathcal{P}_{\leq\lambda_i^+}(A)\colon X/B \text{ free}\} \neq \phi \bmod E^{\lambda_i^+}_{\lambda_i^+}(A)$.*

Remark: The theorem was proved with more axioms (I^*, V) in [6]. The author then eliminated I^* and this is presented in [1]. Later (see [7]) the author found a simpler proof; and both new parts avoid Axiom V. In Hodges [3] this is presented (and more is proved) in a different, but equivalent axiomatic treatment.

Here we wonder what occurs for regular λ. The following examples show that there may be very different behaviors, hence it is reasonable to demand more in proving equivalence.

Example 0.5 Let U be the disjoint union of $P \cup Q$, $R \subseteq P \times Q$, and call A/B free if there is a one-to-one function g $Dom\ g = Q \cap (A - B)$, $(gx)Rx$ and $Rang(g) \subseteq A - B$. This easily satisfies the axioms (but note $A' \subseteq A$ and A/ϕ free $\not\Rightarrow A'/\phi$ is free). Let $\chi = \aleph_0$.

Let λ be any cardinal, and suppose $UA = \{a_i: i < \lambda^+\}$ $a_i \in P \Leftrightarrow i < \lambda$, $a_i\ R\ a_j$ when $i < \lambda \leq j < \lambda^+$. Then A/ϕ is not free. Is it λ^+-free? Yes, iff the Chang conjecture fails for (λ, λ^+), i.e. if for the $\aleph_0$-majority of $X \in \mathcal{P}_{\leq\lambda}(A) |X \cap \{a_i: i < \lambda\}| = |X|$. We can let $S \subseteq \lambda^+ - \lambda$ be stationary, $A_i = \{a_i: i < \lambda^+\}$, $P \cap A = \{a_{i+1}: i < \lambda^+\} \cup \{a_0\}$ $a_i\ R\ a_j \Leftrightarrow$ (j limit, i not, $i < j$, $j \in S$). Then the Chang conjecture has to be strengthened by reflecting that S is stationary in an appropriate sense.

You may prefer examples which are varieties.

Example 0.6 We define a variety; with function F, G, H (1-place, 1-place, 2-place respectively) and the equation

(i) $F(H((x, y)) = F(x)$
(ii) $G(H(x, y)) = y$
(iii) $H(H(y, x), z) = H(y, z)$.

For P, Q, R, A, λ as in the previous example let M be the free algebra generated freely by $\{x_i: i < \lambda^+\} \cup \{y_{i,j}: i < \lambda \leq j < \lambda^+\}$ subject only to the (equation of the variety and) relations:

(*) $F(y_{i,j}) = x_i,\ G(y_{i,j}) = x_j,\ y_{i,j} = H(y_{i,\gamma}, x_j)$.

Let $\lambda + \lambda < \alpha < \lambda^+$, $\{\beta_i: i < \lambda\} = (\lambda, \alpha)$, M_α the subalgebra, generated by $\{x_i: i < \alpha\} \cup \{y_{i,j}: i < \lambda \leq j < \alpha\}$. M_α is free: it is freely generated by $Y = \{y_{i,\beta_i}: i < \lambda\}$. Clearly Y generates M_α as $y_{i,\beta} = H(y_{i,\beta_i}, x_\beta)$, $x_i = F(y_{i,\beta_i})$ $x_{\beta_i} = G(y_{i,\beta_i})$. By the automorphism we can construct, we should consider only the equations between elements of M_α. Then translating the equation in (*) to the members of Y we get

$$F(y_{i,j}) = x_i \Rightarrow F(H(y_{i,\beta_i}, x_j)) = x_i \Rightarrow F(y_{i,\beta_i}) = x_i$$

(the second arrow is by the axiom (i). The result is the definition of x_i in terms of y.)

$$G(y_{i,j}) = x_j \Rightarrow G(H(y_{i,\beta_i}, G(y_{\xi,j}))) = G(y_{\xi,j}) \Rightarrow G(y_{\xi,j}) = G(y_{\xi,j})$$

$$y_{i,j} = H(y_{i,\beta_i}, x_j) \Rightarrow H(y_{i,\beta_i}, G(y_{\xi,j})) = H(H(y_{i,\beta_i}, G(y_{\xi,\gamma})), G(y_{\xi,j}))$$

which holds by equation (ii), (iii) resp.

Similarly, if the Chang Conjecture for λ^+ holds, M is λ^+-free.

But for every δ, $\lambda < \delta < \lambda^+$, M/M_α is not free.

Conclusion Compactness for Abelian groups is not equivalent to compactness for any variety (even $\chi = \aleph_0$).

Historical remark: Sections 1 and 2 were written up together with [6] (in the Spring of 1975) for dealing with the spectrum $\{\lambda$: there is a λ-free not λ^+-free $A\}$. The aim was to prove that for a wide class, few spectrums are possible. In 1981 we returned to the subject, and wrote a proof that the transversals for families of countable sets and the variety of Abelian groups have the same spectrum, (as claimed in [6], although we have had to withdraw the similar claim on the variety of groups). There was a gap in the proof (essentially, indistinction between (B) and (D) in 3.8) which was corrected later.

We are grateful to Alan Mekler for industriously refereeing the paper and for providing the followed specific aid: (A) Originally the dichotomy in 2.8 was carried later, so the nodes of λ-sets were of two kinds; this harms the readability of the proof, and after his suggestion was removed. (B) The proof of 5.2 was marred by jumps and misprints, and Mekler wrote a proof which adds the assumption that freeness is witnessed by bases (and the exchange principle holds). This differs from the author's proof in its proof of Fact C (also, Fact B was replaced by a simpler construction). The proof is included in an appendix by his kind permission. (C) He also suggested writing the equations in the proof of 5.3(1) so as to apply also to groups.

For consistency results (on compactness in regular cardinals (e.g., in $\aleph_{\omega^2+1}$)) see [4] where also, e.g., "G.C.H. $\Rightarrow \neg PT(\aleph_{\omega_1}, \aleph_1)$" hold.

1 A general theorem on lifting incompactness Continuing [6] we shall in this section give additional axioms axiomatizing "free amalgamation" and prove with, and then transfer, theorems of the form "if there is a λ-free not λ^+-free pair A/B, $|A| = \lambda$, then there is a μ-free not μ-free pair A'/B', $|A'| = \mu$".

We shall work in the context of [6], Section 1, but here F will contain not only pairs but also triples $\langle A, B, C\rangle$, and we shall say "A is free over B/C" rather than $\langle A, B, C\rangle \in F$. The meaning for e.g. groups is that the equations holding among elements of $A \cup C$ and of $B \cup C$ generate the equations holding among elements of $A \cup B \cup C$. (Reading [6], Section 1, pp. 324–326, is recommended.) The axioms for $\langle A, B, C\rangle \in F$ are motivated by "the group generated by $A \cup A \cup B \cup C$ is the free product of the groups generated by $A \cup C$ and $B \cup C$ over C" or the properties of nonforking.

We shall have also a fixed cardinal χ_3, $\chi_1 \leq \chi_3 \leq \chi_2$, and assume the following axioms as well as the axioms (conventions and assumptions) of [6], Section 1 (mostly listed in Section 0 for $\chi = \chi_1$ U an algebra on U with χ_0 operations):

Axiom I** *If A/B is free, $A^* \subseteq A$, then A^*/B is free.*

Axiom VIII *A is free over B/C iff $A \cup C$ is free over $B \cup C/C$.*

Axiom VIIIa *A is free over B/B.*

Axiom IX *Commutativity: If A is free over B/C, then B is free over A/C.*

Axiom X *Transitivity: If A_1 is free over B/C and A_2 is free over $B/A_1 \cup C$, then $A_1 \cup A_2$ is free over B/C.*

Axiom XI *Monotonicity: If A is free over B/C, $A' \subseteq A$, $B' \subseteq B$, then A' is free over $B/B' \cup C$.*

Axiom XII *Continuity: If $B_i (i < \alpha)$ is increasing, A free over B_i/C for every $i < \alpha$, then A is free over $\bigcup_{i<\alpha} B_i \Big/ C$.*

Axiom XIII *Existence: If* $|A| + |B| + |C| < \chi_3$, *then for some partial isomorphism f of* $\underline{U}$, *Dom* $f = cl(A \cup C)$ *and f is the identity over C and* $f(A)$ *is free over* B/C *(partial isomorphism means it preserves the operators and relations of* $\underline{U}$*).*

Axiom XIIIa *If f is a partial isomorphism,* $|Dom\, f| < \chi$ *and* $|B| < \chi$, *then for some partial isomorphism g,* $f \subseteq g$ $B \subseteq Dom\, f$.

Axiom XIV *Let A be free over B/C then:*
(1) *A is free over C iff A is free over* $B \cup C$.
(2) *If* $A \cup B/C$ *is free, then* A/C *is also free.*

Axiom XV *Invariance: Partial isomorphism preserves freeness of pairs and triples.*

Axiom XVI *Hereditarity: If* $A, B, C \in N$, $\langle A, B, C\rangle \in F$ *then* $\langle A \cap N, B \cap N, C \cap N\rangle \in F$.

Remark: (1) The only properties of partial isomorphism we need are their use, in Axiom XIII and Axiom XV, the fact that an increasing union of partial isomorphism is a partial isomorphism, and their closeness under compositions. (2) Some times in the text, Axiom I* should be replaced by Axiom I**.

Lemma 1.1 *Suppose* $(T, <)$ *is a tree (i.e., for each* $x \in T$ $\{y \in T: y < x\}$ *is well ordered, of order-type* $h(x)$, *the height of x in T). Assume that for* $x, y \in T$, $x < y \Rightarrow C \subseteq B_x \subseteq B_y$, $|C| + \sum_{x\in T} |B_x| < \chi_3$. *Then:*

(1) *For any well ordering* $<^*$ *of T extending < we can find partial isomorphisms* $g_x (x \in T)$, *Dom* $g_x = B_x$ *such that* $A_x = g_x(B_x)$ *satisfy*

(*) A_x *is free over* $\bigcup_{y<^*x} A_y \Big/ \bigcup_{y<x} A_y \cup C$, *and* $[x < y \Rightarrow g_x \subseteq g_y]$.

(2) *If* (1)(*) *is satisfied then* A_x *is free over* $\bigcup_{x\not\leq y} A_y \Big/ \bigcup_{y<x} A_y \cup C$.

(3) *If* (1)(*) *is satisfied,* $T_l \subseteq T$ *and* $(\forall x)(\forall y)(x \in T_l \wedge y < x \rightarrow y \in T_l)$ *(for* $l = 1, 2$*), then* $\bigcup_{x\in T_1} A_x$ *is free over* $\bigcup_{x\in T_2} A_x \Big/ \bigcup_{x\in T_1\cap T_2} A_x \cup C$.

(4) *If* $B_x \Big/ \bigcup_{y<x} B_y \cup C$ *is free then* $A_x \Big/ \bigcup_{y<x} A_y \cup C$ *is free.*

(5) *If* (1)(*) *is satisfied and* $A_x \Big/ \bigcup_{y<x} A_y \cup C$ *is free (for every x) then* $\bigcup_{x\in T} A_x \Big/ C$ *is free.*

(6) *If* (1)(*) *is satisfied* $A_x^* \subseteq A_x$ *and for any* $y \leq x$, A_x^* *is free over* $\bigcup_{z<y} A_x \Big/ \bigcup_{z<y} A_z^* \cup C$, *then:*

(A) *for any x,* A_x^* *is free over* $\bigcup_{x\not\leq y} A_y \Big/ \bigcup_{y<x} A_y^* \cup C$.

(B) *For any x,* A_x *is free over* $\bigcup_{x\not\leq y} A_y \Big/ A_x^* \cup \bigcup_{y<x} A_y \cup C$.

Proof: (1) Just defined g_x by induction on x in the order $<^*$, using Axioms XIII and XIIIa.

(2) Prove by induction on $z \in T(x \leq^* z)$, in the order $<^*$ that A_x is free over

$$\cup\{A_y: x \not\leq y,\ y <^* z\} \Big/ \bigcup_{y<x} A_y \cup C .$$

For $z = x$ it follows by (1)(*) above. For z limit by Axiom XII. So let z be the successor (by $<^*$) of z_1. If $x \leq z_1$ there is nothing to prove, so assume $x \not\leq z_1$, and let $A^0 = \cup\{A_y: x \not\leq y,\ y < z_1\}$, so by the induction hypothesis $\left\langle A_x, A^0, \bigcup_{y<x} A_y \cup C\right\rangle \in F$. By assumption A_{z_1} is free over $\bigcup_{y<^*z} A_y \Big/ \bigcup_{y<z_1} A_y \cup C$, hence over $A^0 \cup A_x \Big/ \bigcup_{y<z_1} A_y \cup C$ (by Axiom XI) and clearly $\bigcup_{y<z_1} A_y \subseteq A_0$ (as $y < z_1 \rightarrow x \not\leq y$). So by Axiom XI, A_{z_1} is free over $A^0 \cup A_x/A^0 \cup C$, hence over $A_x/A^0 \cup C$. So by Axiom IX A_x is free over $A_{z_1}/A^0 \cup C$. So by Axiom X $\left(A_x, \bigcup_{y<z_1} A_y \cup C, A_0, A_{z_1}\right.$ stand for B, C, A_1, A_2, respectively$\Big)$ A_x is free over $A^0 \cup A_{z_1} \Big/ \bigcup_{y<x} A_y \cup C$.

Adding to $<^*$ a last element ∞, we finish.

(3) Easy using (2) and repeating the argument. We prove by induction on $z \in T \cup \{\infty\}$ that T_1, $(T_1 \cap T_2) \cup \{t \in T_2: t <^* z\}$ satisfy the conclusion (for limit z use Axiom XII, for successor z use part 2) and Axioms IX, X, and XI.

(4) By Axiom XV it is immediate.

(5) It suffices to prove that $A_x \Big/ \bigcup_{y<^*x} A_y \cup C$ is free by Axiom IV. By part (3) A_x is free over $\bigcup_{y<^*x} A_y \Big/ \bigcup_{y<x} A_y \cup C$, so by Axiom XIV(1) we get our conclusion.

(6) (A) For any $x \in T$ by part (2) A_x is free over $\bigcup_{x \not\leq y} A_y \Big/ \bigcup_{y<x} A_y \cup C$, hence by Axiom XI also A_x^* is free over $\bigcup_{y \not\leq x} A_y \Big/ \bigcup_{y<x} A_y \cup C$. By Axioms IX and X A_x^* is free over $\bigcup_{y<x} A_y \Big/ \bigcup_{y<x} A_y^* \cup C$. So by Axiom X A_x^* is free over $\bigcup_{x \not\leq y} A_t \Big/ \bigcup_{y<x} A_y^* \cup C$ (using Axiom IX).

(B) Easy by (6)(A), (2), and Axiom XII.

Definition 1.2 Let $I\,C(\lambda, \delta)$ hold if there is a set S of elements of a tree T, each of height δ, such that

(1) $|S| = \lambda$.
(2) there is no f: $S \rightarrow T$ such that $f(s) < s$ (for each $s \in S$) and for no $s \neq t \in S$ does $f(t) \leq f(s) < t$ (such an f is called a pressing-down function).
(3) But for every $S' \subseteq S$, $|S'| < \lambda$, there is a pressing down function f: $S' \rightarrow T$.

Remark: It is easy to check that by [6], Section 2, $\delta < \lambda$, $I\,C(\lambda, \delta)$ implies λ is regular.

Remark 1: W.l.o.g. each $t \in T - S$ has height $< \delta$, and for some $s \in S$, $t < s$.

Remark 2: The properties $Q_n(\lambda)$ are defined for two purposes: first, to help to prove incompactness cases; second in the hope of proving that when enough axioms are satisfied, $\{n: Q_n(\aleph_0) \text{ holds}\}$ determines the incompactness spectrum.

Definition 1.3 B satisfies $Q_0(\lambda)$ if there is an A, $|A| = \lambda$ such that A/B is λ-free but is not free (in the sense of [6], Definition 1.1(1)!).

Definition 1.4[1]

(1) B satisfies $Q_n(\delta, \lambda)$ $(0 < n < \omega)$ if there are A_i $(i \le \delta)$ and $C_l(2 \le l \le n)$ exemplifying it. They exemplify it if:

(i) A_i is increasing, $A_0 = \emptyset$, $C_l \subseteq A_\delta$ and for limit $j < \delta$, $A_j = \bigcup_{i<j} A_i$, and for some $X \subseteq A_\delta$, $|X| = \lambda$, $A_\delta \subseteq cl(B \cup X)$.

(ii) A_δ is not free over $\bigcup_{j<\delta} A_j \cup \bigcup_{l=2}^{n} C_l \cup B$.

(iii) for $0 \le i \le j \le \delta$, $u \subseteq w = \{k, \ldots, n\}$, $2 \le k \le n+1$.

(α) $A_j \cap \bigcap_{l\in u} C_l$ is free over $\left(A_i \cap \bigcap_{l\in w} C_l\right) \cup B$ $\left(\text{for empty set } v \ \bigcap_{l\in v} C_l = U, \text{ and for } k = n+1, \{k, \ldots, n\} = \emptyset\right)$.

(β) $A_j \cap \bigcap_{l\in u} C_l \Big/ \left(A_j \cap \bigcap_{l\in w} C_l\right) \cup \left(A_i \cap \bigcap_{l\in u} C_l\right) \cup B$ is free.

(iv) for $u \subseteq w = \{k, \ldots, n\}$, $2 \le k \le n$:

(α) $A_\delta \cap \bigcap_{l\in u} C_l$ is free over $\left(\bigcup_{\alpha<\delta} A_\alpha \cap \bigcap_{l\in w} C_l\right) \cup B$.

(β) $A_\delta \cap \bigcap_{l\in u} C_l \Big/ \left(A_\delta \cap \bigcap_{l\in w} C_l\right) \cup \left(\bigcup_{\alpha<\delta} A_\alpha \cap \bigcap_{l\in u} C_l\right) \cup B$ is free.

(v) If $0 \le i \le j \le \delta$, $u \subseteq w \subseteq \{2, \ldots, n\}$, $2 \le k \le n+1$, $w = u \cup \{l: k \le l \le n\}$ then:

(α) $A_j \cap \bigcap_{l\in u} C_l$ is free over $A_i \cap \bigcap_{l=k}^{n} C_l \Big/ \left(A_i \cap \bigcap_{i\in w} C_l\right) \cup B$ and, if $w \ne \emptyset$

(β) $A_\delta \cap \bigcap_{l\in u} C_l$ is free over $\bigcup_{\alpha<\delta} A_\alpha \cap \bigcap_{l=k}^{n} C_l \Big/ \left(\bigcup_{\alpha<\delta} A_\alpha \cap \bigcap_{l\in w} C_l\right) \cup B$.

(2) If $\delta = \lambda \ge \chi_0$ we write $Q_n(\lambda)$ instead of $Q_n(\lambda, \lambda)$.

Claim 1.5

(1) *If B satisfies $Q_n(\delta, \lambda)$, $0 < m \le n$, then B satisfies $Q_m(\delta, \lambda)$. In fact the same A_i's and C_l's exemplify it.*
(2) *If $cf\,\delta = cf\,\delta'$ then B satisfies $Q_n(\delta, \lambda)$ iff B satisfies $Q_n(\delta', \lambda)$.*

Proof: Easy.

Theorem 1.6 *Suppose B satisfies $Q_{n+1}(\delta, \kappa)$, and $\mu > \delta + \kappa + \chi_1$, $\chi_3 > \mu + |B|$ and μ is regular. If $I\,C(\mu, \delta)$ then B satisfies $Q_n(\mu)$.*

Proof: Let $\langle A_i: i \le \delta\rangle$, $\langle C_l: 2 \le l \le n+1\rangle$ exemplify the satisfaction of

$Q_{n+1}(\delta, \kappa)$ by B. Let $S = \{t_i: i < \mu\} \subseteq T$ exemplify the holding of $I\,C(\mu, \delta)$, and assume $s \in T - S$ implies that for some $t \in S$, $s < t$, hence $h(s) < \delta$; so $|T| = \mu$. For each $t \in T$ of height $h(t) = \alpha$ let $B_t = A_{h(t)}$; so we can apply Lemma 1.1 and get partial isomorphisms $g_t (t \in T)$ satisfying the condition (*) there (for any extension of $<$ to a well-ordering). In particular

(α) $t \leq s$ implies g_s extend g_s
(β) $Dom\ g_s = A_{h(s)}$
(γ) $g_t(A_\delta)$ is free over $\bigcup\{g_s(A_\delta): t \not\leq s, s \in T\} \Big/ \bigcup_{s<t} g_t(A_{h(t)}) \cup B$.

Let us define:

1. $A'_\mu = \bigcup_{t\in S} g_t(A_\delta)$
2. for $j < \mu$ $A'_j = \bigcup\{g_t(A_\delta \cap C_{n+1}): t = t_i, i < j\}$ if $n > 0$; and $A'_j = \emptyset$ if $n = 0$
3. for $2 \leq l \leq n$, $C'_l = \bigcup_{t\in S} g_t(A_\delta \cap C_l)$.

We shall prove that $\langle A'_j: j \leq \mu\rangle$, $\langle C'_l: 2 \leq l \leq n\rangle$ exemplify $Q_n(\mu)$ for B (if $n = 0$, A'_μ only is used). Let us check the conditions when $n > 0$.

Condition (i): Trivial.

Condition (ii): Suppose $A'_\mu \Big/ \bigcup_{j<\mu} A'_j \cup \bigcup_{l=2}^{n} C'_l \cup B$ is free, and we shall get a contradiction. Checking the definitions of A'_j, C'_l we see that our hypothesis means that $\bigcup_{t\in S} g_t(A_\delta)$ is free over $\bigcup_{t\in S} g_t\left(A_\delta \cap \bigcup_{l=2}^{n+1} C_l\right) \cup B$. As $\mu > |\delta| + \kappa$, clearly $\mu > |A_\delta|$, and so as $\mu > \chi_1$ we get by [6], Lemma 1.2, that for some closed unbounded subset W of μ, for any $i < j \in W$ (or $i \in W$, $j = \delta$), $\bigcup_{\alpha<j} g_{t_\alpha}(A_\delta)$ is free over $\bigcup_{\alpha<i} g_{t_\alpha}(A_\lambda) \cup \bigcup_{t\in S} g_t\left(A_\delta \cap \bigcup_{l=2}^{n+1} C_l\right) \cup B$.

Let $T_i = \{t \in T: t \leq s_\alpha$ for some $\alpha < i\}$; then clearly for some $i, j \in W$ there is α, $i \leq \alpha < j$ such that $t < t_\alpha \to t \in T_i$ (otherwise S, T will not exemplify $I\,C(\mu, \delta)$). By Lemma 1.1(3) $g_{t_\alpha}(A_\delta)$ is free over $\bigcup_{\alpha\neq\beta<\mu} g_{t_\beta}(A_\delta) \Big/ \bigcup_{\beta<i} g_{t_\beta}(A_\delta) \cup B$.

By using Axiom XI twice, $g_{t_\alpha}(A_\delta)$ is free over $\bigcup_{\alpha\neq\beta<\mu} g_{t_\beta}(A_\delta) \Big/ \left[g_{t_\alpha}\left(A_\delta \cap \bigcup_{l=2}^{n+1} C_l\right)\right] \cup \left[\bigcup_{\alpha\neq\beta<\mu} g_{t_\beta}\left(A_\delta \cap \bigcup_{l=2}^{n+1} C_l\right)\right] \cup \bigcup_{\beta<i} g_{t_\beta}(A_\delta) \cup B$, that is, over $\bigcup_{\alpha\neq\beta<\mu} g_{t_\beta}(A_\lambda) \Big/ \bigcup_{\beta<\mu} g_{t_\beta}\left(A_\delta \cap \bigcup_{l=2}^{n+1} C_\delta\right) \cup \bigcup_{\beta<i} g_{t_\beta}(A_\delta) \cup B$.

As $g_{t_\alpha}(A_\delta)$, $\bigcup_{\alpha\neq\beta<\mu} g_{t_\beta}(A_\delta)$ generate together A'_μ, necessarily by Axiom XIV(2) $g_{t_\alpha}(A_\delta)$ is free over $\bigcup_{\beta<\mu} g_{t_\beta}\left(A_\delta \cap \bigcup_{l=2}^{n+1} C_l\right) \cup \bigcup_{\beta<i} g_{t_\beta}(A_\delta) \cup B$. But by ($\gamma$) $g_{t_\alpha}(A_\delta)$ is free over $\bigcup_{\alpha\neq\beta<\mu} g_t(A_\delta) \Big/ \bigcup_{t<t_\alpha} g_t(A_{h(t)}) \cup B$, hence by Axiom XI

$g_{t_\alpha}(A_\delta)$ is free over $\bigcup_{\alpha \neq \beta < \mu} g_{t_\beta}(A_\delta) \Big/ \bigcup_{t < t_\alpha} g_t(A_{h(t)}) \cup g_{t_\alpha}\left(A_\delta \cap \bigcup_{l=2}^{n+1} C_l\right) \cup B$, hence by Axiom XI $g_{t_\alpha}(A_\delta)$ is free over $\bigcup_{\alpha \neq \beta < \mu} g_{t_\beta}\left(A_\delta \cap \bigcup_{l=2}^{n+1} C_l\right) \cup \bigcup_{\beta < i} g_{t_\beta}(A_\delta) \Big/ \bigcup_{t < t_\alpha} f_t(A_{h(t)}) \cup g_{t_\alpha}\left(A_\delta \cap \bigcup_{l=2}^{n+1} C_l\right) \cup B$. So by Axiom XIV(2) $g_{t_\alpha}(A_\delta)$ is free over $\bigcup_{t < t_\alpha} g_t(A_{h(t)}) \cup g_{t_\alpha}\left(A_\delta \cap \bigcup_{l=2}^{n+1} C_l\right) \cup B$, a contradiction by Axiom XV.

Condition (iii): Assume $u \subseteq w = \{k, \ldots, n\}$, $2 \leq k < n + 1$, $0 \leq i \leq j \leq \delta$ and we should prove that $A'_j \cap \bigcap_{l \in u} C_l$ is free over $\left(A'_i \cap \bigcap_{l \in w} C'_l\right) \cup B$. This means, when $j < \delta$, that $\bigcup_{\alpha < j} g_{t_\alpha}\left(A_\delta \cap \bigcap_{l \in u(*)} C_l\right)$ is free over $\bigcup_{\alpha < i} g_{t_\alpha}\left(A_\delta \cap \bigcap_{l \in w(*)} C_l\right) \cup B$, where $u(*) = u \cup \{n + 1\}$ and $w(*) = w \cup \{n + 1\}$.

By Definition 1.4 (v)(α), for every $t \in T$ (letting $u(*)$, $w(*)$, $n + 1$ stand for u, w, k respectively) $g_t\left(A_{h(t)} \cap \bigcap_{l \in u(*)} C_l\right)$ is free over $g_t(A_{h(t)}) \Big/ \bigcup_{s < t} g_s\left(A_{h(s)} \cap \bigcap_{l \in u(*)} C_l\right) \cup B$.

A similar result holds for $w(*)$. Hence by Lemma 1.1(6), (3), and Definition 1.2(iii) the conclusion follows.

We are left with the case $j = \delta$, but we can prove it similarly, this time $u(*) = u$, $w(*) = w \cup \{n + 1\}$, and we use (v)(α) from Definition 1.2 again.

Condition (iv): The proof is similar to the previous one.

Condition (v): Let us concentrate on the case (v)(α), $j = \mu$; so let $u \subseteq w \subseteq \{2, \ldots, n\}$, $2 \leq k \leq n + 1$, $w = u \cup \{l\colon k \leq l \leq n\}$ and $i < \mu$; we should prove that $A'_\mu \cap \bigcap_{l \in u} C'_l$ is free over $A'_i \cup \bigcap_{l=k}^{m} C'_l \Big/ A'_i \cap \bigcap_{l \in w} C'_l$. This means that $\bigcup_{\beta < \mu} g_{t_\beta}\left(A_\delta \cap \bigcap_{l \in u} C_l\right)$ is free over $\bigcup_{\beta < i} g_{t_\beta}\left(A_\delta \cap \bigcap_{l=k}^{n+1} C_l\right) \Big/ \bigcup_{\beta < i} g_{t_\beta}\left(A_\delta \cap \bigcap_{l \in w} C_l \cap C_{n+1}\right) \cup B$. By Lemma 1.1(6) $\bigcup_{\beta < \mu} g_{t_\beta}\left(A_\delta \cap \bigcap_{l \in u} C_l \cap C_{n+1}\right)$ is free over $\bigcup_{\beta < i} g_{t_\beta}\left(A_\delta \cap \bigcap_{l=k}^{n+1} C_l\right) \Big/ \bigcup_{\beta < i} g_{t_\beta}\left(A_\delta \cap \bigcap_{l \in w} C_l \cap C_{n+1}\right) \cup B$.

By 1.1(6) also $\bigcup_{\beta < \mu} g_{t_\beta}\left(A_\delta \cap \bigcap_{l \in u} C_l\right)$ is free over $\bigcup_{\beta < i} g_{t_\beta}(A_\delta) \Big/ \bigcup_{\beta < \mu} g_{t_\beta}\left(A_\delta \cap \bigcap_{l \in u} C_l \cap C_{n+1}\right) \cup B$.

Hence it is free over $\bigcup_{\beta < i} g_{t_\beta}\left(A_\delta \cap \bigcap_{l=k}^{n+1} C_l\right) \Big/ \bigcup_{\beta < \mu} g_{t_\beta}\left(A_\delta \cap \bigcap_{l \in u} C_l \cap C_{n+1}\right) \cup B$. By Axiom X our conclusion follows.

The other cases are similar.

So we are left with the case $n = 0$. The nonfreeness of A'_μ/B was proved, in fact, when we proved Condition (ii). Let us prove that A/B is μ-free, where $A = A'_\mu$; so let $\{A, B\} \subseteq N$, $\|N\| < \mu$, and so $\mu = |A| \in N$, hence we can assume that $\langle A_i: i < \delta \rangle$, T, S, $\langle g_t: t \in T \rangle$ all belong to N. We can also assume that $N \cap \delta$ is an unbounded subset of δ (otherwise the proof is trivial). Let $T^* = T \cap N$, $S^* = S \cap N$, $A_i^* = A_i \cap N$ (for $i \in N$), and $A^* = A \cap N$. Clearly $A^* = \bigcup_{s \in S^*} g_s(A_\delta^*)$, and $g_s(A_i^*)$ is free over $\bigcup_{\substack{s \neq t \\ t \in S^*}} g_i(A^*_{h(t)}) \Big/ \bigcup_{\substack{t<s \\ t \in S^*}} g_t(A^*_{h(t)}) \cup B$ (by Axiom XVI.) As S, T exemplify $I\,C(\lambda, \delta)$ we can find $f: S^* \to T$ so that for no $s \neq t \in S^*$ does $f(s) \leq f(t) < s$, and as $N \cap \delta$ is unbounded in δ, we can assume $f: S^* \to T^*$. Let $T^+ = \{t \in T^*: \text{for no } s \in S \text{ is } f(s) < t \leq s\} \subseteq T^* - S^*$ and $A_t^+ (t \in T^+)$ be $g_s(A_\delta^*)$ if $t = f(s)$ and $g_t(A_{h(t)})$ otherwise. It is not hard to check that for each $t \in T^+$, A_t^+ is free over $\cup\{A_s^+: t \not\leq s \in T^+\} \Big/ \cup \{A_s^+: s \in T^+, s < t\} \cup B$ and $A_t^+ \Big/ \cup \{A_s^+: s \in T^+, s < t\} \cup B$ is free. Hence $A^* = \bigcup_{t \in T^+} A_t^+$ is free.

Definition 1.7 Let S^λ hold if λ is regular cardinals, and for some $R \subseteq \lambda$, R is stationary but for no $\delta < \lambda$, is $R \cap \delta$ stationary. Let S_κ^λ hold if the above condition holds for some $R \subseteq \{\alpha < \lambda: cf\, \alpha = \kappa\}$.

Lemma 1.8

(1) *If λ is a successor cardinal (or even not Mahlo), then S^λ holds iff S_κ^λ holds for some κ.*
(2) *If S_κ^λ holds then $I\,C(\lambda, \kappa)$ holds provided that* $(\forall \mu < \lambda)\ \mu^{<\kappa} < \lambda$.
(3) *If λ is regular, $S_\lambda^{\lambda^+}$ holds.*

Proof: (1) Because if $R \subset \lambda$ is stationary, then $R = \bigcup_{\kappa < \lambda} R_\kappa$ where $R_\kappa = \{\alpha \in R: cf\, \alpha = \kappa\}$, and as λ is successor, this is a union of $<\lambda$ sets, so at least one R_κ is stationary. As the demands in Definition 1.7 are satisfied by any stationary subset of R, we finish. In general $cf(\delta)$ is a regressive function on R, hence is constant on a stationary subset of R.

(2) Let the tree T consist of all sequences of length $<\kappa$ of ordinals $<\lambda$; ordered by "being an initial segment". For each $\alpha \in R$ (R exemplifying S_κ^λ) let t_α be an increasing sequence of ordinals of length κ whose limit is α, and $S = \{t_\alpha: \alpha \in R\}$. The proof that S, $T \cup S$ exemplify $I\,C(\lambda, \kappa)$ appears in [6].

(3) Let $R = \{\alpha < \lambda^+: cf\, \alpha = \lambda\}$.

Theorem 1.9 *Let $\chi_3 > \lambda > \chi$, $\chi_3 > |B|$, λ is strongly inaccessible, and R exemplify S^λ, and $\alpha \in R$ implies that B satisfy $Q_{n+1}(cf\, \alpha, \mu)$ for some $\mu \leq \alpha$. Then B satisfies $Q_n(\lambda)$.*

Proof: We leave this to the reader.

Remark: For many particular cases, we can demand only "λ is inaccessible" (e.g., Abelian groups).

2 Particular incompactness theorems Each part is a continuation of the corresponding part in [6], Section 2.

Transversals This time we expand U by two one-place relations V, S, and one-place functions $f_i(i < \chi_1)$ so that $x \in V \rightarrow f_i(x) = x$, and $x \in S \Rightarrow x = \{f_i(x): i < \chi_1\}$. Let $\langle A, B, C \rangle \in F$ if $cl(A \cup C) \cap cl(B \cup C) = cl\, C$. Clearly Axioms VIII through XVI hold; more exactly, we can extend (u, V, S) so that they hold (this is needed for Axiom XIV, and is done as in the construction of universal homogeneous models).

Lemma 2.1

(1) *The satisfaction of* $Q_n(\lambda)$ *by* B *does not depend on* B
(2) $Q_0(\lambda)$ *implies* $Q_1(\lambda)$
(3) $Q_1(\delta, \chi_0)$ *holds for* $\delta < \chi^+$, *and* $Q_0(\chi_0^+)$ *holds*
(4) *Theorem 1.8 holds for any inaccessible* λ.

Proof: Left to the reader.

Conclusion 2.2 *If* $\chi_0 = \aleph_\alpha$, *then* $Q_0(\aleph_{\alpha+n})$ *holds. If* $Q_0(\lambda) \wedge S_\lambda^\mu$ *then* $Q_0(\mu)$, *and* $Q_0(\lambda^+)$; *and if* $\lambda > \chi_0$, $S \subset \lambda$ *exemplify* S^λ, *and for each* $\alpha \in S$, $Q_0(cf\, \alpha, cf\, \alpha + \chi_0)$ *holds, then* $Q_0(\lambda)$ *holds.*

Proof: By Lemmas 1.6 and 2.1(2).

Colouring Numbers Let $\langle A, B, C \rangle \in F$ *if no* $a \in A - C$, $b \in B - C$ are connected. Clearly, extending our graph, Axioms VIII–XVI hold (using a 'universal homogeneous graph').

Lemma 2.3

(1) *The satisfaction of* $Q_n(\lambda)$ *by* B *does not depend on* B.
(2) $Q_0(\chi_0^+)$, $Q_1(\chi_0)$ *holds.*
(3) *If* $S_{cf\chi_0}^\lambda$, $\lambda > \chi_0$, *then* $Q_0(\lambda)$.

Proof: Easy.

Free Algebras For a fixed set Γ of identities, by a suitable choice of U, clearly (where $\langle A, B, C \rangle \in F$ iff $cl(A \cup B \cup C)$ is the Γ-free product of $cl(A \cup C)$ and $cl(B \cup C)$ over $cl(C)$, $\chi_0 = \aleph_0 + |\Gamma|$):

Lemma 2.4

(1) *Axioms VIII, IX, X, XII, XV, XVI hold and also XIV*(1).
(2) *If the variety* (= *the class of algebras satisfying* Γ) *has the amalgamation property, then Axioms XI and XIII hold.*
(3) *Axiom I* implies Axiom XIV*(2), *but seemingly not vice versa.*
(4) *If Axioms I*, XI, XIII hold then* $Q_0(\lambda)$ *implies* $Q_1(\lambda)$, *for* $\lambda > |\Gamma| + \aleph_0$ *any B provided that*

(*) *if h is a homorphism from* A_1 *onto* A_0, $B_0 \subseteq A_0$, $B_1 = h^{-1}(B_0)$ *then*: A_1/B_1 *is free iff* A_0/B_0 *is free.*

(5) *For Abelian groups,* $Q_1(\aleph_0)$ *holds, and also* (*) *of* (4).

(6) *When the hypothesis of* (4) *holds, incompactness in* $\lambda(>|P|+\kappa)$ *implies incompactness in* λ^+ (*this is due to Eklof* [2] *for Abelian groups*).

Remark: Mekler [5] showed that $Q_1(\lambda)$ and S^κ_λ implies $Q_1(\kappa)$, $Q_0(\kappa)$ for the variety of groups.

Proof: (1), (2), (3) The reader should be able to check them.

(4) Looking at Definition 1.4, clearly $Q_1(\lambda)$ means:

(*) there are $A_i (i \leq \lambda)$, increasing, continuous for $\delta < \lambda$, $A_0 = \varnothing$ such that A_j/A_i is free for $i \leq j \leq \lambda$ but $A_\lambda \Big/ \bigcup_{j<\lambda} A_j$ is not free; and $|A_\lambda| = \lambda$.

Let B be given and A exemplifies "B satisfies $Q_0(\lambda)$, i.e., A/B is λ-free but not free, $|A| = \lambda$". Let the set of elements of A be $\{a_i: i < \lambda\}$. Let B^* be the algebra generated Γ-freely by $B \cup \{x_i: i < \lambda\}$ with the equations holding in B.

Let h be that following homorphism from B^* onto $cl(A \cup B)$: $h(b) = b$ for $b \in B$ and $h(x_i) = a_i$ for $i < \lambda$ (it is well known that there is a unique such homorphism). Let

$$A_i^0 = \{y : y \in cl(B \cup \{x_\alpha: \alpha < i\}),\ h(y) \in B\}$$
$$A_i^1 = \qquad cl(B \cup \{x_\alpha: \alpha < i\})\ .$$

By [6] there is a closed unbounded subset C of λ, such that for $i \in C$, $\{a_\alpha: \alpha < i\} = cl\{a_\alpha: \alpha < i\}$ is free over B, so by (*) for $i \in C$, A_i^1/A_i^0 is free. By the definition A_i^1/B and A_j^1/A_i^1 are free for $i \leq j \leq \lambda$.

By Axiom III clearly $A_\lambda^1/A_i^0 \cup B$ is free for $i < \lambda$; hence, by Axiom I*, $A_j^0/A_i^0 \cup B$ is free for $i \leq j < \lambda$. Now let A_i be $\varnothing$ for $i = 0$, A_i^0 for $0 < i < \lambda$, A_i^1 for $i = \lambda$. By the above clearly $A_j/A_i \cup B$ is free for $i \leq j \leq \lambda$. (For $i = 0$ use Axiom II and freeness of A_λ^1/B.) Also A_i is increasing, continuous for $\delta < \lambda$, and $A_0 = \varnothing$. Also $A_\lambda \Big/ \bigcup_{i<\lambda} A_i \cup B = A_\lambda^1/A_\lambda^0$ is not free by (*). The last point is that there is $X \subseteq A_\lambda$, $|X| = \lambda$, $A_\lambda \subseteq cl(B \cup X)$, i.e. $X = \{x_i: i < \lambda\}$.

(5) For Abelian groups: Let G_ω be generated freely by $B \cup \{x_i: i \leq \omega\}$ and $\left(x_\omega - \sum_{l=0}^{n} p^l x_l\right) \Big/ p^{n+1}$, and let G_n be the subgroup generated by $B \cup \{x_i: i < n\}$ (where p is any prime natural number).

The proof of (*) is just the classical theory of kernels, normal subgroups, and homorphisms.

Conclusion 2.5 *If* (*) *of* 2.4(4) *holds B satisfies* $Q_0(\lambda)$ *and if* $I\ C(\mu, \lambda)$ *holds then B satisfies* $Q_0(\mu)$.

Proof: By Lemma 2.4(4), and Theorem 1.6.

Theorem 2.6 *The following properties of the pair of regular cardinals* $\lambda > \kappa$ *listed below satisfy*:

$$(A) \Leftrightarrow (B) \Rightarrow (C) \Leftrightarrow (D) \text{ and if } (*) \text{ then } (B) \Leftrightarrow (D)$$

where: (*) *for every* $\mu < \lambda$ *and* $\chi < \kappa$, *the inequality* $\mu^\chi < \lambda$ *holds.*

The properties are:

(A) $I\,C(\lambda, \kappa)$

(B) *One of the following holds*:

(α) *there is a (partial) ordering of* λ, *such that* $(\lambda, <^*)$ *is a tree of height* $\kappa + 1$, $<^* \subseteq <$, $W \subseteq \lambda$ *is stationary, and* $\alpha \in W$ *implies* $\alpha = \bigcup \{\beta: \beta <^* \alpha\}$ *and for every* $\alpha < \lambda$, $W \cap \lambda$ *has a pressing down function.*

(β) *There is an order* $<^*$ *of* $\lambda = \mu^+$, μ *singular*, $cf\ \mu = \kappa$, *such that* $(\lambda, <^*)$ *is a tree of height* $\kappa + 1$, $<^* \subseteq <$, $\mu \leq \alpha < \mu^+$ *implies the height of* α *is* κ, *and for every* $\alpha < \lambda$, $\alpha \cap (\mu, \mu^+)$ *has a pressing-down function and* $(\forall \delta \in W)$ $[cf.\ \delta = \kappa]$.

(C) *There is a graph G with* λ *nodes, which has colouring number* $>\kappa$, *but every* $G' \subseteq G$, $|G'| < \lambda$, *has colouring number* $\leq \kappa$ *(see, e.g.,* [6], *Definition 3.2, p. 336).*

(D) *One of the following holds*:

(α) *there is a stationary set* $W \subseteq \{\alpha < \lambda: cf\ \alpha = \kappa\}$, *and sets* $S_\alpha \subseteq \alpha$, $\bigcup S_\alpha = \alpha$, *of order type* κ *for each* $\alpha \in W$, *such that* $(\forall \beta)$ $[\beta \in S_\alpha$ *implies* β *odd*$]$, *and for each* $\gamma < \lambda$ *the graph* $\{(\beta, \alpha): \beta \in S_\alpha, \alpha \in W, \alpha < \gamma\}$ *(this is the set of edges) has colouring number* $\leq\kappa$;

(β) $\lambda = \mu^+$; $cf\ \mu = \kappa$, *and there are sets* $S_\alpha \subseteq \mu$ $(\alpha < \lambda)$ *of order-type* κ *such that for each* $\gamma < \lambda$ *the graph* $G_\gamma = \{(\beta, \alpha): \beta < \mu < \alpha < \gamma, \beta \in S_\alpha\}$ *has colouring number* $\leq\kappa$.

Proof:

$(A) \Rightarrow (B)$: Let the tree T and $S \subseteq T$ exemplify $IC(\lambda, \kappa)$ (see Definition 1.2). W.l.o.g. $(\forall x \in T)(\exists y \in S)\ x \leq y$ hence $|T| = |S| + \kappa = \lambda$.

Let $T = \{a_i: i < \lambda\}$ and w.l.o.g. $T \models a_i < a_j$ implies $i < j$. Let $R = \{i < \lambda: L_i \neq \emptyset\}$ where $L_i = \{j: j \geq i, (\forall x)[T \models x < a_j \Rightarrow x \in \{a_\alpha: \alpha < i\}]\}$.

Case i. For some $i < \lambda$, $|L_i| = \lambda$.

First note that $\lambda = |i|^+$—otherwise there is $L \subseteq L_i$, $|i| < |L| < \lambda$, so there is a pressing down function f on $\{a_j: j \in L\}$, and $\{\{x: f(a_j) < x < a_j\}: j \in L\}$ is a family of $|L|$ pairwise disjoint nonempty subsets of $\{a_\alpha: \alpha < i\}$. So $|i| \geq \kappa$, hence w.l.o.g. i a cardinal, which we call μ. Now replacing S by $\{a_i: \mu \leq i < \mu^+ = \lambda\}$ we can easily get (β) of (B) by identifying i and a_i.

Case ii. Not (i) but R is stationary.

W.l.o.g. $i \in R \Rightarrow i \in L_i$. As not case (i) $C = \{\delta < \lambda: \delta$ limit and $i < \delta \Rightarrow Sup\ L_i < \delta\}$ is a closed unbounded subset of λ. If $i \in R \cap C$, choose $\gamma \in L_i$, then $\{\alpha: T \models a_\alpha < a_\gamma\}$ is necessarily a subset of i (as $\gamma \in L_i$) and is unbounded below i (by C's definition) hence $cf\ i = \kappa$. Now we can easily get (α) of (B).

Case iii. Not (i) and not (ii).

We can show that S has a pressing down function (let $C \subseteq \lambda$ be closed unbounded disjoint from R, $C \cup \{0\} = \{\alpha_i: i < \lambda\}$, α_i increasing and define $f \restriction [\alpha_i, \alpha_{i+1}]$ for each i). $(B) \Rightarrow (A)$ is left to the reader.

$(A) \Rightarrow (C)$: Let $T = (T, <)$, S exemplify $IC(\lambda, \kappa)$, and w.l.o.g. $|T| = \lambda$. We define a graph G: its set of vertices is T, and $E(G) = \{(b, c): T \models b < c\}$.

Fact *G has colouring number* $>\kappa$.

If not there is a well ordering $<^*$ of T such that $|\{b: b <^* c, (b, c) \in$

$E(G)\}| < \kappa$ for every c. As κ is regular, for some b_a, $T \models b_a < a$ and $(\forall c)(T \models b_a \leq c < a \Rightarrow a <^* c)$. Now for every b $|\{a \in S: b_a = b\}| < \kappa$ (otherwise $|\{c: (c, b) \in E(G), c <^* b\} \supseteq \{a \in S: b_a = b\}$ has power $\geq\kappa$).

We define on S a graph: a_1, a_2 are connected if $\{c: b_{a_1} \leq c < a_1\} \cap \{c: b_{a_2} \leq c < a_2\} \neq \varnothing$, and $\{S_t: t \in I\}$ are the components. Clearly in this graph each node has valency $\leq\kappa$, hence $|S_t| \leq \kappa$. Let $S_t = \{a_i^t: i < i_t \leq \kappa\}$ and we define a pressing down f on S_t, by defining $f(a_i^t)$ by induction on i, such that $b_{a_i^t} <_T f(a_i^t) <_T a_i^t$, $\{c: f(a_j^t) \leq c < a_j^t\}$ for $j \leq i$ are pairwise disjoint. This f contradicts the choice of T, S.

Fact *For each $L \subseteq T$, $|L| < \lambda$, $G \restriction L$ has colouring number $\leq\kappa$.*

W.l.o.g. $x < y \wedge y \in L \Rightarrow x \in L$ (true as $\kappa < \lambda$).

We know that there is a pressing down function f: $L \cap S \to T$ as in Definition 1.2(3). Let for $a \in L \cap S$, $K_a = \{c \in T: f(a) \leq c \leq a\}$, for $i < \kappa$, $K_i = \{b \in T$: b is of level $i\} - \bigcup\{K_a: a \in L \cap S\}$. We define $<^*$ on L such that:

for $i < j < \kappa$, $(\forall x \in K_i)(\forall y \in K_j)(x <^* y)$,
for $i < \kappa$, $(\forall x \in K_i)(\forall y \in \bigcup K_a)(x <^* y)$,
for $a \in L \cap S$, $(K_a, <^*)$ has order type κ.

This is possible and is enough.

$(C) \Leftrightarrow (D)$: Similar to the proof of $(A) \Leftrightarrow (B)$.

$(D) \Rightarrow (B)$ *if* (*) *holds:* We can prove that (D) $(\alpha) \Rightarrow (B)$ (α), and (D) $(\beta) \Rightarrow (B)$ (β). (For the latter we can weaken (*).) As the proofs are similar, let us prove the first. Let $T = \{\eta$: η a sequence of ordinal $<\lambda$ of length $<\kappa\} \cup \{\eta_\delta$: $\delta \in W\}$ where η_δ is a sequence of length κ enumerating S_δ. The order is an initial segment and $S = \{\eta_\delta: \eta \in W\}$.

By (*) for $\alpha < \lambda$, $T \cap ({}^{\kappa\geq}\alpha)$ has power $<\lambda$. The rest is easy.

***3 Canonical counterexamples for** $PT(\lambda, \kappa^+)$* It is clear that the $IC(\lambda, \omega)$ (and the related notions, see Theorem 2.6) are in a sense a degenerate case, e.g. (see [1]), it is consistent for them that "$\aleph_2$-free implies free". $PT(\lambda, \aleph_1)$ seems more complicated and may be a representative case of a class of problems. We analyze a possible counterexample of $PT(\lambda, \aleph_1)$ and get a kind of n-dimensional $IC(\lambda, \aleph_0)$ example. We can fix the n and get intermediate notions. If we agree in 3.8 to weaken $(B)_{\rho,i}$ by replacing condition (i) by $(i)'$ "for $\eta \neq v \in S_f$, u_η, u_v, are equal or disjoint and $\{v: u_\eta = u_v\}$ is countable for every η" the proof becomes much shorter, but does not seem sufficient to construct a λ- free non-λ^+-free Abelian group for example.

Our main tools are λ-sets which are in a sense $(<\aleph_0)$-dimensional stationary sets. This analysis makes explicit the feeling that there is an intimate connection between λ-free non-λ^+-free A (for transversals and specific sets). What about $PT(\lambda, \kappa^+)$ $\kappa > \aleph_0$? By 3.6 we can get a canonical counterexample but cannot prove 3.8 and the parallel to 3.7 is problematic. Even if we assume G.C.H., in the case $\lambda(\eta, S) = \mu^+$, μ singular of cofinality $<\kappa$, we cannot get a tree. We can get reasonable canonical form if in the definition of free for $PT(\lambda, \kappa)$ we replace "having a one-to-one choice function" by "having a κ-to-one choice function" (which has the same spectrum of incompactness).

With G.C.H. (and/or no Mahlo cardinals) we can demand more on the canonical counterexamples.

Definition 3.1

(1) For a regular uncountable cardinal $\lambda(>\aleph_0)$ we call S a λ-set if:
  (a) S is a set of strictly decreasing sequences of ordinals $<\lambda$.
  (b) S is closed under initial segments and is nonempty.
  (c) For $\eta \in S$ if $W(\eta, S) =_{df} \{i: \eta \frown \langle i \rangle \in S\}$ is nonempty *then* it is a stationary subset of $\lambda(\eta, S) =_{df} Sup\ W(\eta, S)$ and $\lambda(\eta, S)$ is a regular uncountable cardinal. Also $\lambda(\langle\,\rangle, S) = \lambda$.

(2) For a λ-set S, let S_f (= set of final elements of S) be $\{\eta \in S: (\forall i)\ \eta \frown \langle i \rangle \notin S\}$ and S_i (= set of initial elements of S) be $S - S_f$ so ($S_f = \{\eta \in S: \lambda(\eta, S) = 0\}$).

We sometimes allow $\lambda = 0$. Then the only λ-set is $\{\langle\,\rangle\}$.

(3) For λ-sets S^1, S^2 we say $S^1 \leq S^2$ (S^1 a sub-λ-set of S^2) if $S^1 \subseteq S^2$ and $\lambda(\eta, S^1) = \lambda(\eta, S^2)$ for every $\eta \in S^1$ (so $S_i^1 = S^1 \cap S_i^2$). Clearly $\leq$ is transitive.

Notation: In this section S will be used to denote λ-sets.

Claim 3.2

(1) *If S is a λ-set, $\lambda(\eta, S) > \kappa$ for every $\eta \in S_i$ (holds always for $\kappa = \aleph_0$) and G is a function from S_f to κ then for some $S^1 \leq S$ G is constant on S_f^1.*

(2) *If S is a λ-set, $\eta \in S_i$, then $S^{[\eta]} = \{v: \eta \frown v \in S\}$ is a $\lambda(\eta, S)$-set, and $\lambda(v, S^{[\eta]}) = \lambda(\eta \frown v, S)$.*

(3) *If S is a λ-set, κ a regular cardinal $(\forall \eta \in S)$ $(\lambda(\eta, S) \neq \kappa)$ and G is a function from S to κ then for some $S^1 \leq S$ and $\gamma < \kappa$ for every $\eta \in S^1$, $G(\eta) < \gamma$.*

(4) *If $\lambda > \aleph_0$ is regular, $W \subseteq \lambda$ stationary, for $\delta \in W$ S^δ is a λ_δ-set for some $\lambda_\delta \leq \delta$ or $S^\delta = \{\langle\,\rangle\}$ then $S = \{\langle\,\rangle\} \cup \{\langle\delta\rangle \frown \eta: \eta \in S^\delta\ \delta \in W\}$, is a λ-set, $\lambda(\langle\delta\rangle \frown \eta, S) = \lambda(\eta, S^\delta)$.*

(5) *If S is a λ-set, F a function with domain $S - \{\langle\,\rangle\}$, $F(\eta \frown \langle\alpha\rangle) < 1 + \alpha$ then F is essentially constant for some $S^1 \leq S$ which means $F \restriction \{\eta \in s': l(\eta) = m\}$ is constant for each m.*

(6) *For any λ-set S there is a λ-set $S^1 \leq S$ such that*
  (a) *all $\eta \in S_f$ has the same length k*
  (b) *for each $l < k$ either*
    (i) *every $\eta(l)$ $(\eta \in S_f)$ is an inaccessible cardinal, or*
    (ii) *every $\eta(l)$ $(\eta \in S_f)$ is a singular limit ordinal,*
  (c) *for each $l < k$, either*
    (i) *$\lambda(\eta \restriction (l+1), S) = \eta(l)$ for every $\eta \in S_f$ or*
    (ii) *$\lambda(\eta \restriction (l+1), S) = \lambda_S^{l+1}$ for every $\eta \in S_f$ (for a fixed λ_S^{l+1}).*
  (d) *The truth value of "$cf\ \eta(l) = \lambda(\eta \restriction m, S)$" is the same for all $\eta \in S_f$ (for constant l, m).*

Proof: (2), (4) Easy.

(1) By induction on λ: for each $\langle\alpha\rangle \in S$ there is (by the induction hypothesis) $S^\alpha \leq S^{[\langle\alpha\rangle]}$ such that $G \restriction S_f^\alpha$ is constant and let its value be $\gamma(\alpha)$. As $W(\langle\,\rangle, S)$ is a stationary subset of $\lambda = \lambda(\langle\,\rangle, S)$ and by a hypothesis $\lambda > \kappa$, there is $\gamma^* < \alpha$ such that $W = \{\alpha \in W(\langle\,\rangle, S): \gamma(\alpha) = \gamma^*\}$ is a stationary subset of λ. Now

$$S^1 = \{\langle\,\rangle\} \cup \{\langle\alpha\rangle \frown v: \alpha \in W,\ v \in S^\alpha\}$$

is as required.

(3), (5) A similar proof.

(6) Use (1), (5).

Claim 3.3 *Suppose P is a family of sets which exemplify the failure of $PT(\lambda, \kappa^+)$ (where $\lambda > \kappa$). Then there is a set λ-set S and function F with domain S_f such that*

(a) *For each $\eta \in S_f$, $F(\eta)$ is a subfamily of P of power $\leq\kappa$.*

(b) *For $\eta \in S_i$, $\lambda(\eta, S) > \kappa$.*

(c) *Let for $\eta \in {}^{\omega>}(\lambda+1)$, $F^0(\eta) = \cup\ \{F(\tau): \tau <_{lx} \eta,\ \tau \in S_f\}$, where $<_{lx}$ is the lexicographic order, $F^1(\eta) = \cup\{F^0(\tau):\ \eta \leq \tau \in S_f\}$ and $F^2(\eta) = \cup\{A: A \in F^0(\eta \frown \langle\lambda\rangle)\} - \cup\ \{A: A \in F^0(\eta)\}$.*

Note that for $\eta \in S$, $F^0(\eta \frown \langle\lambda\rangle) = F^0(\eta) \cup F^1(\eta)$.

(d) *For $\eta \in S_f$, $F^1(\eta)/F^0(\eta)$ is not free.*

For $\eta \in S_i$, $F^1(\eta)/F^0(\eta)$ is $\lambda(\eta, S)$-free not free (see the Introduction) and $|F^1(\eta)| = \lambda(\eta, S)$ (this follows as $|\{\tau:\ \eta \leq \tau \in S\}|) = \lambda(\eta, S)$).

(e) *If $\eta \frown \langle\alpha\rangle \in S$ then α is a limit ordinal, $cf\,\alpha \leq \lambda\ (\eta \frown \langle\alpha\rangle, S) + \kappa \leq |\alpha|$ and if $\beta < \lambda(\eta, S)$ is an inaccessible cardinal $(>\aleph_0)$ then $\beta \cap W(\eta, S)$, is not a stationary subset of β.*

(f) *If $\eta \frown \langle\alpha\rangle < v \in S_f$, $cf\,\alpha > \kappa$ then for some k $\eta \frown \langle\alpha\rangle \leq v \restriction k$ and $\lambda(v \restriction k, S) = cf\,\alpha$.*

Proof: This is proved by induction on λ for a somewhat wider context: P/Q is λ-free not free, $|P - Q| = \lambda > \kappa$ and the only change in (a)–(f) is that Q is included in $F_0(\eta)$. As $\lambda > \kappa$, λ is uncountable and λ is regular by the main theorem of [6]. Let $P = \bigcup_{\alpha<\lambda} P_\alpha$, P_α increasing, continuous and $|P_\alpha| < \lambda$. We know that $W = \{\alpha < \lambda: P/P_\alpha \cup Q$ is not λ-free$\}$ is stationary (otherwise P/Q is free, a contradiction). If $W_0 = \{\mu < \lambda: \mu$ an inaccessible cardinal, $W \cap \mu$ is stationary$\}$ is a stationary subset of λ, then for some $\mu \in W_0$ P_μ/Q is not free, a contradiction. So by renaming the P_α's, w.l.o.g. $W_0 = \varnothing$. W.l.o.g. for $\alpha \in W$; $P_{\alpha+1}/P_\alpha \cup Q$ is not free, and $P/P_\alpha \cup Q$ is $|P_{\alpha+1} - P_\alpha|$-free. Now $|P_{\alpha+1}| \leq |P_\alpha| + \kappa$; otherwise $|P_\alpha| + \kappa < |P_{\alpha+1}| < \lambda$ so by [6], 1.3 for some P', $P_\alpha \subseteq P' \subset P$, $P/P' \cup Q$ is λ-free, $|P'| \leq |P_\alpha| + \kappa$, hence P/P_α is λ-free, a contradiction. Hence w.l.o.g. for some closed unbounded set C of $\alpha < \lambda$, $|P_{\alpha+1}| \leq |P_\alpha| \leq |\alpha|$. If $W_1 = \{\alpha \in W: P_{\alpha+1} - P_\alpha$ has power $\leq\kappa\}$ is stationary we let $S = \{\langle\,\rangle\} \cup \{\langle\alpha\rangle;\ \alpha \in W_1\}$, $F(\langle\alpha\rangle) = P_{\alpha+1} - P_\alpha$ for $\alpha \in W_1$, and it is easy to see that we have got (a)–(d). We shall prove later that $W_1^* =_{df} \{\delta \in W_1: cf\,\delta > \kappa\}$ is not stationary, thus finishing. Then we shall use $t = t_\delta = \phi$, $S_f^\delta = \{\langle\,\rangle\}\ f^\delta(\langle\,\rangle) = P_{\alpha+1} - P_\alpha$ for $\delta \in W_1^*$.

Let $\lambda^\alpha = |P_{\alpha+1} - P_\alpha|$, so λ^α is a regular cardinal or is $\leq\kappa$. If W_1 is not stationary $W_2 = \{\alpha \in W \cap C: \alpha \notin W_1,\ \alpha$ a limit ordinal$\}$ is stationary. Apply the induction hypothesis with λ^α, $P_{\alpha+1} - P_\alpha$, $P_\alpha \cup Q$ standing for λ, P, Q and get S^α, F^α. If we then let:

$$S = \{\langle\,\rangle\} \cup \{\langle\alpha\rangle \frown \eta: \alpha \in W_2,\ \eta \in S^\alpha \text{ and } \alpha > \kappa\}$$
$$F(\langle\alpha\rangle \frown \eta) = F^\alpha(\eta) \text{ for } \eta \in S_f^\alpha$$

it is easy to check that S, F satisfies (a)-(d).
Now for each $\delta \in W_2$ we apply Claim 3.2(1) so that for some $S^{\delta,1} \leq S^\delta$, for all $\eta \in S_f^{\delta,1}$ $\{\langle i, l\rangle: l \leq l(\eta), i < 2$ and $[\lambda(\eta \restriction l, S^\delta) = cf\,\delta$ iff $i = 0]\} \cup \{\langle i, l(\eta)\rangle: i < 2, [i = 0$ iff $cf\,\delta \leq \kappa]\}$ is the same t_δ (notice that there are $\leq \aleph_0$ possible t's). Also for some t $W_3 = \{\delta \in W_2: t_\delta = t\}$ is stationary.

If $(\exists l)[\langle 0, l\rangle \in t]$ necessarily $\lambda^\delta \geq cf\,\delta$ for $\delta \in W_3$ and so

$$S^1 = \{\langle\,\rangle\} \cup \{\langle\delta\rangle \frown \eta: \eta \in S^{\delta,1}, \delta \in W_3\}$$
$$F^1 = F \restriction S^1$$

satisfy all the requirements of Claim 3.3. So suppose $(\forall l)[\langle 0, l\rangle \notin t]$. Now if W_1^* is stationary we will let $S^{\delta,1} = \{\langle\,\rangle\}$, $W_4 = W_1^*$; otherwise let $W_4 = W_3$, $S^{\delta,1}$, S, F as above. Clearly for every $\delta \in W_4$, $\eta \in S_f^{\delta,1}$ the set $\bigcup F^\delta(\eta)$ has power $\leq\kappa$, but $cf\,\delta > \kappa$ (as $\langle 0, l(\eta)\rangle \notin t_\delta = t$). Hence, letting $\delta = \bigcup\{j(\gamma, \delta): \gamma < cf\,\delta\}(j(\delta,\gamma) < \delta)$ for some $\gamma_\delta(\eta) < cf\,\delta$ $[\bigcup F^\delta(\eta)] \cap \bigcup P_\delta \subseteq \bigcup P_{j(\gamma_\delta(\eta),\delta)}$. By Claim 3.2(3) for some $S^{\delta,2} \leq S^{\delta,1}$, and some $\gamma_\delta < cf\,\delta$, $(\forall \eta \in S_f^{\delta,2})$ $\gamma_\delta(\eta) < \gamma_\delta$. By Fodor's lemma for some $\gamma^* < \lambda$

$$W_5 = \{\delta \in W_4: j(\gamma_\delta, \delta) = \gamma^*\}$$

is stationary.

Let g be a one-to-one function from W_5 onto $\{\delta < \lambda: cf\,\delta = \aleph_0\}$, and define $P_\alpha^* = P_\gamma \cup \bigcup\{P_{\delta+1}: \delta \in W_5, g(\delta) < \alpha\}$. We could have used the P_α^*'s instead of the P_α to get the result.

Definition 3.4

(1) A λ-system is $\langle B_\eta: \eta \in S_c\rangle$ where:

(a) S is a λ-set, and we let $S_c =_{df} \{\eta \frown \langle i\rangle: \eta \in S_i, i < \lambda(\eta, S)\}$

(b) $B_{\eta\frown\langle i\rangle} \subseteq B_{\eta\frown\langle j\rangle}$ when $\eta \in S_i$, $i < j$ are $<\lambda(\eta, S)$

(c) If δ is a limit ordinal $<\lambda(\eta, S)$ then $B_{\eta\frown\langle\delta\rangle} = \bigcup\{B_{\eta\frown\langle i\rangle}: i < \delta\}$

(d) $|B_{\eta\frown\langle i\rangle}| < \lambda(\eta, S)$ for $i < \lambda(\eta, \delta)$.

(2) The λ-system $\langle B_\eta: \eta \in S_c\rangle$ is called disjoint if the sets $\{B_{\eta\frown\langle\lambda(\eta,S)\rangle}: \eta \in S_i\}$ (see (3) below) are pairwise disjoint.

(3) We let $S_m = S - \{\langle\,\rangle\}$, $B_{\eta\frown\langle\lambda(\eta,S)\rangle} =_{df} B_\eta^* =_{df} \bigcup\{B_{\eta\frown\langle i\rangle}: i < \lambda(\eta, S)\}$ for $\eta \in S_i$.

Claim 3.5 *Suppose λ is a regular uncountable cardinal, $\langle B_\eta: \eta \in S_c\rangle$ a λ-system, and for $\eta \in S_f$ $s_\eta \subseteq \bigcup_{l<l(\eta)} B_{\eta\restriction(l+1)}$. Then $\{s_\eta: \eta \in S_f\}$ has no transversal.*

Proof: Suppose g is a one-to-one function Dom $g = S_f$ and $g(\eta) \in s_\eta$. We prove by induction on $\eta \in S$ that $(\exists v \in S_f)\left(\eta \preccurlyeq v \wedge g(v) \in \bigcup_{l<l(\eta)} B_{\eta\restriction(l+1)}\right)$ (the induction means: prove for η if we know it for every η', $\eta \preccurlyeq \eta' \in S$). In the induction step we use Fodor's Lemma.

Claim 3.6 *Suppose $PT(\lambda, \kappa^+)$ fail. Then there is a disjoint λ-system $\langle B_\eta: \eta \in S_c\rangle$ and sets $s_\eta^l(\eta \in S_f, l < l(\eta))$, and $C_\delta(\delta < \lambda$ limit$)$ such that*

(a) *S satisfies the conclusion of Claims* 3.2(6), 3.3(*e*), *and* 3.3(*f*).

(b) $s_\eta^l \subseteq B_{\eta\restriction(l+1)}$, $0 < |s_\eta^l| \leq \kappa$.

(c) *For every $I \subseteq S_f$, $|I| < \lambda$, $\left\{\bigcup_l s_\eta^l: \eta \in I\right\}$ has a transversal. Moreover, for* *every* $\rho \in S_i$, $I \subseteq \{v: \rho \prec v \in S_f\}$, $|I| < \lambda(\rho, S)$ *the family* $\left\{ \bigcup_{l \geq l(\rho)} s^l_\eta: \eta \in I \right\}$ *has a transversal.*

(d) *If* $s^l_\eta \cap s^m_v \neq \emptyset$ *then* $l = m$, $\rho =_{df} \eta \upharpoonright l = v \upharpoonright l$ *and* $\eta \upharpoonright [l+1, k) = v \upharpoonright [l+1, k)$ *where* $k = l(\eta)$ *and* $\lambda(\eta \upharpoonright i, S) = \lambda(v \upharpoonright i, S)$ *when* $l + 1 < i < k$ *and either* $\lambda(\eta \upharpoonright l+1, S) = \eta(l)$, $\lambda(v \upharpoonright l+1, S) = v(l)$ *are both inaccessible cardinals or* $\lambda(\eta \upharpoonright (l+1), S) = \lambda(v \upharpoonright (l+1), S)$.
(e) $C_{\eta^\frown\langle\delta\rangle}$ *is a closed unbounded subset of* δ, $C_{\eta^\frown\langle\delta\rangle} = \{\zeta(\eta, \delta, i): i < cf\, \delta\}$, $\zeta(\eta, \delta, i)$ *increasing with* i *and if* δ *is an inaccessible cardinal then* $\emptyset = C_{\eta^\frown\langle\delta\rangle} \cap W(\eta, S)$.
(f) *If* $l < m < k = l(\eta)$, $\eta \in S_f$, $cf[\eta(l)] = \lambda(\eta \upharpoonright m, S)$ *then* $s^l_\eta \subseteq B_{\eta \upharpoonright (l+1)} - B_{\eta \upharpoonright l^\frown\langle\zeta\rangle}$ *where* $\zeta = \zeta(\eta \upharpoonright l, \eta(l), \eta(m))$; *i.e.*, ζ *is the* $\eta(m)$*'s member of* $C_{\eta \upharpoonright (l+1)}$. *Moreover if* $s^l_\eta \cap s^l_v \neq \emptyset$, $\eta \neq v$ *then* $\zeta(\eta \upharpoonright l, \eta(l), \eta(m)) = \zeta(v \upharpoonright l, v(l), v(m))$.
(g) *If* $l < l(\eta)$ $\eta \in S_f$, $cf[\eta(l)] \leq \kappa$ *then for no* $\zeta < \eta(l)$ *is* $s^l_\eta \subseteq B_{\eta \upharpoonright l^\frown\langle\zeta\rangle}$.
(h) *For some well ordering* $<^*_\eta$ *of* B^*_η $(\eta \in S_i)$ *if* $\eta \,^\frown \langle i\rangle \leq v \in S_f$, *then* $[cf\, i \geq \kappa \Rightarrow s^{l(\eta)}_v$ *has order type* $\kappa]$ *and* $[cf\, i < \kappa \Rightarrow s^{l(\eta)}_v$ *has order type* $\kappa \times (cf\, |s^{l(\eta)}_f|)]$. (*This is not really used.*)

Proof: Straightforward and in the most important case see 3.7's proof.

Claim 3.7 *Suppose in Claim* 3.6 *that* $\kappa = \aleph_0$. *Then we can add*

(i) *for* $\eta \in S_i$, B^*_η *has the structure of a tree with* ω *levels* (*e.g., is a family of finite sequences, closed under initial segments except that* $\langle\,\rangle \notin B^*_\eta$), *and* $\eta \prec v \in S_f$ *implies* $s^{l(\eta)}_v$ *is a branch* (*of order type* $\leq\omega$) (*a branch is a maximal linearly ordered subset*), *and for* $m < l$, *and* $k < \omega$, *the k'th element of* s^m_v, *together with* $v \upharpoonright l$ *determine the k-th element of* s^l_v.

Remark: In the proof we get that each s^l_v has order type ω.

Proof: Let, for $\eta \in S_c$, $C_\eta =$ the family of nonempty finite sequences from B_η. We assume w.l.o.g. that for $\eta \neq v \in S_i$, $C_{\eta^\frown\langle i\rangle} \cap C_{v^\frown\langle j\rangle} = \emptyset$ for $i < \lambda(\eta, S)$, $j < \lambda(v, S)$. It is clear that $\langle C_\eta: \eta \in S_c\rangle$ is a disjoint λ-system ($|C_{\eta^\frown\langle i\rangle}| < \lambda(\eta, S)$ as $\lambda(\eta, S)$ is uncountable). Let $s^l_\eta = \{a(\eta, l, i): i < \omega\}$, and let $t^l_\eta = \{\langle a(\eta, l, i): i < m\rangle: 0 < m < \omega\}$.

Now $\langle C_\eta: \eta \in S_c\rangle$, $t^l_\eta(\eta \in S_f, l < l(\eta))$ are as required in 3.6 (with C_η, t^l_η replacing B_η, s^l_η respectively). The least trivial is (c). Suppose $I \subseteq S_f$, $|I| < \lambda$, so $\left\{\bigcup_l s^l_\eta: \eta \in I\right\}$ has a transversal, so there is a one-to-one g, Dom $g = I$, $g(\eta) \in \bigcup_l s^l_\eta$. Let $g(\eta) = a(\eta, h(\eta), f(\eta))$. Now we define a function g^*: Dom $g^* = I$, $g^*(\eta) = \langle a(\eta, h(\eta), i): 0 \leq i \leq f(\eta)\rangle$. Clearly g^* is one-to-one, $g^*(\eta) \in \bigcup_l t^l_\eta$.

For (h) use lexicographic order. It is also obvious that (i) holds, except possibly the last phrase; but the correction needed is small so we finish.

Claim 3.8 *Suppose* $\langle B_\eta: \eta \in S_c\rangle$, $s^l_\eta(\eta \in S_f, l < l(\eta))$ *are as in Claims* 3.6, 3.7; *we can omit* 3.6(*h*)).

Then for any $\rho \in S_i$, $m = l(\rho)$, *and* $I \subseteq \{\eta \in S_f: \rho \preccurlyeq \eta\}$ *the following are equivalent*:

$(\mathrm{A})_{\rho,I}$. *The family* $\left\{\bigcup_{l\geq m} s^l_\eta \colon \eta \in I\right\}$ *has a transversal.*

$(\mathrm{B})_{\rho,I}$. *There are a well ordering* $<^*$ *of* I *and* $\{u_\eta \colon \eta \in I\}$ *such that*

(i) *for* $\eta <^* \nu$ *(both in* I*),* $u_\nu \cap \left(\bigcup_{l\leq m} s^l_\eta\right) = \varnothing$.

(ii) *For every* $\eta \in I$ *for some* l, $m \leq l < l(\eta)$, u_η *is an end-segment of* s^l_η.

(iii) *If* $\xi < Min\{\eta(m) \colon \eta \in I\}$ *is given, we can demand that each* $u_\eta (\eta \in I)$ *is disjoint to* $B_{\rho^\frown\langle\xi\rangle}$.

$(\mathrm{C})_{\rho,I}$. *There is no* $\lambda(\rho, S)$*-set* S^* *such that* $\eta \in S^*_f \Rightarrow \rho ^\frown \eta \in I$.

$(\mathrm{D})_{\rho,I}$. *Suppose* $\xi < Min\{\eta(m) \colon \eta \in I\}$, *there are* $u_\eta(\eta \in I)$ *where*

(i) *the* u_η *are pairwise disjoint*

(ii) u_η *is an end segment of some* s^l_η $m \leq l < l(\eta)$

(iii) u_η *is disjoint to* $B_{\rho^\frown\langle\xi\rangle}$.

We first prove

Claim 3.9 *For every* λ*-set* S *and* $\rho \in S_i$, $\underline{R}_\rho =_{df} \{I \subseteq S_f$*: for every* $\eta \in I$, $\rho \lessdot \eta$, *and* $(\mathrm{C})_{\rho,I}$ *holds*$\}$ *is an* $\aleph_1$*-complete ideal.*

Proof: Trivially $I \subseteq J$, $J \in \underline{R}_\rho$ implies $I \in \underline{R}_\rho$. Suppose $I_n \in \underline{R}_\rho$ for $n < \omega$ but $I = \bigcup_{n<\omega} I_n \notin \underline{R}_\rho$, and let S^* exemplify $I \notin \underline{R}_\rho$; i.e., let it exemplify the failure of $(\mathrm{C})_{\rho,I}$. Define $g\colon S^*_f \to \omega$ by $g(\eta) = Min\{n \colon \eta \in I_n\}$. (By the choice of S', g is well defined.) By 3.2(1) g is constant on some λ-set $S^{**} \leq S^*$, contradicting $I_n \in \underline{R}_\rho$.

Proof of 3.8: The proof is by downward induction on $l(\rho)$. Arriving at ρ (letting l, I be fixed), first note that by Claim 3.5, $\neg(C)_{\rho,I} \Rightarrow \neg(A)_{\rho,I}$ hence $(A)_{\rho,I} \Rightarrow (C)_{\rho,I}$. Also $(B)_{\rho,I} \Rightarrow (A)_{\rho,I}$ is clear: each u_η is not empty, (by $(B)_{\rho,I}$ (ii)) let $g(\eta) \in u_\eta$ (for $\eta \in I$), then by $(B)_{\rho,I}$ (i) g is one-to-one, thus finishing. We shall prove $(C)_{\rho,I} \Rightarrow (D)_{\rho,I}$ and $(D)_{\rho,I} \Rightarrow (B)_{\rho,I}$.

PART α: $(C)_{\rho,I} \Rightarrow (D)_{\rho,I}$. Let $I_\delta = \{\eta \in I \colon \rho ^\frown \langle\delta\rangle \lessdot \eta\}$ for $\delta < \lambda(\rho, S)$.

Fact 3.8A *It suffices to prove that for some* $J \subseteq I$ *the following holds*:

(i) *for every* $\delta < \lambda(\rho, S)$, $(C)_{\rho^\frown\langle\delta\rangle, I_\delta - J}$ *holds.*

(ii) $(D)_{\rho,J}$.

Proof: Let $\langle u^*_\eta \colon \eta \in J\rangle$ exemplify $(D)_{\rho,J}$ and let $J_1 = \{\eta \in J \colon u^*_\eta \subseteq B^*_\rho\}$ so for $\eta \in (J \cap I_\delta - J_1)$ u^*_η is an end-segment of some s^l_η, $m < l < l(\eta)$. Clearly $\langle u^*_\eta \colon \eta \in J \cap I_\delta - J_1\rangle$ exemplify $(D)_{\rho^\frown\langle\delta\rangle, J\cap I_\delta - J_1}$ holds. By the induction hypothesis this implies $(C)_{\rho^\frown\langle\delta\rangle, J\cap I_\delta - J_1}$. But by 3.8 A(i) $(C)_{\rho^\frown\langle\delta\rangle, I_\delta - J}$, holds, so by 3.9 we conclude that $(C)_{\rho^\frown\langle\delta\rangle, I_\delta - J_1}$ holds, hence $(D)_{\rho^\frown\langle\delta\rangle, I_\delta - J_1}$ holds, hence some $\langle u^\delta_\eta \colon \eta \in I_\delta - J_1\rangle$ exemplify this. Now define for $\eta \in I$.

$$u_\eta = \begin{cases} u^*_\eta & \text{if } \eta \in J_1 \\ u^\delta_\eta & \text{if } \eta \in I_\delta - J_1,\ \delta \in W(\rho, S) \end{cases}$$

$\left(\text{remember } I = \bigcup_\delta I_\delta\right)$.

Clearly $\langle u_\eta: \eta \in I\rangle$ exemplify $(B)_{\rho,I}$ (i), (ii), (iii).

Fact 3.8B *Let $(D)^*_{\rho,I}$ be defined like $(D)_{\rho,I}$ omitting* (iii). *Then it is enough to prove $(D)^*_{\rho,I}$; in fact for every $I \subseteq \{\eta \in S_f: \rho \preccurlyeq \eta\}$, $(D)^*_{\rho,I} \Rightarrow (D)_{\rho,I}$.*

Proof: By 3.8A, it is enough to prove for $\delta \in W(\rho, S)$ that $(C)_{\rho^\frown\langle\delta\rangle,J}$ holds when $J = \{\eta: \rho^\frown\langle\delta\rangle \preccurlyeq \eta \in I, s^m_\eta$ has no end segment disjoint to $B_{\rho^\frown\langle\xi\rangle}\}$. This holds by conditions (e) (f) of 3.6 and (f) of 3.3.

Let us define $W = \{\delta \in W(\rho, S)$: the condition $(A)_{\rho^\frown\langle\delta\rangle,I_\delta}$ fail$\}$ where if $\rho^\frown\langle\delta\rangle \in S_f$, $(A)_{\rho^\frown\langle\delta\rangle,I_\delta}$ fails means $I_\delta \neq \emptyset$. By the induction hypothesis for $\delta \in W$ there is a $\lambda(\rho^\frown\langle\delta\rangle, S)$-set S^δ such that $(\forall\eta \in S^\delta_f)[\rho^\frown\langle\delta\rangle^\frown\eta \in I_\delta]$. If W is a stationary subset of $\lambda(\rho, S)$ then by Claim 3.2(4), $\{\langle\,\rangle\} \cup \{\langle\delta\rangle^\frown\eta: \eta \in S^\delta_f, \delta \in W\}$ is a $\lambda(\rho, S)$-set, contradicting $(C)_{\rho,I}$. Hence we conclude that there is a closed unbounded subset C of $\lambda(\rho, S)$ disjoint to W. By Fact 3.8A we can ignore $\{\eta \in I: \eta(m) \notin W\}$. As w.l.o.g. $\rho^\frown\langle 0\rangle \notin W$, we can assume $0 = Min\ C$ and clearly it suffices to prove that:

(*) If $\delta_0 < \delta_1$ are successive members of C, then we can choose $\{u_\eta: \eta \in I^*\}$ as required, such that $u_\eta \cap B_{\rho^\frown\langle\delta_0\rangle} = \emptyset$ where $I^* = \{\eta \in I: \delta_0 < \eta(m) < \delta_1\}$.

By Fact 3.8B we can forget the requirement $u_\eta \cap B_{\rho^\frown\langle\delta_0\rangle} = \emptyset$.

So we have reduced our task to proving $(D)^*_{\rho,I}$ when $|I| < \lambda\ (\rho, S)$, let $\delta_1 = \operatorname{Sup}_{\eta\in I} \eta(m)$. By condition (c) (of Claim 3.6) there is a one-to-one function g, Dom $g = I$, $g(\eta) \in \bigcup_{l\geq m} s^l_\eta$. For each $\delta < \delta_1$ let $J_\delta = \left\{\eta \in I: g(\eta) \in \bigcup_{l>m} s^l_\eta, \eta(m) = \delta\right\}$. So, condition $(A)_{\rho^\frown\langle\delta\rangle,J_\delta}$ holds (as $g \restriction J_\delta$ exemplify), hence by the induction hypothesis $(C)_{\rho^\frown\langle\delta\rangle,J_\delta}$ holds. Let $J = I - \bigcup\{J_\delta: \delta_0 < \delta < \delta_1\}$. By Fact 3.8A it suffices to prove $(D)^*_{\rho,J}$.

Case I: $l(\eta) = m + 1$ for every $\eta \in S_f$.

In this case $\bigcup\{s^l_\eta: m \leq l < l(\eta)\} = s^m_\eta$, and necessarily $cf\,\delta = \aleph_0$ for $\delta \in W(\rho, S)$ (essentially as $s^m_{\rho^\frown\langle\delta\rangle}$ is unbounded below δ; more exactly, for $\zeta < \delta$, $s^m_{\rho^\frown\langle\delta\rangle} \cap B_{\rho^\frown\langle\delta\rangle}$ is finite) [see (e) of 3.6]. Let $u^*_\eta = \{x \in B^*_\rho: x \in s^m_\eta, x \geq g(\eta)\}$ ($\geq$-in the tree order of B^*_ρ). On J we define a graph: $\eta_1, \eta_2 \in J$ are connected if $u^*_{\eta_1}, u^*_{\eta_2}$ are not disjoint. Clearly the valency of every η is $\leq\aleph_0$ (as for every η_2 connected to η $g(\eta_2) \in s_{\eta_m}$ and s^m_η is countable). Now we look at each component; we can shrink somewhat the u^*_η to make them pairwise disjoint (by ordering them in length ω and shrinking by induction) provided that $\eta \neq \upsilon \Rightarrow s^m_\eta \neq s^m_\upsilon$, but as $s^m_{\rho^\frown\langle\delta\rangle}$ is "unbounded below δ" this holds.

Case II: Not Case I and all $\delta \in W(\rho, S)$ are singular ordinals.

In this case for some κ, $(\forall\delta \in W(\rho, S))$ $(cf\,\delta = \kappa)$ and hence (see (3.6(a) and through it 3.3(f), 3.2(6)) for some $n > m$ for every $\eta \in S_f$, $\lambda(\eta\restriction n, S) = \kappa$. We define on J a graph:

$\eta_1, \eta_2 \in J$ are connected *if* $\eta_1 \restriction n = \eta_2 \restriction n$ or if $u^*_{\eta_1} \cap u^*_{\eta_2} \neq \emptyset$
(u^*_η is defined as in Case I).

As in Case I the valency of every $\eta \in J$ is $\leq\kappa$. So let $\langle K_\alpha: \alpha < \alpha^*\rangle$ be a list of the components of the graph. For each α let $\{\eta^{\alpha,\zeta}: \zeta < \zeta(\alpha) \leq \kappa\}$ list $\{\eta \restriction n: \eta \in K_\alpha\}$.

Now by conditions (e), (f) of 3.6 clearly,

Fact *For every $\alpha < \alpha^*$, $\zeta < \zeta(\alpha)$, the following set is bounded (below κ):* $\Big\{i < \kappa$: *there are* v_1, $\eta^{\alpha,\zeta} \frown \langle i \rangle \preccurlyeq v_1 \in J$ *and* v_2, $\bigvee_{\beta<\zeta} \eta^{\alpha,\beta} \preccurlyeq v_2 \in K_\alpha$ *such that* $u^*_{v_1} \cap u^*_{v_2} \neq 0\Big\}$.

We can conclude:

Fact *For every δ, $(C)_{\rho\frown\langle\delta\rangle,L}$ hold when $L = \{\eta \in J: \rho \frown \langle\delta\rangle \preccurlyeq \eta$, and for some α, ζ, ξ, v $\eta^{\alpha,\zeta} \preccurlyeq \eta \in K_\alpha$, $\eta^{\alpha,\xi} \preccurlyeq v \in K_\alpha$, $u^*_\eta \cap u^*_v \neq \varnothing$ and $\xi < \zeta\}$.*

By Fact 3.8A it suffices to prove $(D)_{\rho,J-L}$, however $\langle u^*_\eta: \eta \in J - L\rangle$ are as required:

Case III: Not I nor II.

So every $\delta \in W(\rho, S)$ is necessarily an inaccessible cardinal so for $\delta < \lambda(\rho, S)$ which is not inaccessible, $\delta \cap W(\rho, S)$ is not stationary and by condition 3.3(e) (and 3.6(a)) also for no inaccessible $\delta < \lambda(\rho, S)$ is $\delta \cap W(\rho, S)$ a stationary subset of δ. However (really we can get disjoint end segments in our case):

Fact 3.8C

(1) *Observation: If W is a set of ordinals, each of cofinality $>\aleph_0$ and for no δ is $\delta \cap W$ a stationary subset of δ then we can find $\langle c^*_\delta: \delta \in W\rangle$ such that: each c^*_δ is a closed unbounded subset of δ (for $\delta \in W$) and the c_δ's are pairwise disjoint.*

(2) *Moreover if for $\delta \in W$ c_δ is a closed unbounded subset of δ disjoint to W, then we can find pairwise disjoint end-segments (and even omit the demand $\delta \in W \Rightarrow cf\,\delta > \aleph_0$).*

[This can be proved by induction on sup W].

Applying this to $W' = \{\eta(m): \eta \in J\}^2$ we let, for $\delta \in W'$, $J_\delta = \{\eta \in S_f: \eta \in J, \rho \frown \langle\delta\rangle \leq \eta$ and $\eta(m+1) \notin c^*_\delta\}$. Clearly $(C)_{\rho\frown\langle\delta\rangle,J_\delta}$ holds, hence by 3.8A it suffices to prove $(C)_{\rho,J^*}$ where $J^* = J - \bigcup\{J_\delta: \delta \in W'\}$. But $\{u^*_\eta: \eta \in J^*\}$ are as required: $u^*_{\eta_1} \cap u^*_{\eta_2} = \varnothing$ if $\eta_1(m) \neq \eta_2(m)$ as $c^*_{\eta_1(m)} \cap c^*_{\eta_2(m)} = \varnothing$ by condition (e) of 3.6, and $u^*_{\eta_1} \cap u^*_{\eta_2} = \varnothing$ if $\eta_1(m) = \eta_2(m)$ by condition (d) of 3.6.

PART β: $(D)_{\rho,I} \Rightarrow (B)_{\rho,I}$. For notational simplicity we omit condition (iii) of $(B)_{\rho,I}$. The proof is by cases.

Case I: $\rho \frown \langle i \rangle \in S_f$ for $i \in W(\rho, S)$. Easy.

Case II: Every $\delta \in W(\rho, S)$ is a singular ordinal.

In this case for some regular κ, $[\delta \in W(\rho, S) \Rightarrow cf\,\delta = \kappa]$ and $[\eta \in S_f \rightarrow cf[\eta(n)] = \kappa]$. (Note that if $\eta \frown \langle\delta\rangle \in S_f$ then $cf\,\delta = \aleph_0$). Let $\langle u^\kappa_\eta: \eta \in I\rangle$ exemplify $(D)_{\rho,I}$, u^κ_η an end-segment of $s^{h(\eta)}_\eta$. For $\alpha < \kappa$, $i \in W(\rho, S)$ let $J_{i,\alpha} = \{\eta: h(\eta) \neq l(\rho), \eta \in I, \rho \frown \langle i \rangle \preccurlyeq \eta\} \cup \{\eta: \rho \frown \langle i \rangle \leq \eta, \eta(n) < \alpha\}$.

By Claim 3.9 (apply to $\rho \frown \langle i \rangle$) and the induction hypothesis $(B)_{\rho\frown\langle i\rangle, J_{i,\alpha}}$ holds, and let $u_\eta^\alpha(\eta \in J_{i,\alpha})$ $<_{i,\alpha}^*$ (a well ordering of $J_{i,\alpha}$) exemplify it. Let $J_\alpha = \bigcup_{i \in W(\rho,S)} J_{i,\alpha}$ and $<_\alpha^*$ be the well ordering of J_α defined by:

$$\eta <_\alpha^* \nu \text{ iff } (\exists i)[\eta \in J_{i,\alpha} \wedge \nu \in J_{i,\alpha} \wedge \eta <_{i,\alpha}^* \nu] \vee (\exists i < j)[\eta \in J_{i,\alpha} \wedge \nu \in J_{j,\alpha}] \ .$$

(Note that the $J_{i,\alpha}$ are pairwise disjoint for distinct i's and a fixed α, this also explains the notation u_η^α instead $u_\eta^{i,\alpha}$.) Clearly $<_\alpha^*$, $u_\eta^\alpha(\eta \in J_\alpha)$ exemplify $(B)_{\rho,J_\alpha}$. We now define by induction on ξ, a subset L_ξ of I such that:

(a) L_ξ is increasing continuous, $\cup L_\xi = I$.
(b) $L_0 = \varnothing$, $|L_{\xi+1} - L_\xi| \le \kappa$.
(c) if $\eta \in L_{\xi+1} - L_\xi$, $\nu \in I$, and $u_\nu^\kappa \cap \left(\bigcup_l s_\eta^l\right) \neq \varnothing$ then $\nu \in L_{\xi+1}$.
(d) if $\eta \in L_{\xi+1} - L_\xi$, $\nu \in I$, $\eta \restriction n = \nu \restriction n$ then $\nu \in L_{\xi+1}$.
(e) if $\eta \in L_{\xi+1} - L_\xi$, $\alpha < \kappa$, $\nu \in J_\alpha$, $\eta \in J_\alpha$ and $u_\nu^\alpha \cap \left(\bigcup_l s_\eta^l\right) \neq \varnothing$ then $\nu \in L_{\xi+1}$.

We give now a partial information on the u_η, $<^*$ we shall construct: $u_\eta \in \{u_\eta^\alpha\colon \alpha = \kappa \text{ or } \alpha < \kappa, \eta \in J_\alpha\}$ and if $\eta \in L_\xi$, $\nu \notin L_\xi$ (so $\nu \in L_{\zeta+1} - L_\zeta$ for some $\zeta > \xi$) then $\eta <^* \nu$. This guarantees (ii) of $(B)_{\rho,I}$ and also (i) of $(B_{\rho,I})$ except possibly when for some ξ, η, $\nu \in L_{\xi+1} - L_\xi$. So we can restrict ourselves to a fixed $L_{\xi+1} - L_\xi$.

Now the set $\{\eta(m)\colon \eta \in L_{\xi+1} - L_\xi\}$ has power $\le\kappa$, so let it be $\{i_\zeta\colon \zeta < \zeta(*) \le \kappa\}$. We can define by induction on $\zeta < \zeta(*)$ an ordinal $\alpha_\zeta < \kappa$, such that (remembering the second phrase in 3.6(f)):

(*) if $\gamma < \zeta$, $\rho \frown \langle i_\zeta \rangle \preccurlyeq \eta \in S_f$, $\eta(n) \ge \alpha_\zeta$, $\rho \frown \langle \gamma \rangle \preccurlyeq \nu \in S_f$, then $s_\eta^{l(\rho)}$ is disjoint to $s_\nu^{l(\rho)}$.

Now we define u_η: *if* $\eta(m) = i_\zeta$, $\eta(n) \ge \alpha_\zeta \wedge h(\eta) = m$ then $u_\eta = u_\eta^\kappa$ and *if* $\eta(m) = i_\zeta$, $[\eta(n) < \alpha_\zeta \vee h(\eta) \neq m]$ then $u_\eta = u_\eta^{\alpha_\zeta}$. As for the order if $\eta \neq \nu \in L_{\xi+1} - L_\xi$, $i_{\zeta_0} = \eta(m)$, $i_{\zeta_1} = \eta(m)$, then

$[\zeta_0 < \zeta_1 \Rightarrow \eta <^* \nu]$
$[\zeta_0 = \zeta_1 \wedge u_\eta \subseteq s_\eta^m \wedge u_\nu \not\subseteq s_\nu^m \rightarrow \nu <^* \eta]$
$[\zeta_0 = \zeta_1 \wedge u_\eta \not\subseteq s_\eta^m \wedge u_\nu \not\subseteq s_\nu^m \rightarrow \eta <^* \nu \equiv \eta <_{\alpha_\zeta}^* \varphi]$
$[\zeta_0 = \zeta_1 \wedge u_\eta \subseteq s_\eta^m \wedge u_\nu \subseteq s_\nu^m \rightarrow \eta <^* \nu \equiv \eta <_{lx} \nu]$.

We leave the checking to the reader.

Case III: Every $\delta \in W(\rho, S)$ is an inaccessible cardinal. Easy.

Now we have got:

Theorem 3.10 *For every $\lambda > \aleph_0$ the following are equivalent.*

(A) *$PT(\lambda, \aleph_1)$ fail.*
(B) *There is a family of countable sets $\{s_i\colon i < \lambda\}$, which does not have a transversal but for every $I \subseteq \lambda$ of power $<\lambda$ there is a well ordering $<^*$, such that for $i \in I$*

$$s_i \not\subseteq \cup\{s_j\colon j \in I, j <^* i\} \ .$$

$\left[\text{In fact } s_i = \bigcup_{l<n} s_i^l, \ |s_i^l| = \aleph_0, \text{ and for every } I \text{ there is } <^* \text{ (as above) such that}\right.$

$\left.(\forall i \in I)(\exists l < n)[s_i^l \cap (\cup\{s_j: j \in I, j <^* i\}) \text{ is finite.}\right]$

4 Some investigation of PT

Lemma 4.1 *For $\lambda > \kappa$, $PT(\lambda, \kappa)$ is equivalent to $PT(\lambda, \kappa^+)$ provided that $PT(\kappa, \kappa)$ fail.*

Remark: On $PT(\kappa, \kappa)$ see next lemmas.

Proof: Any counterexample to $PT(\lambda, \kappa)$ is a counterexample to $PT(\lambda, \kappa^+)$. So assume $PT(\lambda, \kappa^+)$ fail. So there are S, $\langle B_\eta: \eta \in S_c\rangle$, s_η^l as in 3.6. Let $s_\eta^l = \{a(\eta, l, i): i < \kappa\}$ (as $|s_\eta^l| = \kappa$ by 3.6(h)).

Define $t_{\eta,i}^l = \{\langle a(\eta, l, i), a(\eta, l, j)\rangle: j < i\}$. $B_v' = B_v \times B_v$ (for $\eta \in S_f$, $l < l(\eta)$, $i < \kappa$ and for $\nu \in S_c$). It is easy to check that $\langle B_\eta': \eta \in S_c\rangle$ is a disjoint λ-system.

As $PT(\kappa, \kappa)$ fail let $\{A_i: i < \alpha\}$ be a family of subsets of κ which has no transversal, $|A_i| < \kappa$, but for each $\alpha < \kappa$ $\{A_i: i < \alpha\}$ has a transversal, and let G_α be such a transversal.

Now we define a family which is a counterexample to $PT(\lambda, \kappa)$. It is $E = \{D_{\eta,i}: \eta \in S_f, i < \kappa\}$ where

$$D_{\eta,i} = \bigcup_{l<l(\eta)} t_{\eta,i}^l \cup (\{\eta\} \times A_i)$$

(we assume w.l.o.g. that every $t_{\eta,i}^l$, $\{v\} \times A_i$ are disjoint). Let us check.

First requirement: E is a family of λ sets each of power $<\kappa$.

This is obvious (note that $|t_{\eta,i}^l| \le |i| < \kappa$).

Second requirement: E has no transversal.

Suppose g is a one-to-one function, Dom $g = S_f \times \kappa$, $g(\langle \eta, i\rangle) \in D_{\eta,h(\eta)}$ where $h(\eta) < l(\eta)$. For each η for some $i = i_\eta$ $g(\langle \eta, i\rangle) \notin \{\eta\} \times A_i$ [otherwise letting $g(\langle \eta, i\rangle) = \langle \eta, f_\eta^{(i)}\rangle$, f_η is a transversal of $\{A_i: i < \kappa\}$]. Hence $f(\eta) =_{df} g(\langle \eta, i_\eta\rangle)$ belong to $\bigcup_{l<l(\eta)} t_{\eta,t}^l$. However, as noted above $\langle B_\eta': \eta \in S_c\rangle$ is a λ-system and $t_{\eta,t}^l \subseteq B_{\eta\restriction(l+1)}$. Clearly f is a transversal of $\left\{\bigcup_l t_{\eta,h(\eta)}^l: \eta \in S_f\right\}$ contradicting 3.5.

Third requirement: If $I \subseteq S_f \times \kappa$, $|I| < \lambda$ then $\{D_{\eta,i}: \langle \eta, i\rangle \in I\}$ has a transversal.

W.l.o.g. $I = J \times \kappa$, $J \subseteq S_f$. By the choice of the $\langle B_\eta: \eta \in S_c\rangle$, $s_\eta^l(\eta \in S_f$, $l < l(\eta))$ there is a one-to-one function g, Dom $g = J$, $g(\eta) \in \bigcup_l s_\eta^l$, and let $g(\eta) = a(\eta, m(\eta), j(\eta))$. We now define a function f, Dom $f = I$, $f(\langle \eta, i\rangle)$ is: $\langle a(\eta, m(\eta), i), a(\eta, m(\eta), j(\eta))\rangle$ if $j(\eta) < i$ and $\langle \eta, G_{j(\eta)}(A_i)\rangle$ otherwise.

The checking is straightforward.

Lemma 4.2 *$PT(\kappa, \kappa)$ iff $\kappa = \aleph_0$ or κ is an uncountable inaccessible cardinal such that*

$(*)_\kappa$ *for every stationary subset* W *of* κ *for some inaccessible cardinal* $\mu < \kappa$ $W \cap \mu$ *is a stationary subset of* μ (*so* κ *is a Mahlo* (*inaccessible*) *cardinal*).

Proof: $PT(\aleph_0, \aleph_0)$ is well known. Let $\kappa > \aleph_0$, $(*)_\kappa$ hold, and suppose $A_i \subseteq \kappa$, $|A_i| < \kappa$ satisfies: $|\{A_i: i < \kappa\}$ has no transversal but every $\{A_i: i < \alpha\}$ has transversal for $\alpha < \kappa$. Clearly $\{\delta < \kappa: (\forall i < \delta)\ A_i \subseteq \delta\}$ is closed unbounded. Then $W = \{\delta < \kappa$: for some β; $\delta < \beta < \kappa$ and $\{A_i: \delta \le i < \beta\}$ has no transversal, with range disjoint to $\delta\}$ has to be stationary. Clearly $C = \{\delta < \kappa$: for $i < \delta$ $A_i \subseteq \delta$ and for $\delta_1 \in \delta \cap W$ there is β, $\delta_1 < \beta < \delta$ such that $\{A_j: \delta_1 \le j < \beta\}$ has no transversal with range disjoint to $\delta_i\}$ is closed unbounded in μ. $(*)_\kappa$ provides us with a $\mu \in C$ such that $\{A_i: i < \mu\}$, has no transversal, contradiction.

If κ is singular or a successor cardinal $PT(\kappa, \kappa)$ easily fails. [For $\kappa = \mu^+$ use $\{\alpha: \mu < \alpha < \kappa\}$, and for $\kappa = \sum_{i<\mu} \kappa_i$, $\kappa > \mu = cf\,\kappa$, $\kappa_i < \kappa$, κ_i increasing, continuous $\kappa_0 = 0$ use $\{\{\alpha\}: \alpha < \kappa,\ \alpha \notin \{\kappa_i: i < \mu\}\} \cup \{\{\alpha: \kappa_i \le \alpha < \kappa_{i+1}\}: i < \mu\}$: $\cup\{\kappa_i: i < \mu\}]$.

If κ is inaccessible but $(*)_\kappa$ fails, let W exemplify this, and w.l.o.g. W be a set of limit cardinals, and $E = \{\mu: \mu \in W\}$ exemplify $PT(\kappa, \kappa)$. [As W is stationary it has no transversal. Now prove by induction on $i < \kappa$ that for $j < i$, $\{\mu: j < \mu \le i\}$ has a transversal with range disjoint to j.]

Lemma 4.3 *Suppose* κ *is an uncountable inaccessible cardinal such that* $(*)_\kappa$ (*from* 4.2) *holds. Then for* $\lambda > \kappa$ $PT(\lambda, \kappa)$ *iff* $(\forall \mu < \kappa)$ $PT(\lambda, \mu)$.

Proof: A counterexample to $PT(\lambda, \mu)$, $(\mu < \kappa)$ is a counterexample to $PT(\lambda, \kappa)$. If we have a counterexample P to $PT(\lambda, \kappa)$, use 3.3. By 4.2 $|F^0(\eta)| < \kappa$ for $\eta \in S_f$ and $\lambda(\eta, S) \neq \kappa$ for $\eta \in S_i$. Now by 3.2 (and 3.5) get a counterexample to $PT(\lambda, \mu)$ for some $\mu < \kappa$.

5 Abelian groups

Let M be $(H(x), \epsilon, U)$ for some large enough regular x.

Axiom XVII

(a) *If* $N \prec M$, A, $B \in N$ *then* $A \cap N$ *is free over* $B/B \cap N$. (*By Axiom XI, w.l.o.g.* $B \subseteq A$.)
(b) *If* $B \subseteq A$, $N \prec M$, A, $B \in N$ $A - B \subseteq N$ *then* $A \cap N$ *is free over* $B/B \cap N$.
(c) *If* $B \subseteq A$, $N \prec M$, A, $B \in N$ *and* A/B *is* $\aleph_0$*-free then* $A \cap N$ *is free over* $B/B \cap N$.

Axiom XVIII *If* $N \prec M$, A, $B \in N$, $A \subseteq N$ *then* A/B *is free iff* $A/B \cap N$ *is free iff* A/B' *is free for any* B', $B \cap N \subseteq B' \subseteq B$.

Claim 5.1

(1) *For Abelian group Axiom XVII* (a) *is satisfied.*
(2) *Any variety satisfies Axiom XVII* (c).
(3) *Axiom XVII* (a) *implies Axiom XVII* (c), *also Axiom XVII* (a) *implies Axiom XVII* (b).

Theorem 5.2 *Assume Axiom* I^{**}, *XVII* (a) (*and II-XVI*). *If there is a* λ*-free not* λ^+*-free pair and* $\chi_0, \chi_1 \le \kappa$, *then* $PT(\lambda, \kappa^+)$ *holds.*

Notation: For a sequence $\eta = \langle \alpha_0, \ldots, \alpha_{m-1} \rangle$ of ordinals, $m > 0$ let $\eta^+ = \langle \alpha_0, \ldots, \alpha_{m-2}, \alpha_{m-1} + 1 \rangle$.

Proof: So let A/B be λ-free not free, $|A| = \lambda$, w.l.o.g. $\lambda > \kappa$. By [6] λ is regular. Now like 3.3:

A. Fact *There is a λ-set S and sets B_ν, A_η for $\eta \in S$, $\nu \in S_c$ such that*

(a) *$\langle B_\eta : \eta \in S_c \rangle$ is a λ-system.*
(b) *For $\eta \in S_i$ $\lambda(\eta, S) > \kappa$.*
(c) *We let $B_{\langle\, \rangle} = B$, $A_{\langle\, \rangle} = A$, we stipulate $B_{\langle\, \rangle^+} = A \cup B$; now B_ν, $A_\eta \subseteq A$ (for $\nu \in S_c$, $\eta \in S$), $\lambda(\eta, S) + \kappa = |A_\eta| + \kappa$, $A_\eta = B_{\eta^+} - B_\eta = B_{\eta \frown \langle \lambda(\eta, S) \rangle}$ for $\eta \in S$).*

(d) *For $\eta \in S_i$, $A_\eta \big/ \bigcup_{l \le l(\eta)} B_{\eta \restriction l}$ is $\lambda(\eta, S)$-free not free and for $\eta \in S_f$ $A_\eta \big/ \bigcup_{l \le l(\eta)} B_{\eta \restriction l}$ is not free.*

(e) *If $\eta \frown \langle \alpha \rangle \in S$ then α is a limit ordinal.*

Notation: For $\eta \in S$ $B^*_\eta = \bigcup_{l \le l(\eta)} B_{\eta \restriction l}$.

By Axiom XVII (a), XIV (1) w.l.o.g. $|B| \le \lambda$.

Let $\eta \in S_f$, D_η be a subset of B^*_η of power $\le \kappa$ such that A_η is free over B^*_η / D_η (exists by Axiom XVII (a)). Let $D^*_\eta = D_\eta \cap A$. We shall prove that $E = \{D^*_\eta \times \kappa : \eta \in S_f\}$ exemplifies the failure of $PT(\lambda, \kappa^+)$ thus finishing the proof.

Clearly it is a family of λ sets each of power $\le \kappa$. By 3.5 E has no transversal $\left(\text{as } \langle B_\eta \times \kappa : \eta \in S_c \rangle \text{ is a } \lambda\text{-system, } D^*_\eta \times \kappa \subseteq \bigcup_{l \le l(\eta)} B_{\eta \restriction l} \times \kappa\right)$. So it suffices to prove that if $\alpha < \lambda$ then $E_\alpha = \{D^*_\eta \times \kappa : \eta \in S_f \cap {}^{\omega >}\alpha\}$ has a transversal.

B. Fact *For any cardinals μ and set V of power $\le \kappa$ and cardinal χ such that $\mu, V \in H(\chi)$. Let $R = \{\eta : \eta$ a decreasing sequence of ordinals $< \mu$ such that $\eta(m) < \kappa \Rightarrow m + 1 = l(\eta)\}$. We can define $M_\eta (\eta \in R, \eta(l(\eta) - 1) > 0)$ such that*

(1) *$M_\eta \prec (H(\chi), \in)$, $V \subseteq M_\eta$ and $(\forall i < \|M_\eta\|)[i \in M_\eta]$, $\eta \in M_\eta$.*
(2) *Let $\mu_\eta = \|M_\eta\|$, then η is final in R iff $\mu_\eta \le \kappa$; and $\mu_\eta = Min\{|\eta(l)| + \kappa : l < l(\eta)\}$.*
(3) *For $\eta \in R_i - \{\langle\, \rangle\}$, $\eta^+ \in R$ and $M_{\eta^+} = \cup\{M_{\eta \frown \langle i \rangle} : i < \mu_\eta\}$; if $\eta \frown \langle \delta \rangle \in R$, δ a limit ordinal, then $M_{\eta \frown \langle \delta \rangle} = \bigcup_{i < \delta} M_{\eta \frown \langle i \rangle}$ and if $\eta \frown \langle i \rangle, \eta \frown \langle j \rangle \in R$, $i < j$, then $M_{\eta \frown \langle i \rangle} \prec M_{\eta \frown \langle j \rangle}$.*
(4) *For $l < l(\eta)$, $\eta \in R$ $M_{\eta \restriction l} \in M_\eta$.*
(5) *Stipulating $M_{\langle \mu \rangle} = \bigcup_{\beta < \mu} M_{\langle \beta \rangle}$, $M_{\nu \frown \langle 0 \rangle} = M_{\langle\, \rangle} = \varnothing$, $\left\{M_{\eta^+} - \bigcup_{l \le l(\eta)} M_{\eta \restriction l} : \eta \in R_f\right\}$ is a partition of $M_{\langle \mu \rangle}$.*

(The proof of their existence is by induction on μ, and then we define by induction on $\beta < \mu$, $M_{\beta+1}$, and $\langle M_{\langle \beta \rangle \frown \eta} : \langle \beta \rangle \frown \eta \in R \rangle$ such that $\alpha \in M_{\langle \alpha + 1 \rangle}$.)

We stipulate for nonfinal $\eta \in R$, $M_{\eta \frown \langle \mu_\eta \rangle} = M_{\eta^+}$, $M_{\langle\, \rangle} = \varnothing$, $M_{\langle\, \rangle^+} = \bigcup_{\beta < \mu} M_{\langle \beta \rangle}$.

We use Fact B for χ large enough (and regular) $V = \{A, B\} \cup \{\langle A_\eta, B_\eta, B^*_\eta, D_\eta: \eta \in S_C\rangle, S, A_\alpha\}$ and $\mu = |A_\alpha|$ and get $\langle M_\eta: \eta \in R\rangle$ as above.

Notation: $R_i = \{\eta \in R: (\exists\beta)\eta \frown \langle\beta\rangle \in R\}$, $R_f = R - R_i$.

By Fact B(5) for each $\eta \in S_f \cap {}^{\omega>}\alpha$ there is a unique $v = v_\eta \in R_f$ such that $\eta \in M_{v^+} - \bigcup_{l \leq l(v)} M_{v \restriction l}$. So in order to prove that E_α has a transversal it is enough to prove for each $\eta \in S_f \cap {}^{\omega>}\alpha$.

$$D^*_\eta \not\subseteq \cup\{M_{v_\eta \restriction l}: l \leq l(v_\eta)\}$$

or equivalently

(*) $\qquad D_\eta \not\subseteq B \cup \cup\{M_{v_\eta \restriction l}: l \leq l(v_\eta)\}$

[because $D^*_\eta \in M_{v^+}$ (as $\langle D_{v_\eta}: v \in S_f\rangle \in M_{v_\eta^+}$), and so the power of

$$(D^*_\eta \times \kappa) \cap \left(M_{v_\eta^+} - \bigcup_{l \leq l(v)} M_{v_\eta \restriction l}\right)$$

is κ, and for such $v \in S_f$ there are at most κ such η's].

C. Fact *Let $\eta \in S_f \cap {}^{\omega>}\alpha$, $v = v_\eta$, $k \leq l(v)$.*

Then A_η is free over $B^*_\eta \cup \left(\bigcup_{i \leq k} M_{v \restriction i} \cap A\right)\Big/ D_\eta$. We prove this by induction on k.

For $k = 0$: this means that A_η is free over B^*_η / D_η (as we have stipulated $M_{\langle\,\rangle} = \emptyset$) and this holds by the choice of D_η.

For $k > 0$: Let $l(k)$ be maximal such that $\eta \restriction l(k) \in M_{v \restriction k}$ and let δ be the minimal ordinal in $M_{v \restriction k}$ which is $\geq \eta(l(k))$.

As $\eta \restriction l(k) \in M_{v \restriction k}$ and $S \in V$, clearly $\lambda(\eta \restriction l(k), S) \in M_{v \restriction k}$. So by the choice of $l(k)$ and δ

(i) $\eta(l(k)) < \delta \leq \lambda(\eta \restriction l(k), S)$.

It is also clear that (as $V \subset M_{v \restriction k} \prec (H(\chi), \in)$ and A, B, $\langle B^*_\rho: \rho \in S\rangle \in M_{v \restriction k}$):

(ii) A, B and $B^*_{\eta \restriction l(k) \frown \langle\delta\rangle}$ belong to $M_{v \restriction k}$.

By (4) of Fact B

(iii) for $\rho \in R$, $\bigcup_{i \leq l(\rho)} (M_{\rho \restriction i} \cap A)$ belong to M_ρ.

By (ii) and (iii) (with $v \restriction k$ standing for ρ)

(iv) $B^*_{\eta \restriction l(k) \frown \langle\delta\rangle} \cup \left(\bigcup_{i<k} M_{v \restriction i} \cap A\right)$ belong to $M_{v \restriction k}$.

As $A \in M_{v \restriction k}$, by Axiom XVIIa $\left(\text{with } B^*_{\eta \restriction l(k) \frown \langle\delta\rangle} \cup \left(\bigcup_{i<k} M_{v \restriction i} \cap A\right),\right.$ A, $M_{v \restriction k}$ standing for B, A, N respectively)

(v) the triple $\left\langle M_{\nu\restriction k} \cap A,\ B^*_{\eta\restriction l(k)^\frown\langle\delta\rangle} \cup \left(\bigcup_{i<k} M_{\nu\restriction i} \cap A\right),\ M_{\nu\restriction k} \cap \left(B^*_{\eta\restriction l(k)^\frown\langle\delta\rangle} \cup \left(\bigcup_{i<k} M_{\nu\restriction i} \cap A\right)\right)\right\rangle$ is free.

By the choice of δ, and as the function h: $B_{\eta\restriction l(k)^\frown\langle\delta\rangle} \to \delta$ $h(b) = Min\{i$: $b \in B_{\eta\restriction l(k)^\frown\langle i\rangle}\}$ is definable in $M_{\nu\restriction k}$:

(vi) $B^*_{\eta\restriction(\eta)^\frown\langle\delta\rangle} \cap M_{\nu\restriction k} = B_{(\eta\restriction l(\eta))^\frown\langle\eta(l(k))\rangle} \cap M_{\nu\restriction k}$.

But $\eta \restriction l(\eta) {}^\frown \langle\eta(l(k))\rangle = \eta \restriction (l(k) + 1)$, so

(vii) $\left\langle M_{\nu\restriction k} \cap A,\ B^*_{\eta\restriction l(k)^\frown\langle\delta\rangle} \cup \left(\bigcup_{i<k} M_{\nu\restriction i} \cap A\right),\ M_{\nu\restriction k} \cap \left(B^*_{\eta\restriction(l(k)+1)} \cup \left(\bigcup_{i<k} M_{\nu\restriction i} \cap A\right)\right)\right\rangle$ is free.

Now by monotonicity Axiom XI (and as $B_{\eta\restriction(l(k)+1)} \subseteq B_{\eta\restriction l(k)^\frown\langle\delta\rangle}$ by (i))

(viii) $\left\langle M_{\nu\restriction k} \cap A,\ B^*_{\eta\restriction l(k)^\frown\langle\delta\rangle} \cup \left(\bigcup_{i<k} M_{\nu\restriction i} \cap A\right),\ B^*_{\eta\restriction(l(k)+1)} \cup \left(\bigcup_{i<k} M_{\nu\restriction i} \cap A\right)\right\rangle$ is free.

But $B^*_{\eta\restriction(l(k)+1)} \subseteq B^*_\eta \subseteq B^*_\eta \cup A_\eta \subseteq B^*_{\eta\restriction(l(k)^\frown\langle\delta\rangle}$ so by monotonicity (Axiom XI)

(ix) $\left\langle M_{\nu\restriction k} \cap A,\ B^*_\eta \cup A_\eta,\ B^*_\eta \cup \left(\bigcup_{i<k} M_{\nu\restriction i} \cap A\right)\right\rangle$ is free.

By Axiom VIII

(x) $M_{\nu\restriction k} \cap A$ is free over $A_\eta \Big/ B^*_\eta \cup \left(\bigcup_{i<k} M_{\nu\restriction i} \cap A\right)$.

By the symmetry Axiom IX

(xi) A_η is free over $M_{\nu\restriction k} \cap A \Big/ B^*_\eta \cup \left(\bigcup_{i<k} M_{\nu\restriction i} \cap A\right)$.

By the induction hypothesis

(xii) A_η is free over $B^*_\eta \cup \left(\bigcup_{i<k} M_{\nu\restriction i} \cap A\right) \Big/ D_\eta$.

By (xi), (xii) and the transitivity axiom (Axiom X) (as $D_\eta \subseteq B^*_\eta$) (and two uses of commutativity)

(xiii) A_η is free over $B_\eta \cup \left(\bigcup_{i\le k} M_{\nu\restriction i} \cap A\right) \Big/ D_\eta$.

So we have carried the induction, thus proving fact C.

D. Fact $A_\eta \Big/ \left(B \cup \bigcup_{m\le l(\nu)} (M_{\nu\restriction m} \cap A)\right)$ *is free for any* $\eta \in S_f \cap {}^{\omega>}\alpha$ *and* $\nu \in R$.

But $A_\alpha \in M_{\nu\restriction m}$, A_α/B is free, $M_{\nu\restriction m} \in M_{\nu\restriction(m+1)}$ hence we can prove by induction on m that $A_\alpha/B \cup \bigcup_{l\le m} (M_{\nu\restriction l} \cap A)$ is free (for $m = 0$ this means A_α/B

is free, for $m + 1$ use Axiom VII (see the Introduction). So $A_\alpha/B \cup \bigcup_{m \le l(v)} (M_{v \restriction m} \cap A)$ is free.

Now by Axiom I** as $A_\eta \subseteq A_\alpha$, $A_\eta/B \cup \bigcup_{m \le l(v)} (M_{v \restriction m} \cap A)$ is free, so we have proved Fact D.

Assume now that (*) fails, i.e., $\eta \in S_f \cap {}^{\omega>}\alpha$, $v = v_\eta$ $D_\eta \subseteq B \cup \bigcup_{m \le l(v)} (M_{v \restriction m} \cap A)$ and we shall get a contradiction, thus finishing the proof of Theorem 5.2. As $D_\eta \subseteq B \cup \bigcup_{m \le l(v)} (M_{v \restriction m} \cap A) \subseteq B^*_\eta \cup \bigcup_{m \le l(v)} (M_{v \restriction m} \cap A)$, by Fact C and Axiom XI A_η is free over

$$B^*_\eta \cup \bigcup_{m \le l(v)} (M_{v \restriction m} \cap A)/B \cup \bigcup_{m \le l(v)} (M_{v \restriction m} \cap A) .$$

So by Fact D and Axiom XIV (1) $A_\eta/B^*_\eta \cup \bigcup_{m \le l(v)} (M_{v \restriction m} \cap A)$ is free. By Fact C and Axiom XI A_η is free over $B^*_\eta \cup \bigcup_{m \le l(v)} (M_{v \restriction m} \cap A)/B^*_\eta$ hence by Axiom XIV (1) we get A_η/B^*_η is free contradicting (d) of Fact A.

Lemma 5.3

(1) *If* $PT(\lambda, \aleph_1)$ *fail there is a* λ*-free not free Abelian group of power* λ.
(2) *Moreover we can get a strongly* λ*-free not free Abelian group and group of power* λ.

Proof: (1) By 3.10 there is a λ-system $\langle B_\eta : \eta \in S_c \rangle$ and $s_\eta \subseteq \bigcup_{l \le l(\eta)} B_{\eta \restriction l}$ such that for $I \subseteq S_f$, $|I| < \lambda$ there is a well ordering $<$ such that $s_\eta - \cup \{s_v : v <^* \eta\}$ is infinite. Let $s_\eta = \{a(\eta, i) : i < \omega\}$. We define an Abelian group G: it is generated freely by $\left\{x_r : r \in \bigcup_v B_v\right\} \cup \{y^n_\eta : \eta \in S_f,\ n < \omega\}$ subject only to the relations

$$y^n_\eta = x_{a(\eta,n)} + 2y^{n+1}_\eta .$$

In order to prove that G is not free, let, for $\eta \in S_c$, G_η be the subgroup of G generated by $\{x_r : r \in \cup\{v \in S,\ v \le_{lx} \eta\} \cup \{y^m_\rho : \rho \in S_f,\ \rho \le_{lx} \eta\}$. (So $G_{\langle\,\rangle}$ is trivial, $G_{\langle\lambda\rangle} = G$.) We now prove by induction on $\eta \in S$ (which is a well-founded tree) that G_{η^*}/G_η is not free. No problem arises.

Lastly we have to prove for $\alpha < \lambda$ that $G_{\langle\alpha\rangle}$ is free, so let $<^*$ be a well ordering of $\{\eta \in S_f : \eta(0) < \alpha\}$, for each η $u_\eta = s_\eta - \cup\{s_v : v <^* \eta\}$ is infinite. Let $A_\eta = \{x_r : r \in s_\eta\} \cup \{y^n_\eta : n < \omega\}$.

We prove by induction on $\eta \in I$ that for $v <^* \eta$, $\bigcup_{\tau <^* \eta} A_\tau \Big/ \bigcup_{\tau <^* v} A_\eta$ is free. A trivial case is η an immediate successor of v which is clear.

(2) Similar proofs.

Conclusion 5.4 There is a λ-free not λ^+-free Abelian group iff $PT(\lambda, \aleph_1)$ fail iff there is a strongly λ-free not λ^+-free Abelian group.

Concluding Remarks:

(1) The proofs in Section 3 suggest dealing with $IC_n(\lambda, \omega)$ where

Definition Let $IC_n(\lambda, \delta)$ hold if there is a set S of n-tuples $\bar{s} = \langle s_0, \ldots, s_{n-1} \rangle$, s_l an element of a tree T of height δ such that

(a) $|S| = \lambda$
(b) There are no functions f: $S \to T$ and g: $S \to n$ such that $f(\langle s_0, \ldots, s_{n-1} \rangle) < s_{g(\langle s_0, \ldots, s_{n-1} \rangle)}$ (for each n-tuple from S) and $\{x \in T: f(\langle s_0, \ldots, s_{n-1} \rangle) < x < s_{g(\langle s_0, \ldots, s_{n-1} \rangle)}\}$ (for $\langle s_0, \ldots, s_{n-1} \rangle \in S$) are pairwise disjoint.

(2) For any $S^* \subseteq S$, $|S^*| < \lambda$ there are such f, g.

Note: (a) We can replace δ by $\bar{\delta} = \langle \delta_0, \ldots, \delta_{n-1} \rangle$, (so s_l has height δ_l); (b) we can treat $IC_n(\lambda, \bar{\delta})$ in the context of the Introduction and prove compactness for singular cardinalities. It is natural to conjecture that under the hypothesis of 5.2, there is an $\alpha^* \le \omega$ such that some A (or some A/B) is λ-free not λ^+-free iff for some $n < \alpha^*$, $IC_n(\lambda, \omega)$ holds. However if we look carefully at the analysis in Section 3, we see that for some $l < \omega$ we may demand that the $s^l_\eta(\eta \in S_f)$ are pairwise disjoint or equal; so really we can replace s^l_η by a point (e.g., if $(\forall \mu < \lambda(\eta \restriction l, S))$ $[\mu^{\aleph_0} < \lambda(\eta \restriction l, S)]$). This suggests a finer division (using 3.6 of course to uniformize) and calls for re-examining Section 1 and 5.2 to make them meet; i.e., we will have few quite simple combinatorial properties so that the set $S = \{\lambda$: there is a λ-free not λ^+-free $A\}$ is determined by them; i.e., if λ_0 is in S then for some such property Pr, $Pr(\lambda_0) \wedge (\forall \lambda)(Pr(\lambda) \to \lambda \in S)$.

(3) What about the variety of groups? If we have a λ-free not λ^+-free group, we can repeat the analysis in Sections 5 and 3. We can prove $PT(\lambda, \aleph_1)$ fail if for every $\eta \frown \langle \delta \rangle \in S_i$, $cf\,\delta = \lambda(\eta \frown \langle \delta \rangle, S)$.

Appendix (by A. Mekler)

Theorem *The following are equivalent*:

(A) *There is a family A of countable sets so that*: $|A| = \lambda$; *A does not have a transversal; and every subfamily of cardinality $<\lambda$ has a transversal.* (*We abbreviate this property as* $\neg PT(\lambda)$.))
(B) *There is a family A exemplifying $\neg PT(\lambda)$ such that if $B \subset A$ and $|B| < \lambda$ then B has a large transversal. Here B has a large transversal if there exists a family of pairwise disjoint infinite sets $\{b^*: b \in B\}$ such that $b^* \subseteq b$.*
(C) *There is a λ-free abelian group of cardinality λ which is not free.*

Proof: By [6] any of (A), (B), and (C) imply that λ is regular. Before proving $(A) \Rightarrow (B)$ in general we will consider a special case.

Proposition 1 *Suppose $A = \{a_\alpha: \alpha \in E\}$ exemplifies $PT(\lambda)$ where E is a stationary subset of λ and each $a_\alpha \subseteq \alpha$. Then (B) holds.*

Proof: Note that Fodor's lemma implies A does not have a transversal without loss of generality. We can assume $|a_\alpha| = \omega$ for all α. For each $\alpha \in E$ let $B_\alpha = {}^{<\omega}\alpha$ and let p_α be some enumeration of a_α. Define $t_\alpha = \{p_\alpha \restriction n: n < \omega\}$. Let $C = \{t_\alpha: \alpha \in E\}$. Since for α a limit point of E, $B_\alpha = \bigcup_{\beta<\alpha} B_\beta$, Fodor's lemma again implies C does not have a transversal. It remains to see that if $I \subseteq \lambda$ and $|I| < \lambda$ then $\{t_\alpha: \alpha \in I\}$ has a large transversal.

Let g be a transversal for $\{a_\alpha: \alpha \in I\}$. We can write $I = \bigcup_{\alpha<\mu} I_\alpha$ (continuous) so that for all α: $I_0 = 0$; $|I_{\alpha+1} \backslash I_\alpha| \leq \omega$; and for all $i \in I_\alpha$ and $j \in I$ if $g(a_j) \in a_i$ then $a_j \in I_\alpha$. Fix α and for each $i \in I_{\alpha+1} \backslash I_\alpha$ choose t_i^* so that $\{t_i^*: i \in I_{\alpha+1} \backslash I_\alpha\}$ forms a large transversal for $\{t_i: i \in I_{\alpha+1} \backslash I_\alpha\}$ and if $p_i \restriction n \in t_i^*$ then for some $k < n$ $g(a_i) = p_i(k)$. It remains to see if, $i \neq j$, $t_i^* \cap t_j^* = 0$. By the construction we can assume there is α so that $i \in I_{\alpha+1} \backslash I_\alpha$ and $j \notin A_{\alpha+1}$. Since $g(a_j) \notin a_i$, $t_j^* \cap {}^{<\omega}a_i = 0$. [Note: we didn't have to assume that a_α's were pairwise disjoint. This remark shows the method applies to indexed families.]

The generalization of a stationary subset of λ which we will use is a λ-system.

Definition A λ-*system* is labeled subtree $\langle S, B_\eta, \lambda_\eta: \eta \in S\rangle$ of ${}^{<\omega}\lambda$ satisfying:

(1) $\lambda = \lambda_{\langle\, \rangle}$
(2) for all $\eta \in S$, λ_η is regular
(3) $\eta \in S_f$ (the terminal nodes of S) iff $\lambda_\eta = \omega$
(4) suppose η is not terminal then
  (a) $E = \{i: \eta \frown \langle i\rangle \in S\}$ is stationary in λ_η
  (b) for all $i \in E$, $\lambda_{\eta\frown\langle i\rangle} \subseteq |B_{\eta\frown\langle i\rangle}| < \lambda_\eta$
  (c) if $i < j \in E$, then $B_{\eta\frown\langle i\rangle} \subseteq B_{\eta\frown\langle j\rangle}$
  (d) if $j \in E$ and j is a limit point of E, then $B_{\eta\frown\langle j\rangle} = \cup B_{\eta\frown\langle i\rangle}(i < j, i \in E)$.

To simplify notation we let $\bar{B}_\eta$ denote $\cup B_{\eta\restriction l}(l \leq l(\eta))$.

Proposition 2 *Suppose $\langle S, B_\eta, \lambda_n: \eta \in S\rangle$ is a λ-system $B_{\langle\,\rangle} = 0$ and $\{s_\eta: \eta \in S_f\}$ is a family of countable sets so that for $\eta \in S_f$ $s_\eta \subseteq \bar{B}_\eta$. Then $\{s_\eta: \eta \in S_f\}$ does not have a transversal.*

Proof: Assume g is a transversal. We will find an infinite branch through S. Fodor's lemma implies $\{i < \lambda$: there is $\eta \in S_f(\eta(0) = i)$ and $g(s_n) \in B_{\langle i\rangle}\}$ is nonstationary. Pick some $\langle i_0\rangle \in S$ so that for all $\eta \in S_f$ if $\eta(0) = i_0$ then $g(s_\eta) \notin B_{\langle i_0\rangle}$. Repeating this argument we can find i_1 so that: $\langle i_0, i_1\rangle \in S$; for all $\eta \in S_f$ if $\eta(0) = i_0$ and $\eta(1) = i_1$ then $g(s_\eta) \in B_{\langle i_0\rangle} \cup B_{\langle i_0, i_1\rangle}$. Continuing we get an infinite path through S and hence an infinite descending sequence of cardinals.

Proposition 3 *Suppose $\langle S, B_\eta, \lambda_\eta: \eta \in S\rangle$ and $\{s_\eta: \eta \in S_f\}$ are as above. Further suppose $\{s_\eta: \eta \in S_f\}$ witnesses $\neg PT(\lambda)$. Then there is $\{t_\eta: \eta \in S_f\}$ witnessing (B) and a λ-system $\langle S, C_\eta, \lambda_n: \eta \in S\rangle$ such that $C_{\langle\,\rangle} = 0$ and for all $\eta \in S_f$ $t_\eta \subseteq \bar{C}_\eta$.*

Proof: If $\eta \in S$ let $C_\eta = {}^{<\omega}B_\eta$. Now we can assume for all $\eta \in S_f$ and $0 < l \leq l(\eta)$ that $|S_\eta \cap B_{\eta\restriction l}| = \omega$. Let p^l: $\omega \to B_{\eta\restriction l}$ enumerate $S_\eta \cap B_{\eta\restriction l}$. Let $t_\eta = \{p^l \restriction n$: $l < \eta$ and $n < \omega\}$. The verification that this definition works is similar to the proof of Proposition 1.

Remark: If we wished to we could require that there be some $k < \omega$ such that $\eta \in S_f$ iff $l(\eta) = k$.

Definition Suppose A is a family of countable sets and B is a set. Then A/B is *free* if there is a transversal g of A so that for all $a \in A$ $g(a) \notin B$. Similarly define λ-*free*.

Note that [6] applies to this concept. So if A/B is λ-free, not free and $|A| = \lambda$, then λ is regular. Also suppose A/B is λ-free and $|A| = \lambda$ and λ is regular $\geq \omega$. Let $A = \bigcup_{\alpha<\lambda} A_\alpha$ (continuous) where $|A_\alpha| < \lambda$. Then A/B is not free iff $\{\alpha\colon A\backslash A_\alpha/B \cup B_\alpha$ is not λ-free$\}$ is stationary where $B_\alpha = \cup A_\alpha$.

Proposition 4 *Suppose* $\neg PT(\lambda)$ *holds. There is* $\langle S, B_\eta, \lambda_\eta\colon \eta \in S\rangle$ *and* $\{s_\eta\colon \eta \in S_f\}$ *satisfying the hypotheses of Proposition* 2.

Proof: Let A exemplify $\neg PT(\lambda)$. We will define $\langle S, B_\eta, \lambda_\eta, A_\eta\colon \eta \in S\rangle$ inductively so that: $\langle S, B_\eta, \lambda_\eta\colon \eta \in S\rangle$ is a λ-system; for all $\eta \in S$ $|A_\eta| + \omega = \lambda_\eta$; for all η $A_\eta/\bar{B}_\eta$ is $|A_\eta|$-free but not free; if η, $\eta \frown \langle i\rangle$, $\eta \frown \langle j\rangle \in S$ and $i \neq j$ then $A_\eta \supseteq A_{\eta\frown\langle i\rangle}$ and $A_{\eta\frown\langle i\rangle} \cap A_{\eta\frown\langle j\rangle} = 0$. Let $A_{\langle\,\rangle} = A$, $B_{\langle\,\rangle} = 0$ and $\lambda_{\langle\,\rangle} = \lambda$. In general suppose B_η, A_η and λ_η have been defined and $\lambda_\eta > \omega$. Write $A_\eta = \cup A_i (i < \lambda_\eta)$ (continuous) so that for all i, $|A_i| < \lambda_\eta$; $A_{i+1}\backslash A_i/\bar{B}_\eta \cup B_i$ is always $|A_{i+1}\backslash A_i|$-free and free iff $A\backslash A_i/\bar{B}_\eta \cup B_i$ is λ_η-free. Here $B_i = \cup A_i$. Let $\eta \frown \langle i\rangle \in S$ iff $A_{i+1}\backslash A_i/\bar{B}_\eta \cup B_i$ is not free. In which case let $B_{\eta\frown\langle i\rangle} = B_i$ and $A_{\eta\frown\langle i\rangle} = A_{i+1}\backslash A_i$. Suppose $\eta \in S_f$ and let $s_\eta = \cup A_\eta \cap \bar{B}_\eta$.

View $\{s_\eta\colon \eta \in S_f\}$ as an indexed family (i.e., we view s_η as different from $s_{\eta'}$ if $\eta \neq \eta'$ even if they are equal as sets). We now show $\{s_\eta\colon \eta \in S_f\}$ is λ-free. Suppose $I \subset S_f$ and $|I| < \lambda$. Let g be a transversal for $\bigcup_{\eta\in I} A_\eta$. For each $\eta \in S_f$ $A_\eta/\bar{B}_\eta$ is not free. Hence there is $a \in A_\eta$ so that $g(a) \in S_\eta$. By the construction if $\eta \neq \eta'$ $(\in S_f)$ $A_\eta \cap A_{\eta'} = 0$. So if we let $f(s_\eta) = g(a)$ for some $a \in A_\eta$ so that $g(a) \in s_\eta$, then f is a transversal for $\{s_\eta\colon \eta\ I\}$.

There is one final difficulty. It is possible for $s_\eta = s_{\eta'}$ for some $\eta \neq \eta'$. We can assume $\lambda > \omega_1$, since the result is true for ω_1. By the above paragraph for any countable set $s|\{\eta\colon s = s_\eta\}| \leq \omega$. So we can modify $B_{\langle i_0\rangle}$ by adding ω new elements and using them to distinguish equal s_η. Here i_0 the least i so that $\langle i\rangle \in S$.

$(B) \Rightarrow (C)$. Rather than using (B) we will use the somewhat stronger conclusion to Proposition 3, which is provable from (B) or (A). So assume $\langle S, B_\eta, \lambda_\eta\colon \eta \in S\rangle$ is a λ-system, $\{s_\eta\colon \eta \in S_f\}$ is such that every subset of cardinality $<\lambda$ has a large transversal; and for all $\eta \in S_f$ $s_\eta \subseteq \bar{B}_\eta$. Let $t_\eta\colon \omega \to s_\eta$ be an enumeration of s_η. Let A be the Abelian group generated by $B = \bigcup_{\eta\in S} B_\eta$ and $\{a_\eta^n\colon n < \omega,\ \eta \in S_f\}$ subject to the relations $2a_\eta^{n+1} = a_\eta^n - t_\eta(n)$ $(n < \omega$, $\eta \in S_f)$. This group can be realized as a free product with amalgamation of the group freely generated by B and the groups freely generated $\{a_\eta^n\colon n < \omega\}$ $(\eta \in S_f)$ where for all $\eta \in S_f$ the subgroup $\langle t_\eta(n)\colon n < \omega\rangle$ is identified with the subgroup $\langle a_\eta^n - 2a_\eta^{n+1}\colon n < \omega\rangle$ via $t_\eta(n) \to a_\eta^n - 2a_\eta^{n+1}$ (cf. ([4] 3.6) for a similar construction).

To show A is λ-free we will use the following simple proposition.

Proposition 5 *Suppose* $\{a_n\colon n < \omega\}$ *freely generates a group and* $b_n = a_n - 2a_{n+1}$ $(n < \omega)$. *If* $I \subseteq \omega$ *is infinite, then* $\{b_n\colon n \notin I\} \cup \{a_n\colon n \in I\}$ *freely generates* $\langle a_n\colon n < \omega\rangle$.

Proof: Suppose $m \in I$. Then $\langle\{b_n\colon n \notin I,\ n < m\} \cup \{a_n\colon n \in I,\ n \le m\}\rangle = \langle a_0, \ldots, a_m\rangle$. By the Hopfian property of Abelian groups if m elements generate a free Abelian group of rank m, they freely generate it.

Suppose that $C \subseteq A$ is a subgroup of cardinality $<\lambda$. Choose $I \subseteq S_f$ so that $|I| < \lambda$ and C is contained in the subgroup D generated by B and $\{a^n_\eta\colon \eta \in I\}$. Let $\{s^*_\eta\colon \eta \in I\}$ be a large transversal for $\{s_\eta\colon \eta \in I\}$. It is not hard to show (cf. [4] 3.6) that D is freely generated by $\bigcup_{\eta\in I}\{a^n_\eta\colon t_\eta(n) \in s^*_\eta\} \cup \left(B \setminus \bigcup_{\eta\in I} s^*_\eta\right)$.

Suppose now that A is free. For $i < \lambda$ define A_i to be the group generated by $\cup B_\eta(\eta(0) < i) \cup \{a^n_\eta\colon n < \omega,\ \eta(0) < i\}$. Since $\{i\colon A_{i+1}/A_i \text{ is free}\}$ is a cub, we can choose i_0 so that $\langle i_0\rangle \in S$ and i_0 is the limit of $\{i < i_0\colon \langle i\rangle \in S\}$. Suppose $\langle i_0\rangle = \eta \in S_f$. Since $B_{\langle i_0\rangle} = \bigcup_{i<i_0} B_{\langle i\rangle} \subseteq A_{i_0}$, a^0_η is infinitely divisible by 2 mod A_{i_0}. Since $a^0_\eta \notin A_{i_0}$, this is a contradiction. Continuing we can choose an infinite path through S. This is a contradiction, so A is not free.

Remark: This construction works equally well to construct a λ-free group G of cardinality λ. This group cannot be free, since G/G' is not free (G' is the commutator subgroup).

$(C) \Rightarrow (A)$: Suppose A is a λ-free Abelian group, $|A| = \lambda$ and A is not free. We will define a labeled tree $\langle S, B_\eta, A_\eta\lambda_n\colon \eta \in S\rangle$ so that: $\langle S, B_\eta, \lambda_n\colon \eta \in S\rangle$ is a λ-system; for all $\eta \in S$, B_η and A_η are subgroups of A; for all η, $\lambda_\eta = |A_\eta|$; for all η, $A_\eta/\bar{B}_\eta$ is $|A_\eta|$-free but not free. (Recall A/B is free if $\langle A \cup B\rangle/\langle B\rangle$ is free as an Abelian group.) Let $A_{\langle\,\rangle} = A$, $B_{\langle\,\rangle} = \langle 0\rangle$, and $\lambda_{\langle\,\rangle} = \lambda$. In general suppose B_η, A_η and λ_η have been defined and $\lambda_\eta > \omega$. Write $A_\eta = \cup A_i(i < \lambda_\eta)$ (continuous) so that for all i: $|A_i| < \lambda_\eta$; $A_{i+1}/\bar{B}_\eta \cup A_i$ is always $|A_{i+1}/A_i|$-free and is free iff $A_\eta/\bar{B}_\eta \cup A_i$ is λ_η-free. Let $\eta \frown \langle i\rangle \in S$ if $A_{i+1}/\bar{B}_\eta \cup A_i$ is not free. In which case let $B_{\eta\frown\langle i\rangle} = A_i$ and choose $A_{\eta\frown\langle i\rangle} \subseteq A_{i+1}$ so that $A_{\eta\frown\langle i\rangle}/\bar{B}_\eta \cup A_i$ is $|A_{\eta\frown\langle i\rangle}|$free but not free. We can choose the A_η and B_η to be subgroups. Note: if $\eta,\tau \in S_f$ and $\eta < \tau$ (lexicographically) then $\langle A_\eta \cup \bar{B}_\eta\rangle \subseteq \langle \bar{B}_\tau\rangle$.

For $\eta \in S_f$ choose $s_\eta \subseteq \bar{B}_\eta$ so that $A_\eta \cap \langle \bar{B}_\eta\rangle \subseteq \langle s_\eta\rangle$. Let $t_\eta = s_\eta \times \omega$. By Proposition 2 $\{t_\eta\colon \eta \in S_f\}$ does not have a transversal. We now need to show for any $I \subseteq S_f$ if $|I| < \lambda$ then $\{t_\eta\colon \eta \in I\}$ has a transversal. To simplify notation for $\eta \in S_f$ let $C_\eta = \langle \bar{B}_\eta\rangle$ and $D_\eta = \langle C_\eta \cup A_\eta\rangle$. By a previous remark $\{E_\eta = D_\eta \setminus C_\eta\colon \eta \in S_f\}$ is a pairwise disjoint family of sets. Now pick F a free subgroup of A so that for all $\eta \in I$ $D_\eta \subseteq F$. Choose X a free basis for F.

Now we introduce some ad hoc terminology and note a few facts. If $Y \subseteq X$ say η *depends* on Y if $E_\eta \cap \langle C_\eta \cup Y\rangle \neq 0$. Otherwise η is *independent* of Y.

Fact A *Suppose for some $J \subseteq S_f$ and $Y \subseteq X$ each $\eta \in J$ is independent of Y. If $Y' \subseteq X$ is countable then $|\{\eta \in J\colon \eta$ depends on $Y \cup Y'\}| \le \omega$.*

Proof: In fact, if $a \in \langle Y'\rangle$ then there is at most one $\eta \in J$ such that there exists $e_1 \in E_\eta$, $c_1 \in C_\eta$ and $b_1 \in \langle Y\rangle$ so that $e_1 = a + c_1 + b_1$. Assume for $\tau \in J$ $\tau \neq \eta$ there exist $e_2 \in E_\tau$, $c_2 \in C_\tau$ and $b_2 \in \langle Y\rangle$ so that $e_2 = a + c_2 + b_2$. We can suppose $\tau < \eta$. So $D_\tau \subseteq C_\eta$. By subtracting we have $e_1 = (e_2 + c_1 - c_2) + (b_1 - b_2)$. So η is not independent of Y.

Fact B *Suppose $Y_\alpha (\alpha < \beta)$ is an increasing sequence of subsets of X. If for all $\alpha < \beta$ and $\eta \in J \subseteq S_f$ η is independent of Y_α, then for all $\eta \in J$, η is independent of $\bigcup_{\alpha<\beta} Y_\alpha$.*

Using these two facts we can write $X = \bigcup_{\alpha<\beta} X_\alpha$ so that: $X_0 = 0$; for all α, $|X_{\alpha+1} \backslash X_\alpha| = \omega$; if η depends on X_α then $s_\eta \subseteq \langle X_\alpha \rangle$.

Claim *If η is independent of X_α then $s_\eta \not\subseteq \langle X_\alpha \rangle$.*

Proof (of Claim): First $A_\eta + \langle X_\alpha \rangle / \langle X_\alpha \rangle \subseteq F / \langle X_\alpha \rangle$. So $A_\eta + \langle X_\alpha \rangle / \langle X_\alpha \rangle$ is free. Now $A_\eta \cap \langle s_\eta \rangle = A_\eta \cap C_\eta = A_\eta \cap \langle C_\eta \cup X_\alpha \rangle \supseteq A_\eta \cap \langle s_\eta \cup X_\alpha \rangle$. So $A_\eta + \langle s_\eta \cup X_\alpha \rangle / \langle s_\eta \cup X_\alpha \rangle \cong A_\eta / A_\eta \cap \langle s_\eta \rangle$ is not free.

If $s_\eta \subseteq \langle X_\alpha \rangle$ then $|t_\eta \backslash \langle X_\alpha \rangle \times \omega| = \omega$. Now we pick a transversal for $\{t_\eta : \eta \in I\}$ by induction on $\alpha < \beta$. For each $\alpha < \beta$ choose a transversal g_α for $\{t_\eta : \eta$ depends on $X_{\alpha+1}$ and is independent of $X_\alpha\}$ so that for all such η, $g_\alpha(t_\eta) \notin \langle X_\alpha \rangle \times \omega$. Then $\bigcup_{\alpha<\beta} g_\alpha$ is the desired transversal.

Corollary *If there is a λ-free Abelian group of cardinality λ which is not free, then there is a λ-free group of cardinality λ which is not free.*

Remark: The proof that $(C) \Rightarrow (A)$ can be given an axiomatic treatment.

Axiom XVII *If $|A| \leq \omega$ then for all B there is a countable $B' \subseteq B$ so that A is free over B/B'.*

Axiom XVIII (Existence of a free basis) *If A is free there is $X \subseteq A$ and $\langle\ \rangle$ a closure operation on X: i.e., for all $Y \subseteq X$, $Y \subseteq \langle Y \rangle = U \langle Z \rangle$ (Z a finite subset Y), such that:*

(a) $\langle X \rangle = A$
(b) *for all $Y \subseteq X$, A/Y is free*
(c) *suppose B, $C \subseteq A$, $Y \subseteq X$ and B is free over Y/C, then B is free over $\langle Y \rangle / C$.*

Such an X is called a basis for A.

Theorem (Axioms I**, XVII, XVIII). *If there is a λ-free nonfree A of cardinality λ, then $PT(\lambda)$.*

Proof: The proof follows the proof above that $(C) \Rightarrow (A)$. Define $\langle S, B_\eta, A_\eta, \lambda_\eta : \eta \in S \rangle$ as above. For $\eta \in S_f$ choose a countable $s_\eta \subseteq \bar{B}_\eta$ so that A_η is free over $\bar{B}_\eta / s_\eta$. As before we let $t_\eta = s_\eta \times \omega$ and show: if $I \subseteq S_f$ and $|I| < \lambda$ then $\{t_\eta : \eta \in I\}$ has a transversal. Pick $F \subseteq A$ free so that $F \supseteq \bar{B}_\eta$ for all $\eta \in I$. Let X be a basis for F. For $\eta \in S_f$ define η *depends* on Y (for $Y \subseteq X$) if A_η is not free over Y/B_η. We now must prove Fact A.

Proof (Fact A): Suppose not. So there is $J' \subseteq J$ such that: for all $\eta \in J'$ η depends on $Y \cup Y'$; and the order type of J' is ω_1. By Axiom XVII and monotonicity there is a countable $J'' \subseteq J'$ so that $\bigcup_{\eta \in J'} (A_\eta \cup \bar{B}_\eta) \cup Y$ is free over $Y'/\bigcup_{\eta \in J''} (A_\eta \cup \bar{B}_\eta) \cup Y$. Pick τ so that for all $\eta \in J''$, $\tau > \eta$. Since $\bar{B}_\tau \supseteq A_\eta \cup \bar{B}_\eta$ for all $\eta \in J''$, A_τ is free over $Y'/B_\tau \cup Y$. So A_τ is free over $Y \cup Y'/\bar{B}_\tau \cup Y$. But A_τ is free over $\bar{B}_\tau \cup Y/\bar{B}_\tau$. So by transitivity A_τ is free over $Y \cup Y'/\bar{B}_\tau$.

To finish the proof we need

Claim *For all $\eta \in S_f$ and $Y \subseteq X$ if η is independent of Y then $s_\eta \not\subseteq \langle Y \rangle$.*

Proof: By I** and XVIII A_η/Y is free. Also A_η is free over $\langle Y \rangle/Y$. Hence (XIV), $A/\langle Y \rangle$ is free. As A_η is free over $\bar{B}_\eta/s_\eta$ and A_η is free over $Y/\bar{B}_\eta$, A_η is free over Y/s_η (transitivity). So by XVIII A_η is free over $\langle Y \rangle/s_\eta$. By Axiom XIV A_η/B_η is free iff A_η/s_η is free iff $A_\eta/\langle Y \rangle \cup s_\eta$ is free. But A_η/B_η is not free. Hence neither is $A_\eta/\langle Y \rangle \cup s_\eta$. So $s_\eta \not\subseteq \langle Y \rangle$.

NOTES

1. The reader is advised to skip the proof of 1.6 and maybe the content of Definition 1.4. This certainly will have no effect on reading Sections 3, 4, and 5.
2. Remember $|W'| \leq |J| < \lambda(\rho, S)$.

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Institute of Advanced Studies
Jerusalem, Israel

and

Institute of Mathematics
The Hebrew University
Jerusalem, Israel