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Prime Spectrum of a Tetravalent Modal Algebra

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1 Introduction Tetravalent modal algebras were introduced by Monteiro in 1978 as an example of DeMorgan algebras. They also provide a very interesting generalization of the three-valued Łukasiewicz algebras. The aim of this paper is to characterize the prime spectrum of a tetravalent modal algebra.

2 Tetravalent modal algebras Let us consider the following definition.

2.1 Definition: A tetravalent modal algebra $(A, \wedge, \vee, \sim, \nabla, 1)$, or simply A, is an algebra of type (2,2,1,1,0) which satisfies the following axioms:

A1 $x \wedge (x \vee y) = x$
A2 $x \wedge (y \vee z) = (z \wedge x) \vee (y \wedge x)$
A3 $\sim\sim x = x$
A4 $\sim(x \wedge y) = \sim x \vee \sim y$
A5 $\sim x \vee \nabla x = 1$
A6 $x \wedge \sim x = \sim x \wedge \nabla x$.

It immediately follows that A is a distributive lattice [7] and a DeMorgan algebra [4], [5].

We assume that the reader is familiar with the basic notions of lattice theory.

Of the properties which can be derived from the definition axioms, the following should be retained, since it will be needed later:

B $x \leqslant \nabla x$ (cf. [2])

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A tetravalent modal algebra A that verifies the axiom A7 $\nabla(x \wedge y) = \nabla x \wedge \nabla y$ is a three-valued Łukasiewicz algebra [6].

Following A. Bialynicki-Birula and H. Rasiowa [1], for each prime filter P of A (i.e., each prime filter of the subjacent lattice) we define the prime filter $\Phi(P) = A \backslash {\sim}P$, where $\backslash$ denotes the set-theoretical complement and ${\sim}P = \{{\sim}x : x \in P\}$.

It is easily checked that:

C1 $\Phi(\Phi(P)) = P$ for each prime filter P of A.
C2 If P and Q are both prime filters of A such that $P \subseteq Q$, then $\Phi(Q) \subseteq \Phi(P)$.

I will call this correspondence Φ the *Birula-Rasiowa transformation associated with A.*

2.2 Lemma *Let A be a tetravalent modal algebra, $a \in A$. If P is a prime filter in A, then $\nabla a \in P$ iff $a \in P$ or $a \in \Phi(P)$.*

Proof: $\Rightarrow$: Let $\nabla a \in P$ and suppose $a \notin \Phi(P)$. It follows ${\sim}a \in P$. By using Axiom A6 we have ${\sim}a \wedge \nabla a = {\sim}a \wedge a \in P$, which implies $a \in P$.

$\Leftarrow$: If $a \in P$, from Condition B we get $\nabla a \in P$.

If $a \in \Phi(P)$, then ${\sim}a \notin P$ and since $1 \in P$ and P is a prime filter, from Axiom A5 it follows that $\nabla a \in P$.

2.3 Definition (A. Monteiro) The *prime spectrum* of a tetravalent modal algebra A is the couple $\langle \pi, \Phi \rangle$ where π is the set of all prime filters of A and Φ is the Birula-Rasiowa transformation associated with A.

We are going to describe briefly a representation of a DeMorgan algebra in terms of ordered topological spaces established in [3] and [8].

If A is a DeMorgan algebra, the dual space $\mathscr{J}(A)$ is defined as follows: (a) π is the set of all prime filters of A; (b) if $a \in A$, set $\sigma(a) = \{P \in \pi : a \in P\}$ and $\mathscr{T}$ is the topology having as a subbase the sets of the form $\sigma(a)$ and $\pi \backslash \sigma(a)$ for each $a \in A$; (c) π is ordered by the set-theoretical inclusion; and (d) Φ is the Birula-Rasiowa transformation associated with A.

By [3] $\mathscr{J}(A)$ is a compact totally order disconnected space and Φ is a continuous decreasing map from π into π. Thus, following [8], $\mathscr{J}(A)$ is an Ockham space. Therefore by Theorem 1 of [8], A is isomorphic to the dual lattice $D(A)$ of $\mathscr{J}(A)$; i.e., the lattice of all clopen increasing subsets of $\mathscr{J}(A)$, with the definition, for each $\alpha \in D(A)$:

D ${\sim}\alpha = \Phi^{-1}[\pi \backslash \alpha] = \{P \in \pi : \Phi(P) \notin \alpha\}$.

2.4 Lemma *If A is a DeMorgan algebra, then A can be expanded to a tetravalent modal algebra iff its prime spectrum satisfies the condition:*

R $X \subseteq Y \Rightarrow (\Phi(X) = Y$ or $X = Y)$.

Proof: $\Rightarrow$: Consider A a tetravalent modal algebra and let $X, Y \in \pi$ such that:

(a) $X \subseteq Y$ and $X \neq Y$.

From (a) we get:

(b) $\Phi(Y) \subset \Phi(X)$, where $\subset$ means strict inclusion.

From Lemma 2.2 and (a) it immediately follows:

(c) $\Phi(X) \subseteq Y \cup \Phi(Y)$.

It is easy to check that (c) implies either:

(d) $\Phi(X) \subseteq Y$

or

(e) $\Phi(X) \subseteq \Phi(Y)$.

Since (e) contradicts (b) then (d) holds. Similarly from Lemma 2.2, (b), and C1 we get $Y \subseteq X \cup \Phi(X)$ which implies either:

(f) $Y \subseteq X$

or

(g) $Y \subseteq \Phi(X)$.

Since (f) contradicts (a), we have (g). From (d) and (g) it follows $\Phi(X) = Y$ and condition R is satisfied.

$\Leftarrow$: Consider A a DeMorgan algebra such that its prime spectrum satisfies the condition R. Let $D(A)$ be the lattice of all clopen increasing subsets of $\mathcal{J}(A)$. As we have stated, A is isomorphic to $D(A)$. For each $\alpha \in D(A)$ we define:

(a) $\nabla\alpha = \alpha \cup \Phi[\alpha]$.

Since $\Phi[\alpha] = \{\Phi(P): P \in \alpha\}$, from D and C1 it easily follows:

(b) $\Phi[\alpha] = \pi \backslash {\sim}\alpha$.

Since α is clopen, then ${\sim}\alpha$ is clopen, hence by (b), $\Phi[\alpha]$ is clopen. Thus $\nabla\alpha = \alpha \cup \Phi[\alpha]$ is clopen.

Let us prove now that $\nabla\alpha$ is increasing. Let $P, Q \in \pi$ such that $P \in \nabla\alpha$ and $P \subseteq Q$. By condition R it follows that either:

(c) $P = Q$

or

(d) $Q = \Phi(P)$.

From (c) we get $Q \in \nabla\alpha$. From (d) and C1 we obtain:

$$Q \in \Phi[\nabla\alpha] = \Phi[\alpha \cup \Phi[\alpha]] = \alpha \cup \Phi[\alpha] = \nabla\alpha.$$

Thus $\nabla\alpha$ is increasing and we have $\nabla\alpha \in D(A)$. Finally it is easily checked that the operation ∇ defined in (a) over $D(A)$ satisfies Axioms A5 and A6. Therefore A can be expanded to a tetravalent modal algebra.

3 ***Characterization of the prime spectrum of a tetravalent modal algebra*** In order to be able to characterize the prime spectrum of A, we need to introduce the following ordered set theory definitions of Monteiro.

3.1 Definition Let $\beta(\leqslant)$ be an ordered set. We write $a \| b$; i.e., *a and b are comparable* ($a, b \in \beta$) if $a \leqslant b$ or $b \leqslant a$.

If this condition is not satisfied we say a, b are not comparable.

3.2 Definition We say that *$a \in \beta$ and $b \in \beta$ are joined* ($a \approx b$) if there is a finite sequence $x_1, \ldots, x_n$ of elements of β such that $x_i \| x_{i+1} (i = 1, \ldots, n - 1)$, $x_1 = a$ and $x_n = b$.

3.3 Remark: It is easy to check that $\approx$ is an equivalence relation in β.

3.4 Definition The equivalence classes $|a| (a \in \beta)$ for the relation $\approx$ are called the *connected components of* β.

Now let A be a tetravalent modal algebra whose prime spectrum is $\langle \pi, \Phi \rangle$.

We consider the set π ordered by set-theoretical inclusion.

For each $P \in \pi$, it is well known that if there is an ultrafilter U of A such that $P \subseteq U$, then by Definition 3.1, $P \| U$ (in $\pi(\subseteq)$) and therefore $|P| = |U|$. For this reason, to know each connected component of $\pi(\subseteq)$ it is sufficient to determine the equivalence class of each ultrafilter U of A.

We have the following result:

3.5 Proposition *If $K \subseteq \pi$ is a connected component of $\pi(\subseteq)$, then the set $\Phi(K) = \{\Phi(P_i)\}_{P_i \in K}$ is also a connected component of π.*

Proof: Let $K \subseteq \pi$ be a connected component of $\pi(\subseteq)$. We are going to prove that the set $\Phi(K) = \{\Phi(P_i)\}_{(P_i \in K)}$ is an equivalence class for the relation $\approx$ of Definition 3.2; i.e., that $\Phi(K)$ verifies the following conditions:

(1) $\Phi(P_i), \Phi(P_j) \in \Phi(K)$ implies that $\Phi(P_i) \approx \Phi(P_j)$.

(2) $Q \in \pi$ and $Q \approx \Phi(P_i) \in \Phi(K)$ imply that $Q \in \Phi(K)$.

For (1), let $\Phi(P_i), \Phi(P_j) \in \Phi(K)$. Since K is a connected component of $\pi(\subseteq)$ we have $P_i \approx P_j$; i.e., there is a finite sequence $P'_1, \ldots, P'_n$ of elements of K such that $P'_\gamma \| P'_{\gamma+1} (\gamma = 1, \ldots, n - 1)$, $P'_1 = P_i$ and $P'_n = P_j$.

The above conditions together with property C2 of Section 2, imply that $\Phi(P_i) \approx \Phi(P_j)$.

For (2), let $Q \in \pi$ and assume that $Q \approx \Phi(P_i) \in \Phi(K)$. By the above result (1) and property C1 of Section 2, it follows that $\Phi(Q) \approx \Phi(\Phi(P_i)) = P_i$; i.e., $\Phi(Q) \in K$ and therefore $Q = \Phi(\Phi(Q)) \in \Phi(K)$.

3.6 Definition (A. Monteiro) The sets $K \cup \Phi(K)$ where K is a connected component of $\pi(\subseteq)$, are called *Φ-connected components of* $\langle \pi, \Phi \rangle$.

3.7 Remark: If K is a connected component of $\pi(\subseteq)$ and $K \cap \Phi(K) \neq \phi$, then $K = \Phi(K)$ and it is a Φ-connected component of $\langle \pi, \Phi \rangle$.

According to our aim, we shall prove the main result of this paper:

3.8 Theorem *A DeMorgan algebra A can be expanded to a tetravalent modal algebra iff the Φ-connected components of its prime spectrum $\langle \pi, \Phi \rangle$ are of the following types:*

Type I: $\{U, \Phi(U)\}$ *where U and $\Phi(U)$ are not comparable and they are both ultrafilters and minimal prime filters.*
Type II: $\{U\}$ *where $U = \Phi(U)$ is an ultrafilter and a minimal prime filter.*
Type III: $\{U, \Phi(U)\}$ *where $\Phi(U) \subset U$, U is an ultrafilter, and $\Phi(U)$ is a minimal prime filter.*

Proof: $\Rightarrow$: By Lemma 2.4, $\langle \pi, \Phi \rangle$ satisfies condition R.

We have seen that it is sufficient to determine the Φ-connected components that contain the ultrafilters of A. Let U be an ultrafilter of A and suppose that U and $\Phi(U)$ are not comparable. If $\Phi(U)$ were not an ultrafilter of A, then there would exist a prime filter P of A such that:

(a) $\Phi(U) \subset P$.

From (a) and condition R it follows that $P = \Phi(\Phi(U)) = U$ and so $\Phi(U) \subset U$ which contradicts the hypothesis, therefore $\Phi(U)$ is an ultrafilter of A.

Similarly it is proved that $\Phi(U)$ is a minimal prime filter, using condition R. Thus $\{\Phi(U)\}$ is a connected component of $\pi(\subseteq)$ because $\Phi(U)$ is the only prime filter comparable with itself. Hence, by Proposition 3.5, $\{U\} = \{\Phi(\Phi(U))\}$ is also a connected component of π. Therefore, in this case $\{U, \Phi(U)\}$ is a Φ-connected component of Type I.

If (b) $U = \Phi(U)$ or (c) $\Phi(U) \subset U$, it can be proved that $\Phi(U)$ is a minimal prime filter, in a similar way.

If we have (b) then $\{U\}$ is a Φ-connected component of Type II.

If we have (c), it is easily proved that there is no prime filter P such that $\Phi(U) \subset P \subset U$, using again condition R. Therefore $\Phi(U)$ is the only prime filter strictly contained in U and U is the only one which strictly contains $\Phi(U)$. Thus $\{U, \Phi(U)\}$ is a Φ-connected component of Type III.

$\Leftarrow$: It follows straightforwardly that if these conditions hold, then the prime spectrum of the DeMorgan algebra A satisfies condition R of Lemma 2.4 and thus A can be expanded to a tetravalent modal algebra by the same lemma.

3.9 Corollary *The Φ-connected components of the prime spectrum of any tetravalent modal algebra A are of Types I, II, III of the previous theorem.*

REFERENCES

[1] Bialynicki-Birula, A. and H. Rasiowa, "On the representation of quasi-Boolean algebras," *Bulletin de l'Académie Polonaise des Sciences, CI, III,* vol. 5 (1957), pp. 259-261.

[2] Cignoli, R. and A. Monteiro, "Boolean elements in Łukasiewicz algebras. II," *Proceedings of the Japan Academy,* vol. 41 (1965), pp. 676-680.

[3] Cornish, W. H. and P. R. Fowler, "Coproducts of DeMorgan algebras," *Bulletin of the Australian Mathematical Society,* vol. 16 (1977), pp. 1-13.

[4] Moisil, G., "Recherches sur l'algèbre de la logique," *Annales Scientifiques de l'Université de Jassy,* vol. 22 (1935), pp. 1-117.

[5] Monteiro, A., "Matrices de Morgan caractéristiques pour le calcul propositionnel classique," *Anais da Academia Brasileira de Ciências,* vol. 32, no. 1 (1960), pp. 1-7.

[6] Monteiro, L., "Axiomes indépendants pour les algèbres de Łukasiewicz trivalentes," *Bulletin de la Société des Mathématicians et des Physiciens de la République Populaire de Roumanie, Nouvelle Serie,* Tome 7 (55) (1963), pp. 199-202.

[7] Sholander, M., "Postulates for distributive lattices," *Canadian Journal of Mathematics,* vol. 3 (1951), pp. 28-30.

[8] Urquhart, A., "Distributive lattices with a dual homomorphic operation," *Studia Logica,* vol. 38, no. 2 (1979), pp. 201-209.

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