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Sequential Compactness and the Axiom of Choice

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A theorem is effective iff it is proved in ZF^o, where ZF^o is Zermelo-Fraenkel set theory without the axioms of choice and foundation (regularity). A well known effective theorem of F. Riesz states that a Hilbert space is finite dimensional iff its closed unit ball is compact. This may fail for sequential compactness. If U is a set of urelements, equipped with the structure of l_2, I is the ideal of all finite subsets of U and G is the group of unitary operators, by an argument similar to [7]. In the resulting permutation model $P(U,G,I)$, each orthonormal (= ON) system in U is finite. Therefore U is locally sequentially compact, but there is no ON base for U. A similar situation holds for the Dworetzky-Rogers characterisation of finite dimensional spaces ([9], Theorem 1.c.2). But in combination we get the effective result:

1 Theorem *In ZF^o the following statements are equivalent for a Hilbert space H:*

(a) *The closed unit ball is sequentially compact.*
(b) *Each unconditionally convergent series converges absolutely.*
(c) *Each ON-system is Dedekind-finite.*

Proof: In ZF^o a first countable Hausdorff space X is sequentially compact, iff each closed, discrete set is D-finite. Though only the obvious part of this remark is used, a proof of its nontrivial implication can be given as follows:

Suppose X is not sequentially compact and $x:\underline{w} \to X$ is a sequence without any convergent subsequence. We shall show that $Im(x)$ is closed and discrete but not D-finite (Dedekind-finite). Because X is first countable, for $p \in Im(x)^-$, the closure of $Im(x)$, there is a neighborhood system $(U_k)_{k \in \underline{w}}$ and we set $n(0) = 0$, $n(k + 1) = \min N_k$ if $N_k = \{n > n(k): x_n \in U_k\} \neq \phi$. There is a k such that $N_k = \phi$. If not, the subsequence $(x(n(k)))_{k \in \underline{w}}$ of x converges to p. Hence

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$U_k \cap Im(x)$ is finite, and by T_2 and $p \in Im(x)^-$ there is an open 0, such that $0 \cap Im(x) = \{p\}$. $Im(x)$ is closed and discrete.

Since ON-systems are closed and discrete, by this observation (a) implies (c).

If $(o_n)_{n \in \underline{w}}$, where $\underline{w}$ if the set of finite ordinal numbers, is an ON-sequence, $\sum_n \frac{1}{n} o_n$ is an unconditionally convergent series which does not converge absolutely, whence (c) follows from (b).

Conversely, an application of the Gram-Schmidt orthonormalization to any sequence $(x_n)_n$ produces an ON-sequence $(o_n)_n$ which spans $(x_n)_n$. If (c) holds, $(o_n)_n$ is finite. Since Riemann's theorem (in $\mathbf{R}^n$ a series is absolutely convergent if it converges unconditionally) and the Bolzano-Weierstrass theorem (a closed and bounded subset of $\mathbf{R}^n$ is sequentially compact) are effective (cf. [2]), then (a) and (b) follow from (c).

It is well-known, that in ZF^o a Hilbert space has an ON-base, iff it is isomorphic to $l_2(B)$ for some B (the base). If its closed unit ball is sequentially compact, B is D-finite. The converse fails, as is seen, e.g., in the Fraenkel model (U is a countable union of pairs), where each locally sequentially compact topological vector space is finite dimensional, as follows from the next argument together with the axiom of multiple choice. We next investigate the strength of the converse:

2 Theorem *The axiom $AC^{\underline{w}}_{fin}$ of choice for countable families of nonempty finite sets is effectively equivalent to the assertion that the closed unit ball of each Hilbert space with a Dedekind-finite ON-base is sequentially compact.*

Proof: Assume $AC^{\underline{w}}_{fin}$ and let D be D-finite.

The image of each $x \in l_2(D)$ is D-finite. For if $(i_n)_n$ is an injective sequence in $Im\, x$, then $x^{-1}(i_n)$ is a countable family of disjoint, nonempty, finite sets, and $AC^{\underline{w}}_{fin}$ provides us with a choice function which determines an injective sequence in D—a contradiction.

We first assume that H is a real Hilbert space. Then $Im(x)$ is a D-finite subset of $\mathbf{R}$. If $Im(x)$ is infinite, there is a cluster point c of $Im(x)$ different from zero, for otherwise $A = Im(x) \cup \{0\}$ would be an infinite, D-finite closed set of reals. This is impossible, since the complement of A is a countable union of disjoint open intervals, the left endpoints of which form a one-to-one sequence in A. If $c \neq 0$ is a cluster point of $Im(x)$, there are infinitely many $d \in D$ such that $|x(d)| > \frac{1}{2} c > 0$, whence $x \notin l_2(D)$. Therefore $Im(x)$ is finite for $x \in l_2(D)$ and so x has a finite support $s(x) = \{d \in D\colon x(d) \neq \phi\}$.

If H is a complex Hilbert space the above argument shows that $Im|x|$ is finite, where $|x|(d) = |x(d)|$. Hence if $Im(x)$ is infinite, $A_r = \{x(d)\colon |x(d)| = r\}$ is infinite for some $r > 0$ and $x \notin l_2(D)$.

If $(x_n)_n$ is a sequence in $l_2(D)$, $\bigcup_n s(x_n)$ is finite. Otherwise we could construct a one-to-one sequence in D (using $AC^{\underline{w}}_{fin}$), which is impossible. Since the Bolzano-Weierstrass theorem holds in finite dimensional spaces, $l_2(D)$ is locally sequentially compact.

Assume conversely, that each D-finite dimensional Hilbert space is locally sequentially compact.

Then each countable family $(F_n)_n$ of nonempty finite sets contains an infinite subfamily with a choice function. We may assume that $(F_n)_n$ is a disjointed family. If no infinite subset of $(F_n)_n$ has a choice function, $D = \bigcup_n F_n$ is an infinite, D-finite set. Let y_n be the characteristic function of the set F_n. In $l_2(D)$ $x_n = \frac{y_n}{\|y_n\|}$, $n \in \underline{w}$, forms an ON sequence, contradicting our hypothesis.

From this remark it follows that each sequence $(F_n)_n$ of finite sets $F_n \neq \phi$ has a choice function. Let C_N be the set of all choice functions on $(F_n)_{n \in N}$, $N \in \underline{w}$. Since each F_n is finite, C_N is finite and $(C_N)_{N \in \underline{w}}$ has a "partial choice function": $f_{n(k)} \in C_{n(k)}$, $n(k) < n(k+1)$. We define a choice function f on $(F_n)_n$ by $f(F_n) = f_{n(k+1)}(F_n)$, $n(k) \leqslant n < n(k+1)$.

In the preceding proof we were led to consider a partial axiom of choice. PAC is the axiom that each infinite family of nonempty sets has an infinite subfamily with a choice function. $PAC^{\underline{w}}$ is PAC for countable families, PAC_{fin} is PAC for families of finite sets, and $PAC^{\underline{w}}_{fin}$ is equivalent to $AC^{\underline{w}}_{fin}$. Since under $AC^{\underline{w}}$ each infinite set has a countable subset, PAC follows from $AC^{\underline{w}}$. Conversely as in 2, PAC implies $PAC^{\underline{w}}$ from which in turn $AC^{\underline{w}}$ follows. Clearly PAC_{fin} follows from AC_{fin}, but the converse is not true:

3 *Example:* PAC_{fin} does not imply AC_{fin} or $AC^{\underline{w}}$ effectively, and $AC^{\underline{w}}_{fin}$ does not imply PAC_{fin}.

Proof: It was shown by [1] that the Fraenkel-Halpern permutation model satisfies Ramsey's theorem, which in turn yields PAC_{fin} by [6]. It is well-known that the Fraenkel-Halpern model does not even satisfy AC_2 or $AC^{\underline{w}}$.

In the model $P(U,G,I)$ of [4], where $U = \bigcup P$, P an infinite family of disjoint pairs, G the group of permutations g such that $gP = P$, and I the ideal of finite sets, PAC_{fin} fails, while it follows from standard arguments that $AC^{\underline{w}}_{fin}$ holds (cf. [10], example 4.4).

There is a weaker notion of effectiveness permitting the use of the axiom of foundation. It follows from the transfer results of Pincus (cf. [5]) together with the remarks of [1] that in Example 3 this weaker version of effectiveness may be substituted.

PAC_{fin} was first defined by Kleinberg [6]; $PAC^{\underline{w}}$ was discovered by Hickman (cf. [8]) who showed that it implies D-finite sets are finite.

REFERENCES

[1] Blass, A., "Ramsey's theorem," *The Journal of Symbolic Logic,* vol. 42 (1979), pp. 387-390.

[2] Dunford, N. and J. T. Schwartz, *Linear Operators I,* Interscience Publishers, New York, 1957.

[3] Felgner, U., *Models of ZF,* Springer-Verlag, New York, 1971.

[4] Hickman, J. L., "Groups in models of set theory," *Bulletin of the Australian Mathematical Society,* vol. 14 (1976), pp. 199-237.

[5] Jech, T. J., *The Axiom of Choice,* American Elsevier Publishing Company, New York, 1973

[6] Kleinberg, E. M., "Ramsey's theorem," *The Journal of Symbolic Logic,* vol. 34 (1969), pp. 205-206.

[7] Läuchli, H., "AC in der algebra," *Commentarii Mathematici Helvetici,* vol. 37 (1963), pp. 1-16.

[8] Munro, G. P., "D-finite sets," *Journal of the Australian Mathematical Society,* vol. 19 (1975), pp. 35-46.

[9] Tzafriri, L. and J. Lindenstrauss, *Classical Banach Spaces I,* Springer-Verlag, New York, 1977.

[10] Brunner, N., "D. Endlichkeit," *Monatshefte für Mathematik,* vol. 94 (1982), pp. 9-31.

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