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Generalized Archimedean Fields

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We consider (linearly) ordered fields, F (actually $\langle F, +, x, 0, 1, \leqslant\rangle$). A subset $S \subseteq F$ is *positive* if $x > 0$ for $x \in S$. S is *separated* (of size a) if $a > 0$ and $|x - y| > a$ whenever $x, y \in X$. An easy exercise shows that F is Archimedean if and only if: (1) there is an infinite positive separated subset of F and (2) every bounded positive separated subset is finite.

Let κ be an infinite cardinal number. We define F to be *κ-Archimedean* in case: (1) F has a positive separated subset of cardinality κ, and (2) no positive separated subset of F having cardinality κ is bounded. An ordered field is Archimedean if and only if it is $\aleph_0$-Archimedean.

In [6] Sikorski gives a very natural example of a κ-Archimedean field $\mathcal{Q}_\kappa$. To construct $\mathcal{Q}_\kappa$ and related systems we use ordinal numbers with Hessenberg natural operation $\#$ (addition) and $\divideontimes$ (multiplication). Recall that an ordinal α can be uniquely written in base ω, as $\alpha = \omega^{\alpha_1}\cdot n_1 + \ldots \omega^{\alpha_s}\cdot n_S$ where $\alpha_1 > \alpha_2 > \ldots \alpha_S$ are ordinal numbers, S is finite, $n_1, \ldots, n_S$ are positive integers, "+" and "·" are ordinal addition and multiplication respectively. This representation is called Cantor normal form. The natural sum of two ordinals in Cantor form is obtained by adding them as if they were polynomials in ω. The natural product, likewise, is obtained by multiplying them as polynomials in ω, but with the provision that natural sum is used for addition of exponents. "$\#$" and "$\divideontimes$" are commutative and associative, have identities "0" and "1" respectively. "$\divideontimes$" distributes over "$\#$".

Considering κ to be the set of ordinals less than κ, we define $\mathcal{N}_\kappa = \langle \kappa, \#, \divideontimes, \leqslant\rangle$, $\mathcal{I}_\kappa$ is the ring of differences over $\mathcal{N}_\kappa$, $\mathcal{Q}_\kappa$ is the quotient field of $\mathcal{I}_\kappa$, and $\mathcal{R}_\kappa$ is the real closure of $\mathcal{Q}_\kappa$. In [6] Sikorski proves that $\mathcal{Q}_\kappa$ is a κ-Archimedean field of cardinality κ. Also every ordered field of cofinality κ contains a subfield isomorphic to $\mathcal{Q}_\kappa$.

In case κ is regular we note that any κ-Archimedean field does have cofinality κ, also that $\mathcal{R}_\kappa$ is κ-Archimedean. An Archimedean field of cardinality λ exists if and only if $\aleph_0 \leqslant \lambda \leqslant 2^{\aleph_0}$. For uncountable κ we are interested

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in determining those cardinals λ for which there exists a κ-Archimedean field of cardinality λ. In [1], p. 47, Cowles proves that if F is κ-Archimedean, then $\kappa \leqslant Card\, F \leqslant 2^{\kappa}$.

Related to the notion of κ-Archimedean we have the notion of $BW(\kappa)$ (for Bolzano-Weierstrass). An ordered field F is called $BW(\kappa)$ if: (1) F has cofinality κ, and (2) every bounded subset of F of cardinality κ, has a limit point (using open intervals as a base for open sets). In [6] and [7], Sikorski has shown that for κ regular, $\mathcal{Q}_\kappa$ and $\mathcal{R}_\kappa$ are $BW(\kappa)$ fields. In [6] he asks whether a $BW(\kappa)$ field of cardinality greater than κ exists.

Proposition *If F is a $BW(\kappa)$ field, then F is κ-Archimedian.*

Proof: Since κ is the cofinality of F, κ is regular. By a result of Sikorski, mentioned above, F contains a copy of $\mathcal{Q}_\kappa$ which contains a positive separated subset of cardinality κ. If F were not κ-Archimedean, then F would contain a bounded positive separated subset $\mathcal{J}$ of cardinality κ. $\mathcal{J}$ would not have a limit point.

In order to describe the relationship between $BW(\kappa)$ and κ-Archimedean we need some more concepts. The notation $\langle a_\alpha \rangle_{\alpha \epsilon \kappa}$ is used to denote a κ-sequence (i.e., a function whose domain is κ). The concepts *κ-Cauchy, converge, strictly monotone* are defined for κ-sequences in the usual way. F is called *κ-complete* if the cofinality of F is κ and every *κ-Cauchy* sequence from F converges. F is called *κ-Ramsey* if the cofinality of F is κ and every subset of F, having cardinality κ, contains a strictly monotone κ-sequence.

Theorem *An ordered field is $BW(\kappa)$ iff it is κ-Archimedean, κ-complete, and κ-Ramsey.*

Proof: Suppose F is κ-Archimedean, κ-Complete, and κ-Ramsey. Let S be a bounded subset of cardinality κ. Since F is κ-Ramsey, S contains a strictly monotonic (say increasing) sequence $\langle a_\alpha \rangle_{\alpha \epsilon \kappa}$.

Since F is κ-Archimedean, $\langle a_\alpha \rangle_{\alpha \epsilon \kappa}$ is κ-Cauchy: choose $a > 0$ and suppose for all $\alpha \epsilon \kappa$ that $b < a_\alpha < c$. Let I be a maximal separated subset of size $\frac{a}{2}$ such that $(\forall x \epsilon I)(b < x < c)$. Let $J = \left\{\frac{2(x-b)}{a} \,|\, x \epsilon I\right\}$. Then J is a positive separated set, and so $Card(I) = Card(J) < \kappa$. For each $x \epsilon I$, let $S_x = \left\{\alpha \epsilon \kappa \,\middle|\, |a_\alpha - x| < \frac{a}{2}\right\}$. Since I is maximal, $\kappa = \bigcup_{x \epsilon I} S_x$. Since κ is regular, some S_y has cardinality κ and is therefore cofinal with κ. Let σ be the smallest ordinal in S_y. Then for all ordinals $\gamma \geqslant \sigma$, there is a $\delta \epsilon S_y$ such that $\delta > \gamma$. Now $a_\sigma < a_\gamma < a_\delta$, $|a_\delta - y| < \frac{a}{2}$, and $|a_\sigma - y| < \frac{a}{2}$, so $|a_\gamma - y| < \frac{a}{2}$ and $\gamma \epsilon S_y$. Then for all $\beta, \gamma \geqslant \sigma$, $|a_\beta - a_\gamma| \leqslant |a_\beta - y| + |y - a_\gamma| \leqslant a$. Thus $\langle a_\alpha \rangle_{\alpha \epsilon \kappa}$ is a κ-Cauchy sequence.

Since F is κ-complete, $\langle a_\alpha \rangle_{\alpha \epsilon \kappa}$ converges and S has a limit point.

Now suppose F is $BW(\kappa)$. It has been shown that F is κ-Archimedean. To see that F is κ-complete, let $\langle a_\alpha \rangle_{\alpha \epsilon \kappa}$ be a κ-Cauchy sequence of elements from F and let $A = \{a_\alpha | \alpha \epsilon \kappa\}$.

Case 1. Suppose Card$(A) < \kappa$. Let $A_a = \{\alpha \epsilon \kappa | a_\alpha = a\}$. Then $\kappa = \bigcup_{a \epsilon A} A_a$, so at least one A_a has cardinality κ (because κ is regular) and is therefore cofinal with κ. Since $\langle a_\alpha\rangle_{\alpha\in\kappa}$ is κ-Cauchy, there is only one such A_a. Let $B = A - \{a\}$. Then $\text{Card}\left(\bigcup_{b\in B} A_b\right) < \kappa$, so $\bigcup_{b\in B} A_b$ is bounded in κ. Therefore there is a $\beta \in \kappa$ such that $(\forall\gamma > \beta)(a_\gamma = a)$, so $\langle a_\alpha\rangle_{\alpha\in\kappa}$ converges (to a).

Case 2. Suppose $\text{Card}(A) = \kappa$. Since $\langle a_\alpha\rangle_{\alpha\in\kappa}$ is κ-Cauchy, A is bounded: $(\exists\beta\in\kappa)(\forall\gamma\geqslant\beta)(|a_\beta - a_\gamma| < 1)$, so for $\gamma \geqslant \beta$, $|a_\gamma| < \max\{|a_\beta - 1|, |a_\beta + 1|\}$. Since $\text{Card}(\{|a_\beta - 1|, |a_\beta + 1|\} \cup \{|a_\delta| \mid \delta < \beta\}) < \kappa$, there is an element in the copy of $\mathcal{N}_\kappa$ contained in F which bounds $\{|a_\alpha| \mid \alpha \in \kappa\}$ from above.

Since F is $BW(\kappa)$, A has a limit point a. For $b > 0$, let $T_b = \{\alpha \in \kappa \mid a_\alpha \neq a \wedge |a_\alpha - a| < b\}$. Since a is a limit point for A, $\text{Card}(T_b) = \kappa$: If $\text{Card}(T_b) < \kappa$, then $\text{Card}\left(\left\{\frac{1}{|a_\alpha - a|} \;\middle|\; \alpha \in T_b\right\}\right) < \kappa$ and is therefore bounded above by an element c in the copy of $\mathcal{N}_\kappa$ contained in F. Then $T_{\frac{1}{c}} = \phi$, making it impossible for a to be a limit point for A.

To see that $\langle a_\alpha\rangle_{\alpha\in\kappa}$ converges (to a), let $b > 0$. Since $\langle a_\alpha\rangle_{\alpha\in\kappa}$ is κ-Cauchy, there is a $\beta \in \kappa$ such that for all $\gamma \geqslant \beta$ and all $\delta \geqslant \beta$, $|a_\gamma - a_\delta| < \frac{b}{2}$. Since $\text{Card}(T_{\frac{b}{2}}) = \kappa$, $T_{\frac{b}{2}}$ is cofinal in κ, so choose $\sigma \in T_{\frac{b}{2}}$ such that $\sigma \geqslant \beta$. Then for all $\gamma \geqslant \sigma$, $|a - a_\gamma| \leqslant |a - a_\sigma| + |a_\sigma - a_\gamma| < \frac{b}{2} + \frac{b}{2} = b$.

To see that if F is a $BW(\kappa)$-field, then F is κ-Ramsey, let A be a subset of cardinality κ.

Case I. Suppose A is not bounded. Then either $P = \{a \in A \mid a > 0\}$ or $N = \{|a| \mid a \in A \wedge a < 0\}$ is not bounded above. Suppose N is not bounded above. Then the cofinality of N is at least κ: Let $\mathcal{N}$ be the copy of $\mathcal{N}_\kappa$ contained in F. If the cofinality of N is less than κ, then let D be a cofinal subset of N with cardinality less than κ. Let $\mathcal{N}_d = \{n \in \mathcal{N} \mid n < d\}$. Then $\mathcal{N} = \bigcup_{d\in D} \mathcal{N}_d$ which contradicts the regularity of κ.

Case II. Suppose A is bounded. Since F is a $BW(\kappa)$-field, A has a limit point a. For $b > 0$, let $A_b = \{c \in A \mid 0 < |a - c| < b\}$. Since a is a limit point, $\text{Card}(A_b) \geqslant \kappa$: For if $\text{Card}(A_b) < \kappa$, then $B = \left\{\frac{1}{|a - c|} \;\middle|\; c \in A_b\right\}$ has cardinality less than κ. Let $\mathcal{N}$ be the copy of $\mathcal{N}_\kappa$ contained in F. Since κ is regular there is a $\lambda \in \mathcal{N}$ which bounds B from above. Then $A_{\frac{1}{\lambda}} = \phi$ making it impossible for a to be a limit point.

Let $L = \{c \in A \mid c < a\}$ and $U = \{c \in A \mid c > a|$. Either $H = \{\beta \in \mathcal{N} \mid \text{Card}(A_{\frac{1}{\beta}} \cap L) \geqslant \kappa\}$ has cardinality κ or $J = \{\beta \in \mathcal{N} \mid \text{Card}(A_{\frac{1}{\beta}} \cap U) \geqslant \kappa\}$ has cardinality κ. Suppose $\text{Card}(H) = \kappa$. Then H is cofinal in $\mathcal{N}$. Choose the strictly monotonic sequence $\langle h_\alpha\rangle_{\alpha\in\kappa}$ by transfinite recursion: Let $h_0 \in L$. Assume $h_\alpha \in L$ for $\alpha < \sigma \in \kappa$ have been chosen so that for all $\gamma < \sigma$ and $\delta < \sigma$, if $\gamma < \delta$, then $h_\gamma < h_\delta$. Since $\text{Card}\left(\left\{\frac{1}{a - h_\alpha} \mid \alpha < \sigma\right\}\right) < \kappa$, $\left\{\frac{1}{a - h_\alpha} \mid \alpha < \sigma\right\}$ is bounded above by some $\lambda \in \mathcal{N}$. Let β be the first element in $\mathcal{N}$ larger than or equal to λ such that

$A_{\frac{1}{\beta}} \cap L \neq \phi$. Choose $h_\sigma \epsilon A_{\frac{1}{\beta}} \cap L$. Then $a - h_\sigma < \frac{1}{\beta} \leqslant \frac{1}{\lambda} < a - h_\alpha$ for all $\alpha < \sigma$, so $h_\sigma > h_\alpha$ for all $\alpha < \sigma$.

After many unsuccessful attempts, using standard algebraic and logical techniques to construct κ-Archimedean fields with cardinality larger than κ, one has a growing suspicion that either they do not exist or at least their existence is not decided by the usual axioms (ZFC) for set theory. A possible clue appears in a recent paper [2] of Juhász and Weiss: Let κ be a cardinal, let G be an ordered abelian group with a strictly decreasing κ-sequence $\langle g_\alpha \rangle_{\alpha \epsilon \kappa}$ of positive elements converging to $0 \epsilon G$, and let S be a set. A κ-metric on S is a function d: $S \times S \to G^+ = \{g \epsilon G | g > 0\}$ such that for all r, s, and t in S,

(i) $d(s, t) = 0$ iff $s = t$,
(ii) $d(s, t) = d(t, s)$, and
(iii) $d(r, t) \leqslant d(r, s) + d(r, t)$.

A topological space is *κ-metrizable* just in case it has the topology given by some κ-metric. A topological space is *κ-compact* just in case every open cover has a subcover of cardinality less than κ. In [8], Sikorski asked if there are κ-metrizable spaces of cardinality greater than κ which are κ-compact. Juhász and Weiss prove that the existence of $\aleph_1$-compact, $\aleph_1$-metrizable spaces of cardinality greater than $\aleph_1$ is consistent with and independent of the usual axiom of set theory.

The strong connection between the Bolzano-Weierstrass theorem and the Heine-Borel theorem for space of real numbers suggests that bounded and closed intervals of $BW(\kappa)$-fields are κ-compact: An ordered field F is $HB(\kappa)$ (HB for Heine-Borel) just in case its cofinality is κ and the κ-compact subsets coincide with the closed and bounded subsets.

Theorem *An ordered field is $BW(\kappa)$ iff it is $HB(\kappa)$.*

Proof: Suppose F is a $HB(\kappa)$-field and that S is a bounded subset of F with cardinality greater than or equal to κ. Since S is bounded, S is a subset of a closed and bounded interval I. Since F is $HB(\kappa)$, I is κ-compact and S must have a limit point: Suppose that S has no limit point in F. Then for each $x \epsilon I$, there is an $a_x > 0$ such that $(\forall s \epsilon S - \{x\})(|x - s| \geqslant a_x)$. Let I_x be the open interval $(x + a_x, x - a_x) = I_x$. Then $(I_x - \{x\}) \cap S = \phi$, so $\{I_x | x \epsilon I\}$ is an open cover of I with no subcover of cardinality less than κ because no subset of $\{I_x | x \epsilon S\}$ covers S.

Now suppose that F is a $BW(\kappa)$-field. Let $\mathcal{W}$ be a copy of $\mathcal{W}_\kappa$ contained in F. Then every κ-compact subset of F is bounded: If S is not bounded, then $\{(-\alpha, \alpha) | \alpha \epsilon \mathcal{W}\}$ is an open cover of S with no subcover of cardinality less than κ.

Now let S be a κ-compact subset of F. Then $F - S$ is open and S is closed: Let $a \epsilon F - S$. For each $x \epsilon S$, let I_x and $J_x = (b_x, c_x)$ be open intervals such that $x \epsilon I_x$, $a \epsilon J_x$, and $I_x \cap J_x = \phi$. Then $\{I_x | x \epsilon S\}$ is an open cover of S. Let $\{I_x | x \epsilon S'\}$ be a subcover with Card$(S') < \kappa$. Since Card$(\{c_x - a | x \epsilon S'\} \cup \{a - b_x | x \epsilon S'\}) < \kappa$, there is a $\delta \epsilon \mathcal{W}$ such that for each $x \epsilon S'$, $\frac{1}{\delta} < c_x - a$ and $\frac{1}{\delta} < a - b_x$. Thus for $x \epsilon S'$, $\left(a - \frac{1}{\delta}, a + \frac{1}{\delta}\right) \subset J_x$. So $\left(a - \frac{1}{\delta}, a + \frac{1}{\delta}\right) \cap S = \phi$.

We now show that a closed and bounded subset, S, of F is κ-compact.[1] Let U be an open cover of S. We define an equivalence relation "$\sim$" on S by $x \sim y$ iff the closed interval $[x, y]$ can be covered by less than κ elements of U. Let C be an equivalence class. C is open: for $x \in C$, pick $G \in U$ with $x \in G$. Then $G \subseteq C$. The class C is also closed. If x is a limit point of C, then $x \in S$ because S is closed. Pick $G \in U$ with $x \in G$, then there is $y \in C \cap G$, $y \neq x$. $Y \sim x$ so $x \in C$.

We now show that there are less than κ equivalence classes. If that were not so, then we form a set D of cardinality at least κ by choosing one point from each equivalence class. D is bounded and has a limit point x, by $BW(\kappa)$. $x \in S$, so choose an equivalence class C with $x \in C$. But C is open in S and C contains only one point of D.

We next show that each equivalence class C can be covered by fewer than κ elements of U. Pick $x \in C$ and split C into $T = \{y \in C | y \geqslant x\}$ and $B = \{y \in C | y \leqslant x\}$. Let τ be the cofinality of T and choose an increasing sequence $\langle x_\alpha | \alpha < \tau \rangle$ cofinal in C. The interval $[x, x_\alpha]$ can be covered by fewer than κ elements of U and $T \subseteq \bigcup \{[x, x_\alpha] | \alpha < \tau\}$. If $\tau < \kappa$, the conclusion follows. If $\tau \geqslant \kappa$ then the set $\{x_\alpha | \alpha < \tau\}$ has a limit point z. The point $z \in C$ because C is closed. $x_\alpha \leqslant z$ for all α so $T \subseteq [x, z]$ and $[x, z]$ can be covered by fewer than κ elements of U. The construction for B is the same.

Corollary *It is consistent with the usual axioms (ZFC) of set theory plus the existence of an inaccessible cardinal that $BW(\aleph_1)$-fields of cardinality larger than $\aleph_1$ do not exist.*

Proof: According to Juhász and Weiss, the existence of an $\aleph_1$-metrizable space of cardinality larger than $\aleph_1$ which is also $\aleph_1$-compact is independent of the usual axioms of set theory plus the existence of an inaccessible cardinal. Since $BW(\kappa)$-fields are $HB(\kappa)$-fields, the unit interval $[0, 1]$ of a $BW(\aleph_1)$-field is an $\aleph_1$-metrizable space which is $\aleph_1$-compact.

The remainder of this paper is devoted to establishing the stronger result obtained from the corollary by replacing "$BW(\aleph_1)$-fields" by "$\aleph_1$-Archimedean fields." A *tree* is a partial ordering $\langle T, < \rangle$ such that for each $y \in T$, $\hat{y} = \{x \in T | x < y\}$ is well-ordered by $<$. For each ordinal α, the *α-th level* of T is $\{y \in T | \hat{y}$ has order type $\alpha\}$. The *height* of T is the first α such that the α-level of T is empty. T' is a *subtree* of T just in case $T' \subseteq T$ and $(\forall y \in T')(\forall x \in T)(x < y \rightarrow x \in T')$. A subset P of T is a *path* through T iff P is linearly ordered by $<$ and contains exactly one element from each nonempty level of T. For a regular cardinal κ, a *κ-Aronszajn* tree is a tree T of height κ such that all levels of T have cardinality less than κ and T has no paths. A *Kurepa* tree is tree T of height $\aleph_1$ such that all levels of T have cardinality less than $\aleph_1$ and T has more than $\aleph_1$ paths.

Juhász and Weiss show that there is an $\aleph_1$-metrizable space of cardinality greater than $\aleph_1$ which is $\aleph_1$-compact iff there is a Kurepa tree with no $\aleph_1$-Aronszajn subtree.

Proposition *If there is a $BW(\aleph_1)$-field with cardinality larger than $\aleph_1$, then there is a Kurepa tree with no $\aleph_1$-Aronszajn subtree.*

Proof: The unit interval $[0, 1]$ of a $BW(\aleph_1)$-field is $\aleph_1$-compact and $\aleph_1$-metrizable, and in an ordered field any interval with at least two points has the same cardinality as the field.

Since it is known that if there is a κ-Aronszajn tree then there is a linearly ordered set of cardinality κ with no strictly monotone κ-sequence, there is some reason to suspect that the Ramsey property of BW-fields accounts for the lack of Aronszajn subtrees while the Archimedean property is enough to ensure the existence of the Kurepa tree. For regular cardinals κ, a *κ-Kurepa family* is a subset $\mathscr{F}$ of $\mathscr{P}(\kappa)$ such that $\text{Card}(\mathscr{F}) > \kappa$ and for all $\alpha \in \kappa$, $\text{Card}(\{x \cap \alpha | x \in \mathscr{F}\}) \leqslant \text{Card}(\alpha) + \aleph_0$. It is known that there is a Kurepa tree iff there is $\aleph_1$-Kurepa family.

Theorem *If there is a κ^+-Archimedean field of cardinality larger than κ^+, then there is a κ^+-Kurepa family.*

Proof: Let F be a κ^+-Archimedean field, let $\mathscr{N}$ be a copy of $\mathscr{N}_{\kappa^+}$ contained in F, and for $\alpha \in \kappa^+$, let $\bar{\alpha}$ be the corresponding element of $\mathscr{N}$. For each $\alpha \in \kappa^+$, a subset D_α of $I = [0, 1]$ is given by $D_0 = \{0, 1\} = D_{\alpha+1}$, and for limit ordinals α, D_α is a maximal subset of I such that

(1) $\bigcup_{\beta \in \alpha} D_\beta \subset D_\alpha$

(2) D_α is separated of size $\frac{1}{2\bar{\alpha}}$.

Let λ be a limit ordinal:

Lemma $(\forall x \in I)(\exists y \in D_\lambda)\left(|x - y| < \frac{1}{2\bar{\lambda}}\right)$.

Proof: Otherwise $(\exists x \in I)(\forall y \in D_\lambda)\left(|x - y| \geqslant \frac{1}{2\bar{\lambda}}\right)$, contrary to the maximality of D_λ.

Lemma $(\exists x \in D_\lambda)\left(0 < 1 - x < \frac{1}{\bar{\lambda}}\right)$.

Proof: If $1 - \frac{1}{2\bar{\lambda}} \in D_\lambda$, let $x = 1 - \frac{1}{2\bar{\lambda}}$, otherwise, there is a $y \in D_\lambda$ such that $0 < \left|\left(1 - \frac{1}{2\bar{\lambda}}\right) - y\right| < \frac{1}{2\bar{\lambda}}$. Since $y \neq 1$, $y \in D_\lambda$, and $1 \in D_\lambda$; $1 - \frac{1}{2\bar{\lambda}} > y$. Thus $1 - \frac{1}{2\bar{\lambda}}, - y < \frac{1}{2\bar{\lambda}}$ and $1 - y < \frac{1}{\bar{\lambda}}$.

Lemma $(\forall d \in D_\lambda - \{1\})(\exists x \in D_\lambda)\left(0 < x - d < \frac{1}{\bar{\lambda}}\right)$.

Proof: There is a $y \in D_\lambda$ so that $|d + \frac{1}{2\bar{\lambda}} - y| < \frac{1}{2\bar{\lambda}}$. Since $y \neq d$, $|d - y| \geqslant \frac{1}{2\bar{\lambda}}$. If $y \geqslant d + \frac{1}{2\bar{\lambda}}$, then $0 \leqslant y - d - \frac{1}{2\bar{\lambda}} < \frac{1}{2\bar{\lambda}}$ and $0 < y - d < \frac{1}{\bar{\lambda}}$. If $y < d + \frac{1}{2\bar{\lambda}}$, then $d - y < 0$, so $y > d$ and $d + \frac{1}{2\bar{\lambda}} > y > d$. Then $|y - d| < \frac{1}{2\bar{\lambda}}$, contrary to $|y - d| \geqslant \frac{1}{2\bar{\lambda}}$.

Lemma $(\forall d \in D_\lambda - \{0\})(\exists x \in D_\lambda)\left(0 < d - x < \frac{1}{\lambda}\right)$.

Let $\Gamma = \{\alpha \in \kappa^+ | \kappa \leqslant \alpha\}$ and let Δ be the set of all limit ordinals in Γ of the form $\kappa \times \beta$ for some ordinal β. For each $\delta \in \Delta$, D_δ is separated of size $\frac{1}{2\bar{\delta}}$ and bounded, and so $\mathrm{Card}(D_\delta) \leqslant \kappa$. Since $\frac{\beta}{\bar{\delta}} < 1$ for $\beta \in \kappa$, $\mathrm{Card}(D_\delta) \geqslant \kappa$. Thus $\mathrm{Card}(D_\delta) = \kappa$. Let f_δ be a bijection from κ onto $D_\delta - \{1\}$. Define a bijection g_δ from κ onto $D_\delta - \{0\}$ by letting $g_\delta(\sigma)$ be the unique $d \in D_\delta$ such that $0 < d - f_\delta(\sigma) < \frac{1}{\bar{\delta}}$. The lemmas ensure that g_δ is defined and surjective. For $\sigma \in \kappa$, let $I_{\delta \dot{+} \sigma} = [f_\delta(\sigma), g_\delta(\sigma)]$.

Let $I' = I - \bigcup_{\alpha \epsilon \kappa^+} D_\alpha$. Since $\mathrm{Card}(D_\alpha) \leqslant \kappa$ and $\mathrm{Card}(F) > \kappa^+$, $\mathrm{Card}(I') > \kappa^+$. For each $x \in I'$, let $S_x = \{\alpha \in \kappa^+ | x \in I_\alpha\}$ and let $\mathscr{F} = \{S_x | x \in I'\}$. Let $x < y$ be elements of I' and choose $\delta \in \Delta$ so that $\frac{1}{\bar{\delta}} < y - x$. If $S_x = S_y$, then for some $\rho \in \kappa$, both x and y are in $I_{\delta \dot{+} \rho} = [f_\delta(\rho), g_\delta(\rho)]$. Then $y - x < g_\delta(\rho) - f_\delta(\rho) < \frac{1}{\bar{\delta}}$, contrary to the way δ was chosen. Thus $S_x \neq S_y$ and $\mathrm{Card}(\mathscr{F}) > \kappa^+$.

Lemma *For δ and γ in Δ, and for α and β in κ, if $\bar{\delta} > 4\bar{\gamma}$ and $I_{\delta \dot{+} \alpha} \cap I_{\gamma \dot{+} \beta} \neq \phi$, then $I_{\delta \dot{+} \alpha} \subseteq I_{\gamma \dot{+} \beta}$.*

Proof: Let $x \in I_{\delta \dot{+} \alpha} \cap I_{\gamma \dot{+} \beta}$. Then $f_\gamma(\beta) \leqslant x \leqslant g_\gamma(\beta)$ and $f_\delta(\alpha) \leqslant x \leqslant g_\delta(\alpha)$. Since $g_\gamma(\beta) - f_\gamma(\beta) \geqslant \frac{1}{2\bar{\gamma}}$, either $g_\gamma(\beta) - x \geqslant \frac{1}{4\bar{\gamma}}$ or $x - f_\gamma(\beta) \geqslant \frac{1}{4\bar{\gamma}}$. Suppose for example that $x - f_\gamma(\beta) \geqslant \frac{1}{4\bar{\gamma}}$. Since $g_\delta(\alpha) - f_\delta(\alpha) < \frac{1}{\bar{\delta}}$, $x - f_\delta(\alpha) < \frac{1}{\bar{\delta}}$. Hence $f_\gamma(\beta) \leqslant f_\delta(\alpha) \leqslant x$. Since $D_\gamma \subset D_\delta$, $g_\gamma(\beta) \in D_\delta$ and $g_\gamma(\beta) - f_\delta(\alpha) \geqslant \frac{1}{2\bar{\delta}}$. If $g_\gamma(\beta) < g_\delta(\alpha)$, then $g_\delta(\alpha) - g_\gamma(\beta) \geqslant \frac{1}{2\bar{\delta}}$. Then $g_\delta(\alpha) - f_\delta(\alpha) \geqslant \frac{1}{\bar{\delta}}$ contrary to $g_\delta(\alpha) - f_\delta(\alpha) < \frac{1}{\bar{\delta}}$. Therefore $g_\delta(\alpha) \leqslant g_\gamma(\beta)$.

Let $\alpha \in \kappa^+$. If $\alpha \in \kappa$, then $\alpha \cap S_x = \phi$ for each $x \in I'$; so in this case, $\mathrm{Card}(\{\alpha \cap S_x | x \in I'\} = 1 \leqslant \mathrm{Card}(\alpha) + \aleph_0$. Suppose now that $\kappa \leqslant \alpha < \kappa^+$. Let $\delta \in \Delta$ be chosen so that $4\bar{\alpha} < \bar{\delta}$. For each $\beta \in \kappa$, if x and y are both in $I_{\delta \dot{+} \beta}$, then $\alpha \cap S_x = \alpha \cap S_y$: for if $\gamma \in \alpha \cap S_x$, then $\kappa \leqslant \gamma$ and so there are $\delta' \in \Delta$ and $\beta' \in \kappa$ such that $\gamma = \delta' + \beta'$. By the lemma, $I_{\delta \dot{+} \beta} \subset I_\gamma$, so $y \in I_\gamma$ and $\gamma \in \alpha \cap I_y$. Therefore $\mathrm{Card}\{\alpha \cap I_x | x \in I'\} \leqslant \mathrm{Card}\{I_{\delta \dot{+} \beta} | \beta \in \kappa\} = \kappa = \mathrm{Card}\ \alpha$. This then is enough to show that $\mathscr{F}$ is κ^+-Kurepa family.

Corollary *For each infinite cardinal κ, it is consistent with the usual axioms of set theory plus the existence of an inaccessible cardinal that κ^+-Archimedean fields of cardinality larger than κ^+ do not exist.*

Proof: J. Silver has shown [9] relative to the existence of an inaccessible cardinal that for each infinite cardinal κ, it is consistent that there are no κ^+-Kurepa families.

Questions:

(1) Is it consistent that for *all* infinite cardinals κ, κ^+-Archimedean fields of cardinality larger than κ^+ do not exist?

(2) For each regular cardinal κ, is it consistent that κ-Archimedean fields of cardinality larger than κ do not exist?

(3) For what cardinals κ (if any) is it consistent that κ-Archimedean fields of cardinality larger than κ exist? This is related to the following. If a κ-Kurepa family exists, must a κ-Archimedean field of cardinality larger than κ also exist? (The constructible universe is the place to look.)

(4) Is it consistent for κ-Archimedean fields of cardinality larger than κ to exist while $BW(\kappa)$-fields of cardinality larger than κ do not exist?

NOTE

1. Thanks are due to the referee for suggesting the following proof, which is shorter than that proposed by the authors.

REFERENCES

[1] Cowles, J., "Generalized Archimedean fields and logics with Malitz quantifiers," *Fundamenta Mathematicae,* vol. 112 (1982), pp. 45-59.

[2] Juhász, I. and W. Weiss, "On a problem of Sikorski," *Fundamenta Mathematicae,* vol. 100 (1978), pp. 223-227.

[3] Kaplanski, I., "Maximal fields with valuations," *Duke Mathematical Journal,* vol. 9 (1942), pp. 303-321.

[4] Kaplanski, I., "Maximal fields with valuations, II," *Duke Mathematical Journal,* vol. 12 (1945), pp. 243-248.

[5] Lagrange, R. and J. Cowles, "Large fields with the Balzano-Weierstrass property need not exist," *Notices of the American Mathematical Society,* June 1979.

[6] Sikorski, R., "On algebraic extensions of ordered fields," *Polskie Towarzystwo Matematyozne Annales,* vol. 22 (1949), pp. 173-184.

[7] Sikorski, R., "On an ordered algebraic field," *Comptes Rendus Société des Séances des Lettres de Varsovie,* C1 III, 41 (1948), pp. 69-96.

[8] Sikorski, R., "Remarks on some topological spaces of high power," *Fundamenta Mathematicae,* vol. 37 (1950), pp. 125-136.

[9] Silver, J., "The independence of Kurepa's conjecture," pp. 383-390 in *Axiomatic Set Theory, Proceedings of Symposia in Pure Mathematics,* XIII, part 1, ed., D. Scott, American Mathematical Society, Providence, Rhode Island, 1971.

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