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A Calculus of Individuals Based on 'Connection'

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Although Aristotle (*Metaphysics*, Book IV, Chapter 2) was perhaps the first person to consider the part-whole relationship to be a proper subject matter for philosophic inquiry, the Polish logician Stanislow Leśniewski [15] is generally given credit for the first formal treatment of the subject matter in his Mereology.[1] Woodger [30] and Tarski [24] made use of a specific adaptation of Leśniewski's work as a basis for a formal theory of physical things and their parts. The term 'calculus of individuals' was introduced by Leonard and Goodman [14] in their presentation of a system very similar to Tarski's adaptation of Leśniewski's Mereology. Contemporaneously with Leśniewski's development of his Mereology, Whitehead [27] and [28] was developing a theory of extensive abstraction based on the two-place predicate, 'x extends over y', which is the converse of 'x is a part of y'. This system, according to Russell [22], was to have been the fourth volume of their *Principia Mathematica*, the never-published volume on geometry. Both Leśniewski [15] and Tarski [25] have recognized the similarities between Whitehead's early work and Leśniewski's Mereology. Between the publication of Whitehead's early work and the publication of *Process and Reality* [29], Theodore de Laguna [7] published a suggestive alternative basis for Whitehead's theory. This led Whitehead, in *Process and Reality*, to publish a revised form of his theory based on the two-place predicate, 'x is extensionally connected with y'. It is the purpose of this paper to present a calculus of individuals based on this new Whiteheadian primitive predicate.

Although the calculus presented below utilizes most of Whitehead's mereological definitions, it differs substantially from Whitehead's system presented in *Process and Reality*. Whitehead does not axiomatize his theory, but refers to assumptions which include both probable axioms and desirable theorems without any distinction. There is, however, a difficulty with his definitions and assumptions which has led me to revise his system in the

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present axiomatization. From his definition of 'x is a part of y', it follows that '$(x)(x$ is a part of $x)$' (T0.5 below). From this and Whitehead's Assumption 5, '$(x)(y)(x$ is a part of $y \supset x$ is connected to $y)$' (T0.11 below), it follows that '$(x)(x$ is connected to $x)$' (T0.1 below). From this and Whitehead's Assumption 4, '$(x) \sim (x$ is connected to $x)$', a contradiction follows.[2] Also, Whitehead does not include the quasi-Boolean operators in his system. On the contrary, he informally assumes that each individual is continuous—an assumption which I have dropped. Likewise, Whitehead does not introduce the quasi-topological operators and predicates as I have done. Both of these parts are extensions of his mereological definitions.[3]

I have chosen to present the present system as an uninterpreted calculus; however, it will be an aid in reading the axioms, definitions, and theorems to keep a particular interpretation in mind. Following Whitehead we may interpret the individual variables as ranging over spatio-temporal regions and the two-place primitive predicate, 'x is connected with y', as a rendering of 'x and y share a common point'. As a result, 'x is a part of y' becomes a rendering of 'All the points of x are contained in the points of y'; 'x overlaps y' becomes a rendering of 'x and y share a common interior point'; and 'x is externally connected to y' becomes a rendering of 'x and y share a common point, but they share no interior points'; that is, they share only boundary points. In so doing, however, we must remember that the individuals are spatio-temporal regions; the individual variables do not range over points. Whitehead's mereological system was, in fact, constructed in order to define points. Points were defined as certain sets of sets of infinitely converging regions. Thus a two-place predicate, '. . . is incident in . . .,' holding between a point and a region, was then defined. In the present system, due to the presence of the algebraic operators, a simpler definition of a point can be constructed in terms of a modified maximal filter, modified due to the presence of external connectedness. With this definition the two-place predicate, '. . . is incident in . . .,' is definable. With the definition of a point and the definition of this two-place predicate, then the theorem, 'x is connected with y if, and only if, there is a common point incident in both x and y' becomes provable, as do analogous theorems for the above suggested interpretation. Also, an open region will have only its interior points incident in it, while a closed region will have also its boundary points incident in it. This extension of the present system, however, is the subject of another paper,[4] and is suggested here only as an aid in reading the present one. Taken as an uninterpreted calculus, the present system may have a number of different interpretations and it stands on its own.

In the following formulation I am assuming classical first-order quantification theory with identity and some form of set theory, although the use of set theory is minimal and, as I suggest below, can be dispensed with. For convenience I have divided the system into (1) a mereological part, which systematizes the mereological predicates; (2) a quasi-Boolean part, which introduces the Boolean operators and the universal individual, but no zero (or null) element (thus the reason for the term 'quasi'); and (3) a quasi-topological part, which introduces topological operators and predicates, but here again there is no zero (or null) element and no boundary elements (thus the use of the term 'quasi' here).

I Mereological part Taking 'Cx,y' as a rendering of 'x is connected to y', we can introduce a definition of 'DCx,y' (x is disconnected from y) and the standard mereologial definitions of 'Px,y' (x is a part of y), 'PPx,y' (x is a proper part of y), 'Ox,y' (x overlaps y), and 'DRx,y' (x is discrete from y) as follows:

D0.1 'DCx,y' $=_{\text{def}}$ '$\sim Cx,y$'
D0.2 'Px,y' $=_{\text{def}}$ '$(z)(Cz,x \supset Cz,y)$'
D0.3 'PPx,y' $=_{\text{def}}$ '$Px,y \cdot \sim Py,x$'
D0.4 'Ox,y' $=_{\text{def}}$ '$(\exists z)(Pz,x \cdot Pz,y)$'
D0.5 'DRx,y' $=_{\text{def}}$ '$\sim Ox,y$'.

This distinction between 'Cx,y' and 'Ox,y' constitutes the virtue of this new calculus. It gives us the power to define 'ECx,y' (x is externally connected to y), 'TPx,y' (x is a tangential part of y), and '$NTPx,y$' (x is a nontangential part of y) as follows:

D0.6 'ECx,y' $=_{\text{def}}$ '$Cx,y \cdot \sim Ox,y$'
D0.7 'TPx,y' $=_{\text{def}}$ '$Px,y \cdot (\exists z)(ECz,x \cdot ECz,y)$'
D0.8 '$NTPx,y$' $=_{\text{def}}$ '$Px,y \cdot \sim(\exists z)(ECz,x \cdot ECz,y)$'.

Our axiomatization requires only two axioms: a mereological axiom,

A0.1 $(x)[Cx,x \cdot (y)(Cx,y \supset Cy,x)]$

and an axiom involving identity, analogous to the axiom of extension in set theory,

A0.2 $(x)(y)[(z)(Cz,x \equiv Cz,y) \supset x = y]$.

From our definitions and two axioms the mereological theorems listed below are provable. I have listed the theorems in their order of provability, some because of their own intrinsic interest and some because of their simplification of the proofs of later theorems. The proofs of these theorems are on the whole simple and straightforward. In the case of a few more complex ones I have listed the theorems and definitions from which they follow in an order from which a proof might be constructed.

T0.1 $(x)Cx,x$
T0.2 $(x)(y)(Cx,y \equiv Cy,x)$
T0.3 $(x)(y)[(z)(Cz,x \equiv Cz,y) \equiv x = y]$
T0.4 $(x)(y)(\sim DCx,y \equiv Cx,y)$
T0.5 $(x)Px,x$
T0.6 $(x)(y)(z)[(Px,y \cdot Py,z) \supset Px,z]$
T0.7 $(x)(y)[(Px,y \cdot Py,x) \equiv x = y]$
T0.8 $(x)(y)[Px,y \equiv (z)(Pz,x \supset Pz,y)]$
T0.9 $(x)(y)(z)[(Px,y \cdot Cz,x) \supset Cz,y]$
T0.10 $(x)(y)[Cx,y \equiv (\exists z)(Pz,y \cdot Cx,z)]$
T0.11 $(x)(y)(Px,y \supset Cx,y)$
T0.12 $(x)(y)(z)[(Px,y \cdot DCz,y) \supset DCz,x]$
T0.13 $(x)\sim PPx,x$
T0.14 $(x)(y)(PPx,y \supset Px,y)$

T0.15 $(x)(y)(PPx,y \supset {\sim}PPy,x)$
T0.16 $(x)(y)(z)[(PPx,y \cdot PPy,z) \supset PPx,z]$
T0.17 $(x)Ox,x$
T0.18 $(x)(y)(Ox,y \equiv Oy,x)$
T0.19 $(x)(y)(Ox,y \supset Cx,y)$
T0.20 $(x)(y)[(Px,y \cdot Oz,x) \supset Oz,y]$
T0.21 $(x)(y)(Px,y \supset Ox,y)$
T0.22 $(x)(y)({\sim}DRx,y \equiv Ox,y)$
T0.23 $(x)(y)(z)[(Px,y \cdot DRz,y) \supset DRz,x]$
T0.24 $(x){\sim}ECx,x$
T0.25 $(x)(y)(ECx,y \equiv ECy,x)$
T0.26 $(x)(y)(ECx,y \supset Cx,y)$
T0.27 $(x)(y)(ECx,y \supset {\sim}Ox,y)$
T0.28 $(x)(y)[Cx,y \equiv (ECx,y \vee Ox,y)]$
T0.29 $(x)(y)[Ox,y \equiv (Cx,y \cdot {\sim}ECx,y)]$
T0.30 $(x)(y)[{\sim}ECx,y \equiv (Ox,y \equiv Cx,y)]$
T0.31 $(x)(y)\{{\sim}(\exists z)ECz,x \supset [Px,y \equiv (z)(Oz,x \supset Oz,y)]\}$
T0.32 $(x)(y)(TPx,y \supset Px,y)$
T0.33 $(x)(y)[TPx,y \supset (\exists z)(ECz,x \cdot ECz,y)]$
T0.34 $(x)(y)(z)[(TPz,x \cdot Pz,y \cdot Py,x) \supset TPz,y]$
T0.35 $(x)(y)(NTPx,y \supset Px,y)$
T0.36 $(x)(y)[NTPx,y \supset {\sim}(\exists z)(ECz,x \cdot ECz,y)]$
T0.37 $(x)(y)(TPx,y \supset {\sim}NTPx,y)$
T0.38 $(x)(y)[TPx,y \equiv (Px,y \cdot {\sim}NTPx,y)]$
T0.39 $(x)(y)[NTPx,y \equiv (Px,y \cdot {\sim}TPx,y)]$
T0.40 $(x)(y)[Px,y \equiv (TPx,y \vee NTPx,y)]$
T0.41 $(x)(NTPx,x \equiv {\sim}(\exists y)ECy,x)$
T0.42 $(x)(y)(z)[(NTPx,y \cdot Cz,x) \supset Cz,y]$
T0.43 $(x)(y)(z)[(NTPx,y \cdot Oz,x) \supset Oz,y]$
T0.44 $(x)(y)(z)[(NTPx,y \cdot Cz,x) \supset Oz,y]$

Proof: T0.36; D0.6; T0.42; T0.43.

T0.45 $(x)(y)(z)[(Px,y \cdot NTPy,z) \supset NTPx,z]$

Proof: T0.44; T0.9; D0.6; T0.6; T0.35; D0.8.

T0.46 $(x)(y)(z)[(NTPx,y \cdot Py,z) \supset NTPx,z]$

Proof: T0.44; T0.19; D0.6; T0.6; T0.35; D0.8.

T0.47 $(x)(y)(z)[(NTPx,y \cdot NTPy,z) \supset NTPx,z]$.

Theorems 0.1-0.23, except those dealing with '*Cx,y*' and '*DCx,y*' explicitly, are standard theorems of the classical calculus of individuals (or mereology). Theorems T0.24-T0.47 are due to the use of the new primitive '*Cx,y*' and the subsequent definitions which it makes possible. Theorem T0.31 is particularly significant in that it shows the relationship between this new calculus and the classical calculus of individuals. In the absence of external connectedness, the partial-ordering relation, is a part of, reduces to the partial ordering relation in the classical calculus. Again we see, in T0.30, that in the absence of external connectedness, 'Cx,y' and 'Ox,y' become synonymous expressions.

II Quasi-Boolean part In order to introduce the quasi-Boolean operators and the universal individual, we follow Tarski [24] and Leonard and Goodman [14], introducing them by way of a theory for the fusion of sets. In what follows, 'X', 'Y', and 'Z' are taken as variables ranging over sets of individuals, that is, subsets of $\{x: Cx,x\}$. The expression, '$x = f'X$' will be taken as a rendering of 'x is identical to the fusion of the set X' and will be introduced as follows:

D1.1 '$x = f'X$' $=_{\text{def}}$ '$(y)[Cy,x \equiv (\exists z)(z \in X \cdot Cy,z)]$'.

Using this expression, we can define '$x + y$' for the quasi-Boolean sum (join, union, or addition), '$-x$' for the quasi-Boolean negate, or complement, 'a^*' for the quasi-Boolean universal, or all inclusive individual, and '$x \wedge y$' for the quasi-Boolean intersection (meet or multiplication) as follows:

D1.2 '$x + y$' $=_{\text{def}}$ '$f'\{z: Pz,x \vee Pz,y\}$'
D1.3 '$-x$' $=_{\text{def}}$ '$f'\{y: \sim Cy,x\}$'
D1.4 'a^*' $=_{\text{def}}$ "$f'\{y: Cy,y\}$'
D1.5 '$x \wedge y$' $=_{\text{def}}$ '$f'\{z: Pz,x \cdot Pz,y\}$'.

As suggested earlier, the use of set theory can be eliminated. Instead of using a Theory for the Fusion of Sets, one may, as Martin [20], use a theory of virtual classes. Alternatively one can make use of a Russellian Theory of Definite Descriptions. This was suggested by Leonard and Goodman [14] and utilized later by Goodman [4] and Eberle [2]. Utilizing such a theory, the following definitions would be substituted for Definitions 1.2-1.5:

D1.2′ '$x + y$' $=_{\text{def}}$ '$(\imath z)\{(w)[Cw,z \equiv (Cw,x \vee Cw,y)]\}$'
D1.3′ '$-x$' $=_{\text{def}}$ '$(\imath y)\{(z)(Cz,y \equiv \sim Pz,x)\}$'
D1.4′ 'a^*' $=_{\text{def}}$ '$(\imath y)\{(z)Cz,y\}$'
D1.5′ '$x \wedge y$' $=_{\text{def}}$ '$(\imath z)\{(w)[Cw,z \equiv (Cw,x \cdot Cw,y)]\}$'.

In place of the definition of 'the fusion of the class' we could substitute a definitional schema so that we could still speak of the sum of all the individuals satisfying a certain predicate as follows:

D1.1′ '$s' \ldots$' $=_{\text{def}}$ '$(\imath x)\{(y)[Cy,x \equiv (\exists z)(\ldots z \cdot Cy,z)]\}$',

where some predicate is to be written in for the ellipsis.

Either of these techniques of introducing the quasi-Boolean operators, however, encounters a problem with reference to the classical rules of Universal Instantiation and Existential Generalization. With reference to the Theory of Definite Descriptions, this has been pointed out since Carnap [1]. It is simply that without some restriction on these rules, one can make the following valid inferences for any definite description.[5] Let '$(\imath x)\phi x$' be some definite description in the theory, then

1. $(x)x = x$	Identity Theory
2. $(\imath x)\phi x = (\imath x)\phi x$	Universal Instantiation
3. $(\exists y)y = (\imath x)\phi x$	Existential Generalization

And, of course, the same would be the case with a Theory for the Fusion of Sets, where '$f'\{x: \phi x\}$' is an expression of the theory. If we had a null element in our Calculus of Individuals, we could simply let '$f'\{x: \phi x\} = 0^*$' be the case whenever $\{x: \phi x\}$ is empty. Likewise in our Theory of Definite Descriptions, we could let '$(\iota x)\phi x = 0^*$' be the case whenever '$(\iota x)\phi x$' has no referent or is not unique—a technique suggested by Frege [3], Carnap [1], and Martin [18] and [19]. One of the proposed virtues, however, of the Calculus of Individuals is that it does not have a null element.

It appears that the most convenient way for us to handle this problem here is to revise our underlying quantification theory along the following lines, where 'α' and 'β' are names of arbitrary individual variables and 'η' is either the name of some arbitrary individual variable or the name of some operator expression, in our case, some expression of the form, '$f'\{\ldots:\ldots\}$' or some expression introduced by way of an expression of this form:

$\ulcorner((\exists\alpha)\alpha = \eta \cdot (\beta)\phi) \supset \psi,\urcorner$ where ψ is like ϕ or differs from ϕ in containing η where ϕ contains some free occurrence of β.

Since such a revision would also limit instantiation in our underlying identity theory, we need to revise it also by adding an axiom:

$\ulcorner(\exists\alpha)\alpha = \beta.\urcorner$

This would, in effect, allow us to continue instantiating with individual variables and allow all of our theorems, T0.1-T0.47, to continue as theorems. Our revision, in effect, only limits Universal Instantiation where expressions of the form, '$f'\{\ldots:\ldots\}$,' and expressions introduced by expressions of this form are concerned.

In addition to the definitions, D1.1-D1.5, we need the following axiom:

A1.1 $(X)(\sim X = \Lambda \supset (\exists x)x = f'X)$.

If we were to utilize the Theory of Definite Descriptions, then we would want to replace A1.1 with the following two axioms:

A1.1′ $(x)(y)(\exists z)z = x + y$
A1.1″ $(\exists x)x = a^*$.

From D1.1 and A1.1 the following theorems concerning the fusion of sets are provable:

T1.1 $(X)\{\sim X = \Lambda \supset (x)[Cx,f'X \equiv (\exists y)(y \in X \cdot Cx,y)]\}$
T1.2 $(X)(\sim X = \Lambda \equiv (\exists x)x = f'X)$
T1.3 $(X)(x)(x \in X \supset Px,f'X)$
T1.4 $(X)(Y)[\sim X = \Lambda \cdot X \subseteq Y) \supset Pf'X,f'Y]$
T1.5 $(X)(Y)[(\sim X = \Lambda \cdot X = Y) \supset f'X = f'Y]$
T1.6 $(x)x = f'\{x\}$
T1.7 $(x)x = f'\{y: Py,x\}$
T1.8 $(x)f'\{x\} = f'\{y: Py,x\}$.

The existence of the sum of any two individuals, T1.9, and the existence of the universal, or all-inclusive, individual, T1.23, follow from A1.1. Consequently, the following theorems are provable without any qualification. I continue my practice for the more complicated proofs of listing the theorems and definitions in an order from which a proof might be constructed. (I.T. indicates that Identity Theory is used.)

T1.9 $(x)(y)(\exists z)z = x + y$

T1.10 $(x)(y)(z)\{Cz,x + y \equiv (\exists w)[(Pw,x \vee Pw,y) \cdot Cz,w]\}$

T1.11 $(x)(y)(z)[Cz,x + y \equiv (Cz,x \vee Cz,y)]$

T1.12 $(X)(Y)[(\sim X = \Lambda \cdot \sim Y = \Lambda) \supset f'X \cup Y = f'X + f'Y]$

T1.13 $(x)(y)x + y = f'\{x\} \cup \{y\}$

T1.14 $(x)x + x = x$

T1.15 $(x)(y)x + y = y + x$

T1.16 $(x)(y)(z)\ (x + y) + z = x + (y + z)$

T1.17 $(x)(y)Px,x + y$

T1.18 $(x)(y)(z)[(Pz,x \vee Pz,y) \supset Pz,x + y]$

T1.19 $(x)(y)(z)(Px,y \supset Px,y + z)$

T1.20 $(x)(y)(z)(x = y \supset z + x = z + y)$

T1.21 $(x)(y)(Px,y \equiv Px + y,y)$

T1.22 $(x)(y)(Px,y \equiv y = x + y)$

T1.23 $(\exists x)x = a*$

T1.24 $(x)[Cx,a* \equiv (\exists y)(Cy,y \cdot Cx,y)]$

T1.25 $(x)Px,a*$

T1.26 $(x)Cx,a*$

T1.27 $(x)Ox,a*$

T1.28 $(x)x + a* = a*$

T1.29 $(x)((y)Py,x \equiv x = a*)$

T1.30 $(x)((y)Cy,x \equiv x = a*)$

Proof: I.T.; T1.26; T1.25; D0.2; T0.7.

T1.31 $(x)\sim ECx,a*$.

A theorem asserting the existence of the negate of any individual is not provable, since the negate of the universal individual does not exist. Thus, the following theorems, T1.32-T1.41, concerning the negate of an individual are all conditional upon the existence of that negate. We can, however, prove that there exists a negate of an individual if, and only if, that individual is not the universal individual, T1.32.

T1.32 $(x)((\exists y)y = -x \equiv \sim x = a*)$

T1.33 $(x)\{(\exists z)z = -x \supset (y)[Cy,-x \equiv (\exists z)(\sim Cz,x \cdot Cy,z)]\}$

T1.34 $(x)[(\exists z)z = -x \supset (y)(Cy,-x \equiv \sim Py,x)]$

T1.35 $(x)((\exists z)z = -x \supset x = --x)$

T1.36 $(x)[(\exists z)z = -x \supset (y)(\sim Cy,x \equiv Py,-x)]$

T1.37 $(x)((\exists z)z = -x \supset \sim Cx,-x)$

T1.38 $(x)[(\exists z)z = -x \supset (y)(x = y \supset -x = -y)]$

T1.39 $(x)((\exists z)z = -x \supset (y)Py,x + -x)$

Proof: T0.11; T1.34; T1.11; D0.2.

T1.40 $(x)(y)[((\exists z)z = -x \cdot (\exists z)z = -y) \supset (Px,y \equiv P-y,-x)]$

Proof: T0.6; T1.34; D0.2; T0.9; T1.34; T1.36; D0.2.

T1.41 $(x)((\exists z)z = -x \supset x + -x = a^*)$.

Likewise, in the absence of a null individual, we cannot prove the existence of the intersection of any two individuals. We can only prove that the intersection of two individuals exists if, and only if, the two individuals overlap, T1.42.

T1.42 $(x)(y)((\exists z)z = x \wedge y \equiv Ox,y)$
T1.43 $(x)(y)((\exists w)w = x \wedge y \supset (z)\{Cz,x \wedge y \equiv (\exists w)[(Pw,x \cdot Pw,y) \cdot Cz,w]\})$
T1.44 $(x)(y)\{(\exists w)w = x \wedge y \supset (z)[Cz,x \wedge y \supset (Cz,x \cdot Cz,y)]\}$

Proof: T1.43; T0.10.

T1.45 $(x)(y)\{(\exists w)w = x \wedge y \supset (z)[(Pz,x \cdot Pz,y) \equiv Pz,x \wedge y]\}$

Proof: T1.43; D0.2; T1.44; D0.2.

T1.46 $(x)(y)\{[((\exists z)z = -x \cdot (\exists z)z = -y) \cdot (\exists z)z = x \wedge y] \supset x \wedge y = -(-x + -y)\}$

Proof: I.T.; D1.5; T1.34; T1.11.

T1.47 $(x)x \wedge x = x$

Proof: T0.5; T1.45; D0.2; T0.7; T0.17; T1.42.

T1.48 $(x)(y)((\exists z)z = x \wedge y \supset x \wedge y = y \wedge x)$
T1.49 $(x)(y)(z)\{[((\exists w)w = x \wedge y \cdot (\exists w)w = y \wedge z) \cdot (\exists w)w = (x \wedge y) \wedge z] \supset (x \wedge y) \wedge z = x \wedge (y \wedge z)\}$

Proof: T0.5; T1.45; T1.50; T0.7; T0.7.

T1.50 $(x)(y)((\exists z)z = x \wedge y \supset Px \wedge y,x)$
T1.51 $(x)(y)[(\exists z)z = x \wedge y \supset (Px,y \equiv x = x \wedge y)]$
T1.52 $(x)(y)[(\exists w)w = x \wedge y \supset (z)(Px,y \supset Px \wedge z,y)]$
T1.53 $(x)(z)[(\exists w)w = x \wedge z \supset (y)(x = y \supset x \wedge z = y \wedge z)]$
T1.54 $(x)(y)\{(\exists w)w = x \wedge y \supset (z)[NTPz,x \wedge y \supset (NTPz,x \cdot NTPz,y)]\}$

Proof: T0.46; T1.50; I.T.; T1.48; T1.42; T1.18; T1.42.

T1.55 $(x)x \wedge a^* = x$
T1.56 $(x)(y)\{[((\exists z)z = -x \cdot (\exists z)z = -y) \cdot \sim ECx,-y] \supset (-x + y = a^* \equiv Px,y)\}$.

It should be pointed out that Theorems T1.47 and T1.55 are provable without any existential conditions since every individual overlaps itself and every individual overlaps the universal individual, a^*. T1.56 is of special importance. It has been pointed out, since Leonard and Goodman [14] and Tarski [26], that the linguistic domain of a classical calculus of individuals can be characterized as a Boolean algebra with the null individual removed. Consequently, Theorems T1.9-T1.55 likewise hold for the linguistic domain of a classical calculus of individuals. This present calculus of individuals, however, not only has the null element missing from its linguistic domain, there are certain other elements missing; and this is indicated by the additional condition in T1.56. For '$-x + y = a^* \equiv Px,y$' holds for any x and y in our linguistic domain only on the condition that the negate of x and the negate of y are members of the domain, as in the classical calculus, but also only on the condition that x and the negate of y are not externally connected. This additional condition in T1.56, as we shall see below, is due to the fact that there are no boundary elements in the linguistic domain of this new calculus. Since T1.56 is such a key theorem in characterizing the linguistic domain of this new calculus, I shall include a proof for it. In the following proof, it shall be understood that everything on the right-hand side of the vertical line is conditioned upon the existence of the individuals on the left-hand side of the line, and S.L. and Q.T. indicate that the step in the proof makes use of the underlying sentential logic or quantification theory, as revised above.

1. $(\exists z)z = -x$		Assumption
2. $(\exists z)z = -y$		Assumption
3.	$(\exists z)z = -x + y$	T1.9, 1
4.	$[(\exists z)z = -x + y \cdot (z)((w)Cw,z \equiv z = a^*)] \supset$ $((w)Cw,-x + y \equiv -x + y = a^*)$	Q.T.
5.	$(w)Cw,-x + y \equiv -x + y = a^*$	3 and T1.26, S.L.
6.	$(w)(Cw,-x \vee Cw,y) \equiv -x + y = a^*$	5, T1.11, 1, S.L.
7.	$(w)(\sim Pw,x \vee Cw,y) \equiv -x + y = a^*$	6, T1.34, 1, S.L.
8.	$(w)(\sim Pw,x \vee \sim Pw,-y) \equiv -x + y = a^*$	7, T1.36, 2, S.L.
9.	$(w)\sim(Pw,x \cdot Pw,-y) \equiv -x + y = a^*$	8, S.L.
10.	$\sim(\exists w)(Pw,x \cdot Pw,-y) \equiv -x + y = a^*$	9, Q.T.
11.	$\sim Ox,-y \equiv -x + y = a^*$	10, D0.4, 2, S.L.
12.	$(\sim ECx,-y \cdot \sim Ox,-y) \equiv \sim Cx,-y$	T0.28, S.L., 2
13.	$(\sim ECx,-y \cdot -x + y = a^*) \equiv \sim Cx,-y$	12 and 11, S.L.
14.	$(\sim ECx,-y \cdot -x + y = a^*) \equiv Px,y$	13, T1.34, S.L.
15.	$\sim ECx,-y \supset (-x + y = a^* \equiv Px,y)$	14, S.L.
16. $((\exists z)z = -x \cdot (\exists z)z = -y) \supset$ $[\sim ECx,-y \supset (-x + y = a^* \equiv Px,y)]$		1-15
17. $[((\exists z)z = -x \cdot (\exists z)z = -y) \cdot \sim ECx,-y] \supset$ $(-x + y = a^* \equiv Px,y)$.		16, S.L.

III Quasi-Topological part It is this third part which constitutes the main advantage of this present calculus of individuals. We saw earlier that by beginning with 'C' as our primitive, we were then able to distinguish between 'C' and 'O', and consequently to define 'EC', 'TP', and 'NTP'. This latter predicate and our Theory for the Fusion of Sets, enables us to introduce the quasi-topological operators, 'ix' for the interior of x, 'cx' for the closure of x, and 'ex' for the exterior of x, and to define such quasi-topological predicates as 'OPx', a rendering of 'x is open', and 'CLx', a rendering of 'x is closed'. The definitions are as follows:

D2.1 'ix' $=_{\text{def}}$ '$f'\{y: NTPy,x\}$'
D2.2 'cx' $=_{\text{def}}$ '$f'\{y: \sim Cy,i{-}x\}$'
D2.3 'ex' $=_{\text{def}}$ '$f'\{y: NTPy,-x\}$'
D2.4 'OPx' $=_{\text{def}}$ '$x = ix$'
D2.5 'CLx' $=_{\text{def}}$ '$x = cx$'.

If we wanted to utilize the Theory of Definite Descriptions in our new calculus, we could, along the lines of our definitional schema for the sum of all the individuals satisfying a certain predicate, D1.1′, substitute the following for D2.1-D2.3.

D2.1′ ‘ix’ $=_{\text{def}}$ ‘$(\iota y)\{(z)[Cz,y \equiv (\exists w)(NTPw, x \cdot Cz,w)]\}$’
D2.2′ ‘cx’ $=_{\text{def}}$ ‘$(\iota y)\{(z)[Cz,y \equiv (\exists w)(\sim Cw, i\text{-}x \cdot Cz,w)]\}$’
D2.3′ ‘ex’ $=_{\text{def}}$ ‘$(\iota y)\{(z)[Cz,y \equiv (\exists w)(NTPw, -x \cdot Cz,w)]\}$’.

For the quasi-topological part of the calculus, we shall need an additional axiom:

A2.1 $(x)((\exists z)NTPz,x \cdot (y)(z)\{[(Cz,x \supset Oz,x) \cdot (Cz,y \supset Oz,y)] \supset (Cz,x \wedge y \supset Oz,x \wedge y)\})$.

The first half of the main conjunct in the axiom assures us that each individual has an interior and the second half will assure us that the intersection of two open individuals (that is, individuals not containing their boundaries) is itself likewise open.

Given A2.1 and D2.1 the following theorems concerning the interiors of individuals become provable:

T2.1 $(x)(\exists y)y = ix$
T2.2 $(x)(y)[Cy,ix \equiv (\exists z)(NTPz, x \cdot Cy,z)]$
T2.3 $(x)(y)(NTPy,x \supset Py,ix)$
T2.4 $(x)Pix,x$
T2.5 $(x)(y)(Cy,ix \supset Oy,x)$
T2.6 $(x)(y)(ECy,x \supset \sim Cy,ix)$
T2.7 $(x)(y)(ECy,x \supset \sim ECy,ix)$
T2.8 $(x)(y)(Py,ix \supset Py,x)$
T2.9 $(x)NTPix,x$
T2.10 $(x)\sim TPix,x$
T2.11 $(x)(y)(Py,ix \equiv NTPy,x)$
T2.12 $(x)(y)(z)[(NTPx,y \cdot Cz,x) \supset Cz,iy]$
T2.13 $(x)(y)(z)[(NTPx,y \cdot Oz,x) \supset Oz,iy]$
T2.14 $(x)(y)(Px,y \supset Pix,iy)$
T2.15 $(x)(y)(x = y \supset ix = iy)$
T2.16 $(x)ix + x = x$
T2.17 $(x)ix \wedge x = ix$

Proof: T1.51; T0.21; T2.4; T1.42.

T2.18 $(x)(NTPx,x \equiv ix = x)$

Proof: T2.3; T2.4; T0.7; T2.11; T0.7.

T2.19 $(x)(y)(Ox,y \equiv Oix,iy)$

Proof: T0.43; T0.43; T2.11; A2.1; D0.4; T2.4; T0.19; T2.4.

T2.20 $(x)(y)(Ox,y \equiv Ox,iy)$

Proof: T2.21; T0.19; T2.4; T0.17; T0.19; T2.4.

T2.21 $(x)(y)(Cx,iy \equiv Ox,y)$
T2.22 $(x)(y)(Cx,iy \equiv Ox,iy)$
T2.23 $(x)(y)(Cix,iy \equiv Oix,iy)$
T2.24 $(x)(y)((\exists z)z = x \wedge y \equiv (\exists z)z = ix \wedge iy)$
T2.25 $(x)(y){\sim}ECx,iy$
T2.26 $(x)Pix,iix$

Proof: T2.24; T2.22; T0.18; D0.2.

T2.27 $(x)iix = ix$
T2.28 $ia^* = a^*$

Proof: T2.27; T1.25; T1.31; D0.8; T2.11; T0.7.

Since the first half of A2.1 assured us that every individual has an interior, T2.1, then all the theorems, T2.2-T2.28, are provable without an existential condition. Theorems T2.4, T2.27, and T2.28 give us three of the four standard properties of an interior operator. When, however, we come to the fourth standard characteristic of an interior operator, that is, that the interior of the intersection of two individuals is identical to the intersection of their interiors, we run into a condition; namely, the condition that the intersection of the two individuals exists, T2.32.

T2.29 $(x)(y)((\exists z)z = x \wedge y \supset Pix \wedge iy,x \wedge y)$

Proof: T2.8; T2.8; T1.43; D0.2; T2.24.

T2.30 $(x)(y)((\exists z)z = x \wedge y \supset Pi(x \wedge y), ix \wedge iy)$

Proof: T1.54; T2.11; T0.10; T1.43; D0.2; T2.24.

T2.31 $(x)(y)\{(\exists w)w = x \wedge y \supset (z)[(NTPz,x \cdot NTPz,y) \equiv NTPz,x \wedge y]\}$

Proof: T1.54; A2.1; D0.6; T1.45; D0.8.

T2.32 $(x)(y)((\exists z)z = x \wedge y \supset ix \wedge iy = i(x \wedge y))$

Proof: T2.31; T2.11; T1.45; T0.8; T2.30; T0.7; T2.24.

Likewise, since not every individual has a negate, all our theorems concerning the closure of an individual are conditional upon the existence of the negate of that individual, T2.35. In fact, at this point the proofs of the theorems become exceedingly complex due to the need to prove that upon the existence of the negate of the given individual, the other individuals used in the instantiations of the needed theorems for the proof likewise exist.

T2.33 $(x)((\exists z)z = cx \equiv (\exists y){\sim}Cy,i{-}x)$
T2.34 $(x)\{(\exists z)z = cx \supset (w)[Cw,cx \equiv (\exists y)({\sim}Cy,i{-}x \cdot Cw,y)]\}$
T2.35 $(x)((\exists z)z = -x \supset (\exists z)z = cx)$

Proof: T2.1; T0.6; T2.4; T1.25; T0.7; T1.30; T2.33; T1.30; T1.34; T0.5.

T2.36 $(x)[(\exists z)z = -x \supset (w)(Cw,cx \equiv {\sim}NTPw,-x)]$

Proof: T2.35; T2.34; T0.2; D0.2; T2.11.

T2.37 $(x)((\exists z)z = -x \supset cx = -i-x)$

Proof: T2.36; T2.11; T1.34; T0.3; T2.1; T2.35; T2.33; T1.30; T1.32.

T2.38 $(x)((\exists z)z = -x \supset i-x = -cx)$

Proof: T2.1; T2.35; T2.33; T1.30; T1.32; T0.5; T2.35; T2.37; T1.30; T1.32; T2.2; T2.36; T1.33; T0.3.

T2.39 $(x)((\exists z)z = -x \supset c-x = -ix)$

Proof: T1.35; T1.30; T0.5; T1.34; T1.32; T2.34; I.T.; T1.33; T2.35; T1.32; T0.6; T2.4; T1.25; T0.7; T1.32; T0.3.

T2.40 $(x)((\exists z)z = -x \supset ix = -c-x)$

Proof: T1.30; T0.5; T1.34; T1.32; T2.39; T1.35; I.T..

T2.41 $(x)((\exists z)z = -x \supset Px,cx)$

Proof: T1.30; T0.5; T1.34; T1.32; T2.1; T2.35; T2.33; T1.30; T1.32; T2.35; T1.35; T2.4; T1.40; I.T.; T2.37.

T2.42 $(x)((\exists z)z = -x \supset ccx = cx)$

Proof: T2.1; T2.1; T2.35; T0.5; T2.35; T2.37; T1.30; T1.32; T2.1; T2.35; T2.33; T1.30; T1.32; T2.35; T0.6; T2.4; T1.25; T0.7; T1.32; T1.32; T2.1; T0.6; T2.4; T1.25; T0.7; T1.30; T1.34; T1.32; T2.27; T1.38; T2.37; T2.38; T2.37.

T2.43 $(x)(y)\{[((\exists z)z = -x \cdot (\exists z)z = -y) \cdot (\exists z)z = -x \wedge -y] \supset$
$cx + cy = c(x + y)\}$

Proof: T1.35; T1.30; T0.5; T1.34; T1.32; T1.35; T1.30; T0.5; T1.34; T1.32; T2.35; T2.35; T1.37; T1.37; T0.9; T0.9; T1.50; T1.48; T0.2; T0.2; T1.11; T1.30; T1.9; T1.32; T2.35; T2.31; T1.46; T2.36; T1.11; T0.3.

T2.44 $(x)(y)[((\exists z)z = -x \cdot (\exists z)z = -y) \supset (Px,y \supset Pcx,cy)]$

Proof: T2.1; T2.35; T2.33; T1.30; T1.32; T2.1; T2.35; T2.33; T1.30; T1.32; T2.35; T2.35; T2.14; T1.40; T1.40; T2.37.

T2.45 $(x)((\exists z)z = -x \supset ex = i-x)$.

Theorems T2.37-T2.40 give us the standard relationships between the closures and interiors of individuals, albeit conditioned. And theorems T2.41-T2.43 give us three of the four standard characteristics of a closure operator, albeit conditioned. Since we have no null individual we cannot prove the fourth characteristic, namely, that the closure of the null individual is identical to the null individual.

We mentioned earlier, with reference to T1.56, that the linguistic domain of the present calculus of individuals not only lacked a null element, as does the classical calculus, but it also lacks boundary elements. A boundary element is generally characterized in this way: x is a boundary element if, and only if, $ix = 0^*$. Since every element in this calculus has an interior, T2.1, and we have no null element, it follows that we can have no boundary elements. Thus just as the linguistic domain of the classical calculus of individuals is a Boolean algebra with the null element removed, our theorems indicate that the domain of the present calculus is a closure algebra[6] with the null element and the boundary elements removed. It is interesting, however, that so much topology can be reflected under these conditions and with such minimal assumptions. This should be particularly interesting to those in the Leśniewski tradition. Likewise, it should also be of interest to those interested in Whitehead's Theory of Extensive Connection, for it bodes well for the success of his over-all project to found geometry on such a basis.

NOTES

1. For an exposition of Leśniewski's system, see [16] and [23].
2. This contradiction appears to have been noted first by Palter [21].
3. Dwight Van de Vate utilized an axiomatization of Whitehead's mereological system in his Yale dissertation, *The Formalization of Certain Aristotelian Concepts*, 1957. His axiomatization differs from the present one in taking '$(x)(y)[(z)(Oz,x \supset Oz,y) \supset Px,y]$' as an axiom. This makes it possible to prove as theorems: '$(x)[Px,y \equiv (z)(Oz,x \supset Oz,y]$' and '$(x)ix = x$'. As a consequence we have the unfortunate result that the calculus is reduced to the classical calculus. Instead of Van de Vate's result, we have T0.31 as a theorem. It was, however, Van de Vate's work that first got me interested in the present calculus of individuals.
4. See my "Individuals and Points," forthcoming. This extension of the system, however, requires that the system be nonatomic.
5. For a treatment of this general problem and some selected solutions, see [2], [5], [6], and [8]-[13].
6. For a thorough discussion of closure algebras and their relation to topology, see [17].

REFERENCES

[1] Carnap, R., *Meaning and Necessity*, University of Chicago Press, Chicago, Illinois, 1947.

[2] Eberle, R. A., *Nominalistic Systems*, D. Reidel, Dordrecht, 1970.

[3] Frege, G., "Über sinn und bedeutung," *Zeitschrift für Philosophie und philosophisch Kritik*, vol. 100 (New Series, 1892), pp. 25-50.

[4] Goodman, N., *The Structure of Appearance*, Harvard University Press, Cambridge, Massachusetts, 1951.

[5] Hintikka, J., "Existential presuppositions and existential commitments," *The Journal of Philosophy*, vol. 56 (1959), pp. 125-137.

[6] Hintikka, J., "Towards a theory of definite descriptions," *Analysis*, vol. 19 (1959), pp. 79-85.

[7] de Laguna, T., "Point, line and surface, as sets of solids," *The Journal of Philosophy*, vol. 19 (1922), pp. 449-461.

[8] Lambert, K., "Singular terms and truth," *Philosophical Studies*, vol. 10 (1959), pp. 1-5.

[9] Lambert, K., "Notes on 'E!': II," *Philosophical Studies*, vol. 12 (1961), pp. 1-5.

[10] Lambert, K., "Notes on E!: III: a theory of descriptions," *Philosophical Studies*, vol. 13 (1962), pp. 51-59.

[11] Lambert, K., "On logic and existence," *Notre Dame Journal of Formal Logic*, vol. 6 (1965), pp. 135-141.

[12] Leblanc, H. and T. Hailperin, "Nondesignating singular terms," *Philosophical Review*, vol. 68 (1959), pp. 239-243.

[13] Leonard, H. S., "The logic of existence," *Philosophical Studies*, vol. 7 (1956), pp. 49-64.

[14] Leonard, H. S. and N. Goodman, "The calculus of individuals and its uses," *The Journal of Symbolic Logic*, vol. 5 (1940), pp. 45-55.

[15] Leśniewski, S., "O podstawack matematyki," *Prezeglad Filosoficzny*, vols. 30-34 (1927-1931).

[16] Luschei, E. C., *The Logical Systems of Leśniewski*, North-Holland, Amsterdam, 1962.

[17] McKinsey, J. C. C. and A. Tarski, "The algebra of topology," *Annals of Mathematics*, vol. 45 (1944), pp. 141-191.

[18] Martin, R. M., "A homogeneous system for formal logic," *The Journal of Symbolic Logic*, vol. 8 (1943), pp. 1-23.

[19] Martin, R. M., "Of time and the null individual," *The Journal of Philosophy*, vol. 62 (1965), pp. 723-736.

[20] Martin, R. M., *Events, Reference, and Logical Form*, Catholic University of America Press, Washington, D.C., 1978.

[21] Palter, R. M., "Preface," *Whitehead's Philosophy of Science*, Second Impression, University of Chicago Press, Chicago, Illinois, 1970.

[22] Russell, B., "Preface," *Our Knowledge of the External World*, G. Allen & Unwin, Ltd., London, 1952.

[23] Sobociński, B., "L'analyse de l'antinomie russellienne par Leśniewski," *Methodos*, vol. I (1949), pp. 94-107, 220-228, 308-316; vol. II (1950), pp. 237-257.

[24] Tarski, A., "Appendix E" in *The Axiomatic Method in Biology*, J. H. Woodger, Cambridge University Press, Cambridge, 1937.

[25] Tarski, A., "Foundations of the geometry of solids," in *Logic, Semantics, Metamathematics*, transl. J. H. Woodger, Oxford University Press, Oxford, 1956.

[26] Tarski, A., "On the foundations of Boolean algebra," in *Logic, Semantics, Metamathematics*, transl. J. H. Woodger, Oxford University Press, Oxford, 1956.

[27] Whitehead, A. N., *An Enquiry Concerning the Principles of Natural Knowledge*, Cambridge University Press, Cambridge, 1919.

[28] Whitehead, A. N., *The Concept of Nature*, Cambridge University Press, Cambridge, 1920.

[29] Whitehead, A. N., *Process and Reality*, The MacMillan Company, New York, 1929.

[30] Woodger, J. H., *The Axiomatic Method in Biology*, Cambridge University Press, Cambridge, 1937.

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