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Cofinal Extensions of Nonstandard Models of Arithmetic

C. SMORYŃSKI

A relatively neglected aspect of the study of nonstandard models of arithmetic is the study of their cofinal extensions.* These extensions certainly do not present themselves to the intuition as readily as do their more popular cousins the end extensions; but they are not exactly shrouded in mystery or unnatural objects of study either. They are equal partners with end extensions in the construction of general extensions of models; they offer both special advantages and disadvantages worthy of our interest; and, occasionally, they are useful in understanding the generally more simply behaved end extensions. Cofinal extensions deserve more attention than they have traditionally received.

1 The splitting theorem The fundamental theorem on cofinal extensions is Gaifman's Splitting Theorem, which not only establishes their existence but also reveals one of their most basic properties. Briefly, the Splitting Theorem asserts that every extension of nonstandard models splits into an *elementary* cofinal extension followed by an end extension. In particular, it follows that cofinal extensions are always elementary.

Unfortunately, the Splitting Theorem is language dependent. If we add a few new relations to the language of arithmetic, the theorem could well become false. For this reason, there are two additional versions of the theorem. The simplest form it takes, valid for any language (provided full induction is assumed), is the following:

1.1 Elementary Splitting Theorem *Let $\mathfrak{M} \prec \mathfrak{N}$ be models of arithmetic. There is another model $\mathfrak{M}^{cf}$ of arithmetic satisfying*

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$$\mathfrak{M} \prec_c \mathfrak{M}^{cf} \prec_e \mathfrak{N},$$

i.e., $\mathfrak{M}^{cf}$ *elementarily cofinally extends* $\mathfrak{M}$ *and* $\mathfrak{N}$ *elementarily end extends* $\mathfrak{M}^{cf}$.

Proof: Let $|\mathfrak{M}^{cf}| = \{a \in |\mathfrak{N}|: \exists b \in |\mathfrak{M}|(a < b)\}$. This is closed under $+$, $\cdot$, $'$ since these functions are monotone. It is closed under application of parameter-free Skolem functions since they can be majorized by functions under which $\mathfrak{M}$ is closed: If Fv is a parameter-free Skolem function, $a \in |\mathfrak{M}^{cf}|$ and $b \in |\mathfrak{M}|$ is such that $b > a$, then

$$Fa \leqslant Gb = \sup\{Fv: v \leqslant b\}.$$

1.2 Corollary *Every nonstandard model of arithmetic has a proper elementary cofinal extension.*

The corollary is probably best credited to Rabin who, in [15], first announced the existence of such extensions under an irrelevant hypothesis. The origin of the Elementary Splitting Theorem is not known to the author. Although it would seem to have been needed by Rabin in order to deduce the corollary, he makes no mention of it and, indeed, it seems only to have been stated in the literature (e.g., in [3] and [6]) after the full Splitting Theorem was known.

Rabin did, however, include in his paper a rudimentary form (essentially Corollary 1.4, under the same irrelevant hypothesis) of the nonelementary Splitting Theorem. This was further discussed by Adler ([1]) and Chudnovsky ([4]), and, finally, by Gaifman in [5] where the full result was proven:

1.3 Splitting Theorem *Let* $\mathfrak{M} \subseteq \mathfrak{N}$ *be models of arithmetic in the usual arithmetic language. There is another model* $\mathfrak{M}^{cf}$ *of arithmetic satisfying*

$$\mathfrak{M} \prec_c \mathfrak{M}^{cf} \subseteq_e \mathfrak{N},$$

i.e., $\mathfrak{M}^{cf}$ *elementarily cofinally extends* $\mathfrak{M}$ *and* $\mathfrak{N}$ *end extends* $\mathfrak{M}^{cf}$.

1.4 Corollary *In the usual arithmetic language, cofinal extensions are always elementary.*

Since the Skolem functions of $\mathfrak{M}$ and $\mathfrak{N}$ can differ, Theorem 1.3 must be given a different proof from that of its special case Theorem 1.1. $|\mathfrak{M}^{cf}|$ is defined as before and the closure of $|\mathfrak{M}^{cf}|$ under a few encoding and decoding functions is proven similarly. With coding ability, it is not hard to use the cofinality of $\mathfrak{M}$ in $\mathfrak{M}^{cf}$ to bound quantifiers and make, as it were, all sentences Δ_0. Since $\mathfrak{M}^{cf}$ is an initial segment of $\mathfrak{N}$, $\mathfrak{M}^{cf}$ and $\mathfrak{N}$ agree on Δ_0 formulas and it suffices to see that $\mathfrak{M}$ and $\mathfrak{N}$ similarly agree. But for this we have the solution to Hilbert's 10th Problem: All Δ_0 formulas are simultaneously purely existential (and so preserved from $\mathfrak{M}$ to $\mathfrak{N}$) and purely universal (and so preserved from $\mathfrak{N}$ to $\mathfrak{M}$).

If one's arithmetic is in an expanded language, the last step of the above proof cannot always be made and one has in this case the following result (cf. [12]):

1.5 General Splitting Theorem *Let* $\mathfrak{M} \subseteq \mathfrak{N}$ *be models of arithmetic. If the extension is* Δ_0*-elementary, there is another model* $\mathfrak{M}^{cf}$ *of arithmetic satisfying*

$$\mathfrak{M} \prec_c \mathfrak{M}^{cf} \subseteq_e \mathfrak{N}.$$

The Splitting Theorem is, particularly for the usual arithmetic language, of supreme methodological importance. First, for this language, it establishes as completely reasonable the dichotomization of research into extensions of nonstandard models into separate, but equal, studies of end and cofinal extensions. Second, since cofinal embeddings in the usual case are always elementary, it suggests that for other languages as well the natural cofinal embeddings to study are the elementary ones. We take this consequence to heart: *Throughout the rest of this survey, we will consider only elementary cofinal extensions*—regardless of the underlying language. Third, and perhaps most vaguely, since cofinal extensions are elementary and end extensions need not be, it points to the difference in roles played by the two kinds of new integers in cofinal and end extensions.

Technically, the Splitting Theorem has not yet been of supreme importance. But it has been of use. Wilkie in [18] showed how, in conjunction with the Löwenheim-Skolem Theorem, it could be used to lift certain results concerning initial segments and end extensions from the countable to the uncountable case. Rather than repeat Wilkie's example (for which Lesan [11] has supplied a lovely direct proof that works in the general case in his thesis), we give a simpler application:

1.6 Application In [7], Guaspari proved the equivalence, for a given recursively enumerable theory $T \supseteq PA$, of the following two assertions:

i. For a sentence ϕ, $T + \phi$ is Σ_1-conservative over T, i.e., for all Σ_1 sentences σ, if $T + \phi \vdash \sigma$, then $T \vdash \sigma$;
ii. For any *countable* model $\mathfrak{N} \models T$, there is an initial segment $\mathfrak{M} \subseteq_e \mathfrak{N}$ such that $\mathfrak{M} \models T + \phi$.

The proof that ii implies i is straightforward and makes no use of the countability of $\mathfrak{N}$. Guaspari's proof of the converse, however, depended on Friedman's initial embeddability criterion and only worked in the countable case. With the aid of the Splitting Theorem, however, the uncountable case reduces quickly to the countable case: Assume ii. Let $\mathfrak{N} \models T$ and let $\mathfrak{N}_0 \prec \mathfrak{N}$ be a countable elementary submodel of $\mathfrak{N}$. Applying ii to $\mathfrak{N}_0$, we get an $\mathfrak{M}_0 \subseteq_e \mathfrak{N}_0$ such that $\mathfrak{M}_0 \models T + \phi$. Simply consider $\mathfrak{M}_0 \subseteq \mathfrak{N}$ and split the extension:

$$\mathfrak{M}_0 \prec_c \mathfrak{M}_0^{cf} \subseteq_e \mathfrak{N}.$$

$\mathfrak{M}_0^{cf}$ is the initial segment $\mathfrak{M}$ desired. (We should mention that Guaspari notes Joe Quinsey has given an alternate proof of the equivalence in the uncountable case.)

2 The greatest common initial segment The greatest common initial segment of the two models of an extension is of importance in more ways than can be conveniently summarized in advance. Let us simply define it cold:

2.1 Definition Let $\mathfrak{M} \prec_c \mathfrak{N}$ be given. The *greatest common initial segment* of $\mathfrak{M}$ and $\mathfrak{N}$, denoted GCIS($\mathfrak{M}, \mathfrak{N}$), is the set

$$I = \{a \in |\mathfrak{M}| : \forall b \in |\mathfrak{N}| - |\mathfrak{M}| (a < b)\}.$$

The obvious significance of the greatest common initial segment of an extension $\mathfrak{M} \prec_c \mathfrak{N}$ is that it determines–partially–where the new elements of the extension begin to appear. We say "partially" because nonstandard models are not well-ordered and there is no guarantee that the new elements occur *immediately* after GCIS($\mathfrak{M}, \mathfrak{N}$):

2.2 Definition Let $\mathfrak{M} \prec_c \mathfrak{N}$ be given and let $I = \text{GCIS}(\mathfrak{M}, \mathfrak{N})$. We say that $\mathfrak{N}$ is a *regular* cofinal extension of $\mathfrak{M}$ iff there is an element $a \in |\mathfrak{N}| - |\mathfrak{M}|$ such that $I < a < |\mathfrak{M}| - I$.

Now the definition of regularity merely requires that new elements begin to appear immediately after the greatest common initial segment. It says nothing about how many such elements immediately appear. But, before we get too deeply embroiled in a maze of complications, let us look at what we already have.

2.3 Lemma *Let $\mathfrak{M} \prec_c \mathfrak{N}$ be given and let $I = GCIS(\mathfrak{M}, \mathfrak{N})$.*

i. $\omega \subseteq I$
ii. *I is closed under addition and multiplication*
iii. *If $\mathfrak{N}$ is a regular extension of $\mathfrak{M}$, then I is also closed under exponentiation.*

Proof: i. Trivial.

ii.a. Addition. Let $a, b \in I$. We want to see that $a + b \in I$. Let $c \in |\mathfrak{N}| - |\mathfrak{M}|$. It suffices to show that $a + b < c$.

Either $c = 2[c/2]$ or $c = 2[c/2] + 1$. In either case, $[c/2] \in |\mathfrak{N}| - |\mathfrak{M}|$, whence $a, b < [c/2]$ and

$$a + b < 2[c/2] \leqslant c.$$

ii.b. Multiplication. Let $a, b \in I$. To see that $a \cdot b \in I$, we let $c \in |\mathfrak{N}| - |\mathfrak{M}|$ and consider $d = [\sqrt{c}]$. Suppose $d \in |\mathfrak{M}|$. There are $2d + 1$ elements e of $|\mathfrak{M}|$ such that $d^2 \leqslant e < (d + 1)^2$. Thus $c = d^2 + e$ for some $e < 2d + 1$. Since $c \notin |\mathfrak{M}|$ and $d \in |\mathfrak{M}|$, it follows that $e \in |\mathfrak{N}| - |\mathfrak{M}|$ and so $[e/2] \in |\mathfrak{N}| - |\mathfrak{M}|$, whence

$$a, b < [e/2] \leqslant d.$$

If $d \notin |\mathfrak{M}|$, then $a, b < d$ automatically. Thus, $a, b < d$, whence

$$a \cdot b < d^2 \leqslant c.$$

iii. Assume the extension is regular, with $b \in |\mathfrak{N}|$ such that

$$I < b < |\mathfrak{M}| - I.$$

Suppose for some $a \in I$, $a^a \notin I$. Then, if

$$c = \max\{d\colon d^d \leqslant b\},$$

we have $c < a$, so $c \in I$. But

$$\begin{aligned}(c + 1)^{(c+1)} &= (c + 1)^c(c + 1) < (2c)^c(c + 1)\\ &< 2^c \cdot c^c \cdot (c + 1)\\ &< c^c \cdot c^c \cdot (c + 1) \in I,\end{aligned}$$

since $c^c \in I$ and I is closed under multiplication.

Parts i and ii of the lemma are best possible; part iii is but an example of the power of regularity. The converse to Lemma 2.3.ii was proven by Paris and Mills in [13] and is only known in the countable case:

2.4 Theorem *Let $\mathfrak{M}$ be a countable nonstandard model and let $I \subseteq_e \mathfrak{M}$ be a nontrivial initial segment. Then the following are equivalent:*

i. *I is closed under addition and multiplication*
ii. *$I = GCIS(\mathfrak{M}, \mathfrak{N})$ for some cofinal extension $\mathfrak{N}$ of $\mathfrak{M}$.*

In conjunction with Lemma 2.3.iii, this theorem tells us that the answer to the silly question of whether or not new elements immediately follow the greatest common initial segment of an extension is a matter of consequence. The full impact of regularity in terms of closure properties of this segment has, again in the countable case, been assessed by Kirby and Paris in [8] and [9]:

2.5 Theorem *Let $\mathfrak{M}$ be a countable nonstandard model and let $I \subseteq_e \mathfrak{M}$ be a nontrivial initial segment. Then the following are equivalent:*

i. *I is regular in $\mathfrak{M}$*
ii. *$I = GCIS(\mathfrak{M}, \mathfrak{N})$ for some regular cofinal extension $\mathfrak{N}$ of $\mathfrak{M}$.*

We refer the reader to the above cited papers for the proof as well as for the definition of the regularity of an initial segment. We note merely that regularity is somewhat stronger than closure under exponentiation, but that we have stressed the latter in Lemma 2.3.iii for the sake of Corollary 2.14, below.

As remarked earlier, the regularity of a cofinal extension merely depends on the immediate appearance of *some* rather than *many* new elements after the greatest common initial segment. To assess the impact of many such new elements, we need some definitions:

2.6 Definitions Let $\mathfrak{M} \prec_c \mathfrak{N}$ and $I \subseteq_e \mathfrak{M}$ be given. The *closures* of I in $\mathfrak{N}$ are the two initial segments of $\mathfrak{N}$ defined by:

$$I^- = I^{cf} = \{a \in |\mathfrak{N}|: \exists b \in I(a < b)\}$$
$$I^+ = \{a \in |\mathfrak{N}|: \forall b \in |\mathfrak{M}| - I(a < b)\}.$$

Note that $I \subseteq I^- \subseteq I^+$ and $I = I^-$ iff $I \subseteq \mathrm{GCIS}(\mathfrak{M}, \mathfrak{N})$. It can happen that I^+ properly extends I^-. Indeed, if $\mathfrak{N}$ is a regular cofinal extension of $\mathfrak{M}$, this happens for $I = \mathrm{GCIS}(\mathfrak{M}, \mathfrak{N})$. Since $I \subseteq_c I^-$, these two segments are closed under the same monotone functions definable in $\mathfrak{M}$. Of greater interest is the fact that I^+ inherits some closure:

2.7 Lemma *Let $\mathfrak{M} \prec_c \mathfrak{N}$ and $I \subseteq_e \mathfrak{M}$ be given. Let F: $\mathfrak{N} \to \mathfrak{N}$ be definable in $\mathfrak{N}$ with parameters from $\mathfrak{M}$ and strictly monotone. If I is closed under F, so is I^+.*

Proof: Since F is strictly monotone, we can define a sort of inverse function

$$Ga = \text{least } b[Fb \leqslant a < F(b+1)].$$

Let $a \in I^+$, $c \in |\mathfrak{M}| - I$. Since I is closed under F, $I < Gc$. But $Gc \in |\mathfrak{M}|$, whence $a < Gc$. But then

$$Fa < FGc \leqslant c.$$

2.8 Corollary *Let $\mathfrak{N}$ be a regular cofinal extension of $\mathfrak{M}$ and $I = GCIS(\mathfrak{M}, \mathfrak{N})$. Then I^+ is closed under addition, multiplication, and exponentiation.*

Proof: For addition, let $Fa = 2a$; for multiplication, $Fa = a^2$; and for exponentiation, $Fa = a^a$.

Not every nice closure property of $I = \mathrm{GCIS}(\mathfrak{M}, \mathfrak{N})$ is inherited by I^+, as the following result of Kirby and Paris (from [8] and [9]) shows:

2.9 Theorem *Let $\mathfrak{M}$ be a countable nonstandard model and let $I \subseteq_e \mathfrak{M}$ be a nontrivial initial segment. Then the following are equivalent:*

i. *I is strong in $\mathfrak{M}$*
ii. *$I = GCIS(\mathfrak{M}, \mathfrak{N})$ and I^+ is semiregular in $\mathfrak{N}$ for some regular cofinal extension $\mathfrak{N}$ of $\mathfrak{M}$.*

The definitions of "strong" and "semiregular" are to be found in the papers cited. We note merely that semiregularity is a strictly weaker property of initial segments than regularity, and regularity is strictly weaker than strength. By this theorem, we see that closure conditions on the set of immediate new elements entail strong closure conditions on the greatest common initial segment.

One more example along these lines is given by the following result from Kirby's dissertation [8]:

2.10 Theorem *Let $\mathfrak{M}$, $\mathfrak{N}$ be countable, $\mathfrak{N}$ a regular cofinal extension of $\mathfrak{M}$, and let $I = GCIS(\mathfrak{M}, \mathfrak{N})$. If I is strong in $\mathfrak{M}$, then $I \prec_e I^+$.*

Evidently, a great deal more can be done along these lines. But there are other lines to follow. Integers code sets and new integers can code new sets. The position where the new integers begin to appear has a direct bearing on what new sets can appear.

2.11 Definitions Let $\mathfrak{M}$ be nonstandard and $I \subseteq \mathfrak{M}$. Any set $X \subseteq I$ is termed an *I-set*. $X \subseteq I$ is an *I-set of* $\mathfrak{M}$ if, for some formula $\phi v v_0$ and some parameter $b \in |\mathfrak{M}|$,

$$X = \{a \in I \colon \mathfrak{M} \models \phi \overline{ab}\}.$$

Finally, an I-set X is a *bounded I-set* if $X < b$ for some $b \in I$.

For $I = \omega$, the I-sets are called *standard* sets and it is well-known that the collection of standard sets does not change in the passage to an end extension. For $I = \mathfrak{M}$, Phillips (in [14]) has defined an extension $\mathfrak{M} \subseteq \mathfrak{N}$ to be *conservative* provided $\mathfrak{M}$ and $\mathfrak{N}$ have the same I-sets. Moreover, he observed that elementary conservative extensions are always end extensions. A close look at his argument (for which cf. [3] or [6]) reveals something more:

2.12 Theorem *Let $\mathfrak{M} \subseteq \mathfrak{N}$ be nonstandard models. The following are equivalent:*

i. *$\mathfrak{M}$ and $\mathfrak{N}$ have the same bounded $\mathfrak{M}$-sets*
ii. *$\mathfrak{M} \subseteq_e \mathfrak{N}$.*

Instead of proving this, we prove the related:

2.13 Theorem *Let $\mathfrak{M} \prec_c \mathfrak{N}$, $I = GCIS(\mathfrak{M}, \mathfrak{N})$, and $J \subseteq_e \mathfrak{M}$.*

i. *Every (bounded) J-set of $\mathfrak{M}$ is a (bounded) J-set of $\mathfrak{N}$*
ii. *If $I \subsetneq J$, then $\mathfrak{N}$ has more bounded J-sets than $\mathfrak{M}$*
iii. *If $J \subseteq I$ and J is closed under exponentiation, then $\mathfrak{M}$ and $\mathfrak{N}$ have the same bounded J-sets.*

Proof: i. Since the extension is elementary, definitions in $\mathfrak{M}$ lift to $\mathfrak{N}$.

ii. Let $b \in |\mathfrak{N}| - |\mathfrak{M}|$, $c \in J$ such that $b < c$, and define

$$X = \{a \in J: a < b\}.$$

Then $X < c$ is a bounded J-set of $\mathfrak{N}$. To see that it is not a J-set of $\mathfrak{M}$, suppose

$$X = \{a \in J: \mathfrak{M} \models \phi a\overline{d}\}$$

for some formula ϕ and some $d \in |\mathfrak{M}|$. But then

$$X = \{a \in |\mathfrak{M}|: \mathfrak{M} \models \phi\overline{ad} \wedge \overline{a} < \overline{c}\},$$

and X is definable in $\mathfrak{M}$. But $0 \in X$ and X is closed under successor, whence induction yields $X = |\mathfrak{M}|$, a contradiction.

iii. Let $J \subseteq I$ be closed under exponentiation and let X be a bounded J-set of $\mathfrak{N}$. Pick $c \in J$ such that $X < c$ and choose $d \in |\mathfrak{N}|$ by Arithmetic Separation such that

$$D_d = X = \{a \in |\mathfrak{N}|: a \in X \wedge a < c\},$$

where D_d is the $\mathfrak{N}$-finite set with canonical index d:

$$D_d = \{d_0 < \ldots < d_{e-1}\}, \text{ where } d = 2^{d_0} + \ldots + 2^{d_{e-1}}.$$

But

$$d \leqslant \sum_{i<c} 2^i < 2^c \in J.$$

Thus $d \in J$ and X is a bounded J-set of $\mathfrak{M}$ defined in $\mathfrak{M}$ by the formula

$$\phi v\overline{d}: v \in D_{\overline{d}}.$$

2.14 Corollary *Let $\mathfrak{M}$ be given, $\mathfrak{N}$ a regular cofinal extension of $\mathfrak{M}$, and $I = GCIS(\mathfrak{M}, \mathfrak{N})$. Then $\mathfrak{M}$ and $\mathfrak{N}$ have the same bounded I-sets.*

Proof: By Lemma 2.3.iii, I is closed under exponentiation.

What about unbounded I-sets? The prototype is Phillips' notion of conservative extension. Not every elementary end extension is conservative; indeed, Kirby notes in his thesis that, if $\mathfrak{N}$ is a conservative elementary extension of $\mathfrak{M}$, then $\mathfrak{M}$ is strong in $\mathfrak{N}$. A related result for cofinal extensions is the following result of Kirby and Paris from [8] and [9]:

2.15 Theorem *Let $\mathfrak{M}$ be a countable nonstandard model and let $I \subseteq_e \mathfrak{M}$ be a nontrivial initial segment. The following are equivalent:*

i. *I is strong in* $\mathfrak{M}$
ii. $I = GCIS(\mathfrak{M}, \mathfrak{N})$ *for some regular extension* $\mathfrak{N}$ *of* $\mathfrak{M}$ *which possesses the same* I*-sets as* $\mathfrak{M}$.

3 Short models Psychologists assure us that tall people command more attention and respect than short ones. In as anthropomorphic a field as logic, it would follow that taller concepts excite more imagination than shorter ones. Thus, more is written about tall end extensions than stubby cofinal ones, and, asked for a preference between tall and short models, most logicians would make the tall choice. Such high-minded strategy might work well in the short run; but in the long run we must pay everything its due.

3.1 Definition Let $\mathfrak{M}$ be nonstandard. $\mathfrak{M}$ is called a *simple* model if $\mathfrak{M}$ is generated by one element by means of its parameter-free Skolem functions. If $a \in |\mathfrak{M}|$ is such a generator, we write $\mathbf{N}[a]$ for $\mathfrak{M}$.

The notation $\mathbf{N}[a]$ is rather bad: it falsely suggests that $\mathbf{N}[a]$ is an elementary extension of the standard model $\mathbf{N}$. However, the author has grown used to this notation and is not willing to give it up just because of a minor flaw.

3.2 Definitions Let $\mathfrak{M}$ be nonstandard. $\mathfrak{M}$ is *short* if it has a cofinal simple submodel, i.e., if there is an $a \in |\mathfrak{M}|$ such that $\mathbf{N}[a] \prec_c \mathfrak{M}$. If $\mathfrak{M}$ is not short, $\mathfrak{M}$ is called *tall.*

The tall/short distinction is fine so long as one does not try to turn it into a comparative distinction, i.e., tall*er vs* short*er*. The following simple lemma supporting the noncomparative notion can be proven by considerations similar to those of the proof of Theorem 1.1:

3.3 Lemma *Let* $\mathfrak{M} \prec_c \mathfrak{N}$ *be given.*

i. $\mathfrak{M}$ *is short iff* $\mathfrak{N}$ *is short*
ii. $\mathfrak{M}$ *is tall iff* $\mathfrak{N}$ *is tall.*

With this lemma, we see that the distinction between short and tall models has at least one property testifying to its coherence. The following result of Blass (from [2]) offers even more convincing evidence of the significance of the distinction and suggests some importance for short models:

3.4 Theorem *Let* $\mathfrak{N}$ *be a short nonstandard model and* $\mathfrak{M}_0$, $\mathfrak{M}_1$ *cofinal submodels. Then:* $\mathfrak{M}_0 \cap \mathfrak{M}_1 \prec_c \mathfrak{N}$.

We note the following: if we do not assume the cofinality of $\mathfrak{M}_0$, $\mathfrak{M}_1$ in $\mathfrak{N}$, it need not follow that $\mathfrak{M}_0 \cap \mathfrak{M}_1$ is a model of *PA* (as remarked already by Rabin in [15]). By the cofinality of the extensions, we are dealing with elementary submodels and it follows that $\mathfrak{M}_0 \cap \mathfrak{M}_1 \prec \mathfrak{N}$. But, unless $\mathfrak{N}$ is short, it need not be the case that the intersection is cofinal in $\mathfrak{N}$: Blass has constructed tall models $\mathfrak{M}_0$, $\mathfrak{M}_1$, $\mathfrak{N}$ with each $\mathfrak{M}_i$ cofinal in $\mathfrak{N}$ and with $|\mathfrak{M}_0 \cap \mathfrak{M}_1| = \omega$. Thus, the theorem exhibits a property peculiar to short models.

Blass's Theorem, or rather the work behind it, has an interesting application to recursively saturated models of arithmetic (solving, incidently, an open problem of [16]):

3.5 Theorem *Let $\mathfrak{M}$ be a countable recursively saturated model of arithmetic. Then $\mathfrak{M}$ has a countable infinity of nonisomorphic elementary initial segments.*

It is an elementary observation that $\mathfrak{M}$ has continuum many distinct elementary initial segments. Slightly less elementary is the fact that there is not much variety among them; obviously there are two types of such initial segments—tall ones and short ones. The tall ones inherit the recursive saturation of $\mathfrak{M}$ and, thus, are all isomorphic to $\mathfrak{M}$ (cf., e.g., [16]); the short ones, as we shall see, need not be isomorphic, but they are determined by their simple cofinal submodels and so are countable in number.

Just before we finished checking the final stages of our proof of Theorem 3.5, we heard from Kotlarski that he had proven that the short elementary initial segments need not be isomorphic. Presumably his proof yields the actual cardinal calculation as well and he intends to publish his results. This happy coincidence allows us to be quite sketchy in the exposition of our proof, which is largely Blass's proof of Theorem 3.4 and in print anyway.

We begin with two lemmas directly from [2]:

3.6 Lemma *Let $c \in \mathbf{N}[a]$. The following are equivalent:*

i. *For some F: $\mathbf{N}[a] \to \mathbf{N}[a]$ definable without parameters,*
   a. $\mathbf{N}[a] \models F$ *is finite-to-one*
   b. $c = Fa$
ii. $\mathbf{N}[c] \prec_c \mathbf{N}[a]$.

The point here is that every $c \in \mathbf{N}[a]$ is of the form Fa. If F is finite-to-one, while it does not necessarily have an actual inverse, it does have enough of an inverse to capture something of the magnitude of a from c, and thus to guarantee the cofinality of $\mathbf{N}[c]$ in $\mathbf{N}[a]$.

3.7 Lemma *Let $\mathbf{N}[a], \mathbf{N}[b] \prec_c \mathfrak{K}$. There is a $c \in |\mathfrak{K}|$ such that*

$$\mathbf{N}[c] \prec_c \mathbf{N}[a] \text{ and } \mathbf{N}[c] \prec_c \mathbf{N}[b].$$

This lemma, the proof of which is based on Lemma 3.6, is the secret behind the proof of Theorem 3.4. The fact is that only one of the initially chosen cofinal submodels needs to be simple in its proof. Thus, if $\mathfrak{N}$ is short and $\mathfrak{M}_0$, $\mathfrak{M}_1$ are cofinal submodels, $\mathfrak{M}_0$ and "the" simple submodel witnessing the shortness of $\mathfrak{N}$ have a common simple cofinal submodel, which shares a similar submodel with $\mathfrak{M}_1$, whence Theorem 3.4.

Our reason for stating the weak form of the lemma is to pair it with the following converse.

3.8 Lemma *If $\mathbf{N}[c] \prec_c \mathbf{N}[a]$ and $\mathbf{N}[c] \prec_c \mathbf{N}[b]$, there is a model $\mathfrak{K}$ such that*

$$\mathbf{N}[a] \prec_c \mathfrak{K} \text{ and } \mathbf{N}[b] \prec_c \mathfrak{K}.$$

To prove this, simply let $\mathfrak{N}$ be an $\aleph_0$-saturated elementary extension of $\mathbf{N}[c]$ and split:

$$\mathbf{N}[c] \prec_c \mathfrak{K} \prec_e \mathfrak{N}.$$

The saturation of $\mathfrak{N}$ allows both $\mathsf{N}[a]$ and $\mathsf{N}[b]$ to be embedded in $\mathfrak{N}$. Since $\mathsf{N}[c] \prec_c \mathsf{N}[a]$ and $\mathsf{N}[c] \prec_c \mathsf{N}[b]$, it follows that $\mathsf{N}[a] \prec_c \mathfrak{K}$ and $\mathsf{N}[b] \prec_c \mathfrak{K}$.

Putting Lemmas 3.6-3.8 together, we get the following characterization.

3.9 Theorem *Let $T = Th(\mathsf{N}[a]) = Th(\mathsf{N}[b])$. The following are equivalent:*

i. *There is a $\mathfrak{K}$ such that*

$$\mathsf{N}[a] \prec_c \mathfrak{K} \text{ and } \mathsf{N}[b] \prec_c \mathfrak{K}$$

ii. *There are functions F, G provably total in T such that*

a. $T \vdash F, G$ *are finite-to-one*
b. $t_{Fa}v = t_{Gb}v$, *where*

$$t_c v = \{\phi v \colon \mathsf{N}[c] \models \phi\bar{c}\}$$

is the type of c.

This theorem furnishes us with the tool needed to construct the non-isomorphic elementary initial segments promised in the statement of Theorem 3.5. To find nonisomorphic $\mathfrak{M}_0, \mathfrak{M}_1 \prec_e \mathfrak{M}$, where $\mathfrak{M}$ is a given recursively saturated model, we merely diagonalize to find $a, b \in |\mathfrak{M}|$ such that, for all F, G which $Th(\mathfrak{M})$ declares to be finite-to-one, we have $t_{Fa}v \neq t_{Gb}v$. To obtain a countable infinity of nonisomorphic such segments, it suffices to find, for all $n \in \omega$, $a_0, \ldots, a_{n-1} \in |\mathfrak{M}|$ such that, for all $F_0, \ldots, F_{n-1}$ which $Th(\mathfrak{M})$ declares finite-to-one, we have $t_{F_i a_i}v \neq t_{F_j a_j}v$ for $i \neq j$. The trick to finding the elements $a_0, \ldots, a_{n-1}$ is to diagonalize on all tuples $(F_0, \ldots, F_{n-1})$ to construct the *types* $t_{a_i}v$ of the a_i's. To guarantee these types to be realized in $\mathfrak{M}$, it suffices to make the types standard sets of $\mathfrak{M}$ (cf., e.g., [16]).

We sketch the diagonalization. For notational convenience, we consider the binary case. Let $\mathfrak{M}$ be given and $T = Th(\mathfrak{M})$. Let $(F_0, G_0), (F_1, G_1), \ldots$ be an enumeration of all pairs of T-provably finite-to-one functions. We construct types $t_a v$, $t_b v$ by stages. At the n^{th} stage, we guarantee $F_n a$ and $G_n b$ to be of different types by finding T-provably unbounded sets A_n, B_n such that $F_n'' A_n \cap G_n'' B_n = \phi$ and letting $t_a v$ say $v \in A_n$ and $t_b v$ say $v \in B_n$. (To guarantee consistency, one makes descending choices $A_0 \supseteq A_1 \supseteq \ldots$ and $B_0 \supseteq B_1 \supseteq \ldots$. The construction is not problematic for F_n and G_n are finite-to-one and avoiding overlaps is easy.)

After ω steps, one has partial types

$$t^A v = \{v \in A_n \colon n \in \omega\},\ t^B v = \{v \in B_n \colon n \in \omega\}$$

recursive in T, hence coded as standard sets of $\mathfrak{M}$. One now simply chooses $t_a v$ and $t_b v$ to be completions of $t^A v$ and $t^B v$, respectively, coded in $\mathfrak{M}$.

4 Cofinal extensions and saturation In the last section, we gave a nice application of cofinal extensions to the study of recursively saturated models. The significance of cofinal extensions for recursive saturation is even more direct, as the following result of Smoryński and Stavi (from [17]) shows.

4.1 Theorem *Let $\mathfrak{M} \prec_c \mathfrak{N}$.*

i. *If $\mathfrak{M}$ is recursively saturated, so is $\mathfrak{N}$*
ii. *If $\mathfrak{M}$ is $\aleph_0$-saturated, so is $\mathfrak{N}$.*

The proof is based on two simple ideas. First, if $\mathfrak{M}$ is recursively saturated, there is a recursive type realized arbitrarily highly in $\mathfrak{M}$ that asserts any element realizing it to encode a truth definition for all formulas with small parameters. The arbitrary high realizability of such a type (actually, any type) is inherited by cofinal extensions. Now, in $\mathfrak{N}$, the elements realizing this type work to reduce the complexities of formulas in arbitrary recursive types. But it has long been known (if not actually enunciated) that recursive types consisting of formulas of bounded complexity are realized. Hence every recursive type over $\mathfrak{N}$ is realized and we have Theorem 4.1.i. Assertion 4.1.ii reduces readily to 4.1.i—some parameter from $\mathfrak{M}$ will make any type over $\mathfrak{N}$ recursive.

Kotlarski, in [10], has shown Theorem 4.1 to be best possible:

4.2 Theorem *$\aleph_1$-saturation is not preserved under cofinal extensions.*

Kotlarski notes, however, that a *simple* cofinal extension of an $\aleph_1$-saturated model is again $\aleph_1$-saturated. Also, if $\mathfrak{M}$ is saturated, some simple cofinal extension of $\mathfrak{M}$ is saturated. These results compare favorably with knowledge of end extensions: simple end extensions are short and, hence, not even recursively saturated.

Another sense in which Theorem 4.1 is best possible is that it is saturation and not resplendence that the extension preserves.

4.3 Example Resplendence is not preserved under cofinal extensions.

A counterexample is based on two facts: (i) there are countable resplendent models; and (ii) resplendent models are not two-cardinal models. Let $\mathfrak{M}$ be a countable resplendent model and let I be any initial segment other than ω closed under addition and multiplication. Following Paris and Mills, Theorem 2.4 can be iterated $\aleph_1$ times to obtain $\mathfrak{M} \prec_c \mathfrak{N}$, with $I = \mathrm{GCIS}(\mathfrak{M}, \mathfrak{N})$ and $|\mathfrak{N}|$ of cardinality $\aleph_1$. Since I remains countable, any definable initial segment $\{a\colon a < b\}$, with $b \in I$ nonstandard, is countable and $\mathfrak{N}$ is a two-cardinal model.

Despite this failure, Kotlarski [10] has shown some expandability to be preserved. For example, he has proven the following:

4.4 Theorem *If $\mathfrak{M}$ is expandable to third-order arithmetic, then every cofinal extension $\mathfrak{N}$ of $\mathfrak{M}$ is expandable to second-order arithmetic.*

The situation is reminiscent of that in Section 2, where very strong properties of $I = GCIS(\mathfrak{M}, \mathfrak{N})$ entailed moderately strong properties of I^+.

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Westmont, Illinois