Notre Dame Journal of Formal Logic
Volume XVIII, Number 4, October 1977
NDJFAM

ON RAMSEY'S THEOREM AND THE AXIOM OF CHOICE

GABRIELE LOLLI

It is known that Ramsey's theorem cannot be proved in **ZF** without the axiom of choice (see, e.g., Kleinberg [2]) but there does not seem to exist in the literature, or at least be widely recognized, a clear cut statement of the exact relationship between this combinatorial result and the principle of choice (in Drake [1], p. 72, the problem is mentioned but only a partial answer is given). The aim of this note* is to write down a proof of the

Proposition *Ramsey's theorem is equivalent to the axiom of choice for countable families of finite sets.*

For a set X, let $[X]^2$ be the set of unordered pairs from X; if $f: [X]^2 \rightarrow 2$ is a partition of $[X]^2$ into two disjoint sets, a set $Y \subseteq X$ is said to be homogeneous for f if $f \upharpoonright [Y]^2$ is constant. Then by Ramsey's theorem we mean the statement

(**RT**) *Any partition $f: [X]^2 \rightarrow 2$ of an infinite set X possesses an infinite homogeneous set*

which is the crucial step of Ramsey [3].

We abbreviate with (**CCF**) the axiom of choice for countable families of non-empty finite sets; (**CCF**) is equivalent in **ZF** to König's lemma

(**KL**) *Any infinite finitary tree has an infinite branch*

and also (**KL**) $\Rightarrow$ (**RT**) (see, e.g., Drake [1], p. 203). It remains to be shown that (**RT**) $\Rightarrow$ (**KL**); we prove it in a roundabout way through the following weak form of compactness for propositional logic

(**CPL**) *Let S be a countable set of propositional sentences over an infinite set of propositional letters; then S has a model iff every finite subset of S has a model.*

---

*The author is associated to Consiglio Nazionale delle Recerche, GNSAGA, Section 5.

Received November 29, 1976

The fact must be stressed that the set of propositional letters is not necessarily outright countable, otherwise (**CPL**) is a theorem of **ZF** alone, hence some care must be put in the definition of the propositional sentences: more precisely conjunction and disjunction should be construed as operators over unordered finite sets of sentences.

We divide the proof into two steps:

(**RT**) $\Rightarrow$ (**CPL**): let S be given by the enumeration $A_1, A_2, \ldots$, assume that every finite subset of S has a model and define $B_1 = A_1$, $B_{n+1} = \& \{B_n, A_{n+1}\}$; hence every B_n has a model; we identify its models with finite functions g from the propositional letters of B_n into 2. Let X the set of all those g which are models of some B_n, define a partition f by putting $f(g_1, g_2) = 0$ iff g_1 and g_2 agree on the common part of their domain and let Y be an infinite homogeneous set for f. If $f \restriction [Y]^2$ has value 0 we are done, since the union of Y gives a model for the whole S. But this is the only possible way for Y to be homogeneous, for suppose the contrary: given any g_1 there is a largest n such that g_1 is a model of B_n, otherwise g_1 itself is a model of the whole S; hence there are only a finite number of ways in which any other g_2 can differ from g_1 over its finite propositional letters and Y cannot be infinite.

(**CPL**) $\Rightarrow$ (**KL**): let $\langle T, \leq_T \rangle$ be an infinite finitary tree, that is a partially ordered set, whose elements are called nodes, such that

(i) for each $x \in T$ the set $\{y \in T: y \leq_T x\}$ is well ordered by $\leq_T$,
(ii) there is only one first element

and

(iii) each node has only finitely many immediate successors;

a branch is a linearly ordered subset of T containing the first element; the height of a node is the length of the branch connecting it to the bottom of the tree and the level L_n of the tree is the set of all nodes of height n.

For every natural number n the level L_n is not empty; introduce a propositional letter $p_{n,a}$, of level n, for each node $a \in L_n$. We shall define a set S of propositional sentences over the $p_{n,a}$'s such that any model of S determines an infinite branch of T, given by those nodes a for which the corresponding $p_{n,a}$ has value 1 in the model, and we shall show that S has a model.

So let us first write down that at each level L_n one and only one node belongs to the intended branch: this can be assured by a disjunction A_n each of whose disjuncts is a conjunction of propositional letters of level n and of negations of propositional letters of level n, in which all propositional letters of level n occur, but one and only one is unnegated, and A_n must have so many disjuncts that all propositional letters of level n occur exactly once in A_n unnegated. Although we are not allowed to write down on the paper A_n unless we simultaneously choose a particular ordering of L_n, the definition of A_n is legitimate.

Next for each level L_n let us say that it is possible to reach that level from the bottom along one of the actual finite branches of T; so let B_n again be a disjunction, each disjunct describing a branch of length n: each disjunct must be a conjunction of n propositional letters such that for all $i \leqslant n$ one and only one letter of level i occur in the conjunction and whenever $p_{i,a}$ and $p_{i+1,b}$ occur in it then $a \leqslant_T b$.

To any model of B_n there corresponds some branch of T of length n, possibly more than one; any model of B_n and of $\{A_i: i \leqslant n\}$ describes exactly one such branch, and there exists one such model; in any model of say B_n, $\{A_i: i \leqslant n\}$ and B_m with $m < n$, the branch associated to B_m is an initial segment of the branch associated to B_n. Hence if S is the set of all A_n and B_n, n ranging over the natural numbers, S is finitely satisfiable in T, hence by (**CPL**) it has a model, which is easily seen to determine an infinite branch of T, and the proof is complete.

REFERENCES

[1] Drake, F. R., *Set Theory*, North Holland Co., Amsterdam (1974).

[2] Kleinberg, E. M., "The independence of Ramsey's theorem," *The Journal of Symbolic Logic*, vol. 34 (1969), pp. 205-206.

[3] Ramsey, F. P., "On a problem of formal logic," *Proceedings of the London Mathematical Society*, Second Series, vol. 30 (1930), pp. 264-286.

Istituto di Scienze dell'Informazione
Università di Salerno, Salerno, Italia