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THE ENTAILMENT OPERATOR

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Finding an adequate definition of the entailment operator is a fundamental concern of logicians. Such an explication would, hopefully, reveal that entailment, long assumed to be a logical relationship, can satisfactorily be treated in some truth-tabular way. This paper attempts to follow out a few heartfelt and deeply entrenched assumptions of logicians on the basis of some recently advanced hypotheses concerning the nature of entailment. It is also one exemplification of the treatment of this relation by means of a many-valued, modal, truth-tabular system of logic. The application of these techniques to this set of assumptions and hypotheses should yield a plenary set of tables which will be paradigmatic and, thus, definitive of the entailment operator.

1 *Starting Points* Let us make the following assumptions:

(1) Entailment is a logical relation; that is, one that is not essentially dependent upon factors which cannot be formalized.
(2) Entailment is a relation that obtains between statements.
(3) Entailment is, as such, amenable to truth-tabular treatment.
(4) A relation is amenable to truth-tabular treatment if, and only if, it can be expressed as a logical operator such that its function is capable of being characterized by a set of matrices.
(5) Every statement has a logically manipulatable "value", ("true", "false", "0", "1", "2", etc.) which can perform as either an argument or a value in a logical function.

Let us operate with the following hypotheses:

(6) Entailment is not adequately expressed by any two-valued modal or non-modal truth-tabular set of matrices.
(7) An adequate analysis of the entailment relationship is:

A statement S entails a statement E if, and only if, S's being true is a

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sufficient condition for E's being true, and E's being false is a sufficient condition for S's being not true.[1]

Assumptions (1), (2), (3), and (4) seem to be dogma, with, perhaps, only the conditions set forth in (4) being open to some question because of their strictness. Assumption (5) is a weaker version of the familiar claim that every proposition (or, at least every one that logicians deal with) has a truth value—generally, either "true" or "false". Let me offer my apologies to those logicians who have, or feel they have, analyses and definitions which falsify hypothesis (6). Frankly, though, I doubt that such analyses will ultimately turn out to be adequate. The use of concepts like "sufficient condition" and "necessary condition" in almost any treatment of entailment strongly suggest that a modal explication is required.[2] Further, the distinction between "p is false" and "p is not true", which we will come to presently, strongly suggests that we will need a minimum of three values in our matrices.

Ginsberg's analysis, captured in (7), seems to suggest an easy modal treatment. Let '$(A \nabla B)$' be read "A entails B". We might be tempted to render the analysis as:

(8) $(S \nabla E) \equiv ((S ⓢ E) \& (\sim E ⓢ \sim S))$[3]

But, such a rendition would neglect the very point upon which Ginsberg's paper turns, viz. his warning that we should not confuse the difference between "p is false" and "p is not true". If p is not true, it may be either false or "truthvalueless". Similarly, if p is not false, it may be either true or "truthvalueless". In order to preserve (5) and still to make some sense of "truthvaluelessness" I will introduce a three-valued system. The claim that p is truthvalueless shall amount to saying that it is neither true ("0"), nor false ("1"), but something neutral, different, and indeterminate ("2").

2 *System* **E*** System **E*** will be a three-valued system of propositional logic. It shall have the normal stock of symbols and the standard formation rules. A formula $\ulcorner\sim A\urcorner$ shall have the value 1 if A is 0, 0 if A is 1, and otherwise 2. A formula $\ulcorner(A \& B)\urcorner$ shall be 0 if both A and B are 0, 1 if either A or B is 1, and otherwise 2. A formula $\ulcorner(A \vee B)\urcorner$ shall be 0 if either A or B is 0, 1 if both A and B are 1, and otherwise 2. A formula $\ulcorner(A \supset B)\urcorner$ shall be 0 if either A is 1 or B is 0, 1 if A is 0 and B is 1, and otherwise 2. A formula $\ulcorner(A \equiv B)\urcorner$ shall be 0 if both A and B are 0 or both are 1, 1 if A is 0 and B is 1 or if A is 1 and B is 0, and otherwise 2.

1. Ginsberg, Mitchell, "The entailment-presupposition relationship," *Notre Dame Journal of Formal Logic*, vol. XIII (1972), pp. 511-516.

2. Facione, Peter A., "A modal truth-tabular interpretation for necessary and sufficient conditions," *Notre Dame Journal of Formal Logic*, vol. XIII (1972), pp. 270-272.

3. Read '$(A ⓢ B)$' as "A is a sufficient condition for B".

On the basis of the above we can set down the paradigm tables for the five familiar logical operators.

E* Paradigm Tables for '~', '&', 'v', '⊃', and '≡'

A	$\sim A$
0	1
1	0
2	2

A	B	$(A \& B)$	$(A \vee B)$	$(A \supset B)$	$(A \equiv B)$
0	0	0	0	0	0
0	1	1	0	1	1
0	2	2	0	2	2
1	0	1	0	0	1
1	1	1	1	0	0
1	2	1	2	0	2
2	0	2	0	0	2
2	1	1	2	2	2
2	2	2	2	2	2

In E* the following pairs of schema are interchangeable, since they have identical values under identical interpretations:

$(A \& B)$, $(B \& A)$
$(A \vee B)$, $(B \vee A)$
$(A \equiv B)$, $(B \equiv A)$
$(A \& B)$, $\sim(\sim A \vee \sim B)$
$(A \supset B)$, $(\sim A \vee B)$
$(A \equiv B)$, $((A \supset B) \& (B \supset A))$

We must expand E* beyond these more familiar operations in order to handle six new truth-functions: assertion of truth, assertion of falsehood, assertion of truth-value indeterminateness, denial of truth, denial of falsehood, and denial of truth-value indeterminateness. For the time being let us designate these six operations by '$*_1$', '$*_2$', . . ., '$*_6$'. Their paradigm tables are derived from these rules:

$\ulcorner *_1 A \urcorner$ is 0 if A is 0, otherwise 1.
$\ulcorner *_2 A \urcorner$ is 0 if A is 1, otherwise 1.
$\ulcorner *_3 A \urcorner$ is 0 if A is 2, otherwise 1.
$\ulcorner *_4 A \urcorner$ is 0 if A is 1 or 2, otherwise 1.
$\ulcorner *_5 A \urcorner$ is 0 if A is 0 or 2, otherwise 1.
$\ulcorner *_6 A \urcorner$ is 0 if A is 0 or 1, otherwise 1.

E* Paradigm Tables for '$*_1$', '$*_2$', '$*_3$', '$*_4$', '$*_5$', and '$*_6$'

A	$*_1 A$	$*_2 A$	$*_3 A$	$*_4 A$	$*_5 A$	$*_6 A$
0	0	1	1	1	0	0
1	1	0	1	0	1	0
2	1	1	0	0	0	1

We can make these star functions more perspicuous if we adopt certain notational conventions. Let us write $\ulcorner \mid A \urcorner$ for the assertion of A as true.

Let $\ulcorner \nmid A \urcorner$ assert A as false. Let $\ulcorner \overline{A} \urcorner$ assert that A is value-neutral or indeterminate. With this notation and with the use of the paradigm tables we can determine that the following are interchangeable pairs:

$$
\begin{array}{ll}
*_4A, & (\nmid A \vee \overline{A}) \\
*_5A, & (\mid A \vee \overline{A}) \\
*_6A, & (\mid A \vee \nmid A) \\
\sim *_1A, & \sim \mid A \\
\sim *_2A, & \sim \nmid A \\
\sim *_3A, & \sim \overline{A} \\
\sim \mid A, & (\nmid A \vee A) \\
\sim \nmid A, & (\mid A \vee \overline{A}) \\
\sim A, & (\mid A \vee \nmid A)
\end{array}
$$

3 *First Try* Having now formulated System **E*** we can return to examine the analysis suggested in (7). As a first try let us render (7) in a non-modal truth-functional way, as

(9) $(S \nabla E)$ if, and only if, $((\mid S \supset \mid E) \ \& \ (\nmid E \supset *_4S))$[4]

The values of the schema $\ulcorner(S \nabla E)\urcorner$ treated non-modally are:

S	E	$\mid S$	$\mid E$	$(\mid S \supset \mid E)$	$\nmid E$	$*_4S$	$(\mid E \supset *_4S)$	$((\mid S \supset \mid E) \ \& \ (\nmid E \supset *_4S))$ [= $(S \nabla E)$]
0	0	0	0	0	1	1	0	0
0	1	0	1	1	0	1	1	1
0	2	0	1	1	1	1	0	1
1	0	1	0	0	1	0	0	0
1	1	1	1	0	0	0	0	0
1	2	1	1	0	1	0	0	0
2	0	1	0	0	1	0	0	0
2	1	1	1	0	0	0	0	0
2	2	1	1	0	1	0	0	0

Unfortunately there are several reasons why this first attempt is unacceptable. First of all we can note that '∇' is like '$\supset$' in that it is 0 if either S, the first component, is 1 or E, the second component, is 0; and, it is 1 when S is 0 and E is 1. But the function never turns out to have the value 2. We might expect that the last row, at least, where both S and E are 2, would yield a 2, or perhaps a 1. But, the 0 is rather unexpected there. We should, rather, deny that two statements of indeterminate value entail one another. Secondly, this non-modal treatment fails to take into account the modal aspects of the concept of "sufficient conditions". Third, these results do not accord well with Ginsberg's own conclusions. For example, when S is 1 $\ulcorner(S \nabla E)\urcorner$ should be 0 only if E is 0 or 1, but not when

4. (9) could also be "$((S \nabla E) \equiv ((\mid S \supset \mid E) \ \& \ (\nmid E \supset \sim \mid S)))$.

E is 2. Also, when S is 2 there should not be enough information to determine the value of $\ulcorner(S \nabla E)\urcorner$. In such cases *no* value should appear. Apparently Ginsberg would have us devise a system which would provide for cases of total lack of value (being totally without any value) as distinct from cases of truth-value indeterminateness (being neither true nor false but something neutral). Yet, this cannot be done, if we are to preserve (5). In this we have a challenge to other assumptions as well. If we preserve (5) this is done knowing the threat to the workability of (4) and the truth of (1).

4 *Second Try* The first objection to (9) may amount to no more than a note of surprise at the direction in which our intuitions, once systematized, have led. The second objection can be taken care of by moving to a modal approach and determining the set of plenary tables which will be paradigmatic for '∇' as a modal operator. Each table in the set represents one possible world and the interpretations of S and E that can exist in that world. Because there are 511 tables in the full set, we will forego constructing them. Let us render (7) as

(10) $(S \nabla E)$ if, and only if, $((\mid S \circledS \mid E)$ & $(\dashv E \circledS *_4 S))$[5]

'$A \circledS B$' is read "A is a sufficient condition for B". '$(A \circledS B)$' is primarily designed to be equivalent to '$(\Box(A \supset B))$' in S5. This, when expanded into our three-valued system, generates tables based on the following rule: In any given table **T**, the value of '$(A \circledS B)$' shall be 1 on every row of **T** if A has the value 0 and B has the value 1 on any row of **T**, the value of '$(A \circledS B)$' shall be 0 on every row of **T** if on every row of **T** either the value of A is 1 or the value of B is 0, otherwise the value of '$(A \circledS B)$' shall be 2 on every row of **T**. From here on it would be a purely mechanical procedure to crank out the 511 tables which characterize '$\circledS$' and '∇'. And so, if everything else is shipshape, we will have a set of tables which characterize entailment.

We can expect that on every row of every table in the set we will find a 0 or 1 as the value of '$(S \nabla E)$'. The value 2 will not appear since the operators, '$*_1$', '$*_2$', . . ., '$*_6$', take all values into either 0 or 1. Thus, the outcome of '$((\mid S \circledS \mid E)$ & $(\dashv E \circledS *_4 S))$ will always be either 0 or 1, given the **E*** paradigm tables for '&' and because '$(\mid S \circledS \mid E)$' and '$(\dashv E \circledS *_4 S)$' will come out to be 0 or 1 on every row of every table. This is the case because the values of '$\mid S$', '$\mid E$', '$\dashv E$', and '$*_4 S$' are all always 0 or 1. Moreover, on the great majority of tables the value of '$(S \nabla E)$' will be 1 (false) because if either '$(\mid S \circledS \mid E)$' or '$(\dashv E \circledS *_4 S)$' is 1, due to $\langle 0, 1\rangle$ or $\langle 0,2\rangle$ occurring as a value of 'S' and 'E' on any row of **T**, then their conjunction will be false on every row of **T**. Thus, in at least the majority of problematic cases where the values of 'S' and 'E' are either $\langle 0, 2\rangle$, $\langle 2, 1\rangle$, or $\langle 2, 2\rangle$, the value of $\ulcorner(S \nabla E)\urcorner$ will be 1. This result seems to be preferable to the result obtained in our first, non-modal try at interpreting

5. (10) could also be more simply put by replacing '$*_4 S$' with '$\sim\mid S$'.

$\ulcorner(S \nabla E)\urcorner$ when, in the three final rows, it was 0; provided that one is biased in favor of "true" and is, thus, prone to be cautious in assenting to any proposition—and, especially one asserting entailment.

5 *Conclusion* It would seem that we have done what we set out to do—we have defined the relation of entailment having made certain assumptions and acted under certain hypotheses. One interesting and unexpected result of this is that given any two statements p and q we can always say, for any possible world, whether or not p entails q, the value of '$(p \nabla q)$' will never be indeterminate but always either 0 or 1 (true or false) and most often 1. But this "wonderful" result seems too good to be true. To say that entailment can always be determined positively to obtain or not obtain does not sit well with our experience of the difficulty of making that determination in practice. For example. Let 'p' be "God, a being who knows all truth and never intends to deceive, says that the earth is flat". Let 'q' be "the earth is flat". Does 'p' entail 'q'? Further, consider the complexity of $\ulcorner(A \nabla B)\urcorner$. Its characteristic set of tables contains 511 members! This strongly suggests that the operator '∇' is difficult to manage. This hardly seems to be what one might originally have expected when one innocently accepted (1), (2), and (3). But, then, travelling to the moon is a lot more difficult, and interesting, than many a science-fiction author had imagined too.

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