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GENERALIZING KÖNIG'S INFINITY LEMMA

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1 *Introduction* D. König's famous lemma on trees has many applications; in graph theory it is used to extend certain results from finite to infinite graphs (see Nash-Williams [7]); in logic it can be used to prove that a denumerable set of propositional formulas is satisfiable if every finite subset is (see, for example, Van Fraassen [9]). This last result, known as the compactness theorem for propositional logic, is even true when "denumerable" is replaced by "infinite" and thus it seems reasonable to ask whether the stronger form can be obtained from a generalized König's lemma. We shall show that this is indeed the case.

2 *Preliminary definitions* By a (rooted) *tree* we shall mean a connected undirected graph without circuits, one of whose vertices is designated the *origin* (see Ore [8]). The number of vertices on the unique path connecting a vertex v with the origin is called the *level* of v, $\mathsf{l}(v)$. Thus the set of vertices of a tree is decomposed into an at most denumerable set of levels with the origin being the sole vertex at level one. A vertex v' is a *successor* of a vertex v if v and v' are connected by an edge and $\mathsf{l}(v') = \mathsf{l}(v) + 1$.

A tree will be called *finite* if the set of its vertices is finite, otherwise *infinite*; it will be called *locally finite* if each vertex has only finitely many successors. Equivalently, a tree is locally finite iff each of its levels is finite. A *branch* of a tree is any maximal path beginning at the origin. If vertices v and v' are on the same branch, then v' *dominates* v if $\mathsf{l}(v') \geqslant \mathsf{l}(v)$. König's lemma states that any locally finite infinite tree has an infinite branch. Of course "infinite" in this context means denumerable since locally finite trees have at most denumerably many vertices. Finally, for any set A, $\overline{\overline{A}}$ denotes the cardinality of A.

3 *Main results* Let $\mathcal{T}$ be a collection of locally finite trees.[1] By a *vertex* or a *level of* $\mathcal{T}$ we mean a vertex or a level of some tree in $\mathcal{T}$. Also, if v, v' are vertices of $\mathcal{T}$, then v' *dominates* v *in* $\mathcal{T}$ if v' dominates v in some

1. It is *not* assumed that the trees in $\mathcal{T}$ are pairwise disjoint.

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tree of $\mathcal{T}$. Let S be a set of vertices of $\mathcal{T}$. S will be said to *pierce* a level l of $\mathcal{T}$ if $\overline{\overline{S \cap l}} = 1$. S is *consistent in* $\mathcal{T}$ if for every v, v' in S, there is a v'' which dominates them both in $\mathcal{T}$. The first theorem is our generalization of König's lemma.

Theorem 1 *Let $\mathcal{T}$ be a collection of locally finite trees such that for any finite set of levels of $\mathcal{T}$ there is a consistent set of vertices piercing those levels. Then there is a consistent set of vertices piercing the entire set of levels of $\mathcal{T}$.*

Proof: Call a set G of vertices of $\mathcal{T}$ *good* if for every finite $G_0 \subset G$ and every finite set of levels, L, there exists a consistent set of vertices G', with $G' \supset G_0$, and G' pierces the levels in L.

The hypothesis implies that the empty set of vertices is good. Also "goodness" is a property of finite character; hence, by Tukey's lemma (see [3], p. 31) , there exists a maximal good set G^*. The proof will be completed by showing G^* pierces all levels of $\mathcal{T}$, since any good set is clearly consistent.

Suppose not, suppose G^* fails to pierce level $l = \{v_1, \ldots, v_n\}$. Since G^* is maximal, none of $G^* \cup \{v_i\}$ can be good, $1 \leqslant i \leqslant n$. Therefore there exists for each such i, a finite $G_i \subset G^*$ and a finite set of levels L_i such that no consistent G, with $G \supset G_i \cup \{v_i\}$, pierces the levels in L_i. We show this is impossible. Let $G' = \bigcup_{i=1}^{n} G_i$ and $L' = \bigcup_{i=1}^{n} L_i$. Since G^* is good and G' is a finite subset of G^*, there exists a consistent G, with $G \supset G'$, which pierces the levels in $L' \cup \{l\}$. If $G \cap l = \{v_j\}$, then $G \supset G_j \cup \{v_j\}$ and G pierces L_j! Contradiction. Therefore G^* pierces all levels of $\mathcal{T}$.

Corollary (D. König) *A locally finite, infinite tree has an infinite branch.*

Proof: Let $\mathcal{T}$ be the set consisting of the tree itself; then a set of vertices of $\mathcal{T}$ is consistent iff the vertices in the set belong to the same branch of the tree. Because the tree is infinite it has, for any finite set of levels, a branch whose vertices pierce these levels. Theorem 1 now gives a consistent set of vertices piercing all levels of $\mathcal{T}$, that is, an infinite branch.

Next we derive the compactness theorem for propositional logic from Theorem 1.

Theorem 2 *Let K be an infinite set of propositional formulas, every finite subset of which is satisfiable. Then K is satisfiable.*

Proof: For any finite $W \subset K$, let P_W be the propositional variables occurring in W and let V_W be the finite set of valuations of P_W which satisfy W. We define a set, $\mathcal{T}$, of locally finite trees as follows. The levels of $\mathcal{T}$ shall consist of the V_W together with $\{\emptyset\}$, where $\emptyset$ is the empty set. If $W \subset W'$, where W, W' are finite subsets of K we form a tree with levels $\{\emptyset\}$, V_W, $V_{W'}$ in the following way. The origin is $\emptyset$ and every valuation in

V_W is connected to $\emptyset$; whereas f in $V_{W'}$ is connected to $f \upharpoonright P_W$[2] in V_W. $\mathcal{T}$ shall consist of all such trees. Given any finite set of levels $\{V_{W_1}, \ldots, V_{W_n}\}$, let $f \in V_{W_1 \cup \ldots \cup W_n}$; then $\{f \upharpoonright P_{W_1}, \ldots, f \upharpoonright P_{W_n}\}$ is consistent (since any pair is dominated by f in $\mathcal{T}$) and pierces the V_{W_i}, $1 \leq i \leq n$. By Theorem 1, there exists a set of valuations which satisfy the finite subsets of K and such that any two are the restrictions of a single valuation, that is, they must agree on common variables. Therefore we have a single valuation satisfying all of K.

An examination of the above proof shows that it really has very little to do with propositional logic per se; if we abstract what is required to allow the proof to go through we obtain the following compactness result for finite sets of functions. The proof is virtually identical to the above and so is omitted.

Theorem 3 *Let $\langle W, \leq \rangle$ be a directed*[3] *partially ordered set and for each $w \in W$, let F_w be a finite, non-empty, set of functions with domain U_w. Suppose $w_1 \leq w_2$ and $f \in F_{w_2}$ implies $f \upharpoonright U_{w_1} \in F_{w_1}$. Then there exists a function f such that $f \upharpoonright U_w \in F_w$ for every $w \in W$.*

Theorems 1-3 are equivalent[4] to each other and to **P.I.**, the prime ideal theorem for Boolean algebras; this is easy to establish using our results in [1] where some strong forms of Rado's selection lemma were shown to be equivalent to **P.I.** Moreover, the proof of Theorem 2 from Theorem 1 only required a consideration of trees with three levels and thus we could restrict $\mathcal{T}$ in Theorem 1 to trees with at most three levels without limiting its potency. There is another natural way of restricting $\mathcal{T}$ in Theorem 1 and the rest of this paper will be devoted to its consideration.

If $\mathcal{T}$ is a collection of trees, we define the *order* of $\mathcal{T}$, $o(\mathcal{T})$, to be the least cardinal $\mathfrak{K}$ such that no tree in $\mathcal{T}$ contains a vertex with more than $\mathfrak{K}$ successors. If $\mathcal{T}$ consists of locally finite trees, $o(\mathcal{T}) \leq \aleph_0$. Let T_n be the assertion of Theorem 1 only for $\mathcal{T}$ with $o(\mathcal{T}) = n$, n a positive integer. Clearly $T_m \rightarrow T_n$, $m \geq n$. We shall show that $T_3 \rightarrow$ **P.I.**

Let $\{A_i\}_{i \in I}$ be a collection of sets and S, a symmetric binary relation on $\bigcup_{i \in I} A_i$. A choice function, f, for $\{A_i\}_{i \in I}$ is said to be *S-consistent* if $f(i)\, S f(j)$, for all i, j in I, with $i \neq j$. Łoś and Ryll-Nardzewski [5] derived the following finite consistent choice principle from a consistent choice principle for compact spaces.

Theorem 4 *Let $\{A_i\}_{i \in I}$ be a collection of finite sets and S a symmetric binary relation on $\bigcup_{i \in I} A_i$. Suppose that for every finite $W \subset I$, there is an*

2. $f \upharpoonright P_W$ means the function which is the restriction of f to P_W.

3. A partially ordered set is directed iff every pair of elements in the set has an upper bound in the set.

4. Equivalent, in this context, means without using the Axiom of Choice.

S-consistent choice function for $\{A_i\}_{i\in W}$. Then there is an S-consistent choice function for $\{A_i\}_{i\in I}$.

If in the above theorem, we require $\overline{\overline{A}}_i \leq n$, for all $i \in I$, a restricted finite consistent choice principle is obtained; we denote this restricted principle by F_n. In [6], Łoś and Ryll-Nardzewski show that $F_n \rightarrow$ **P.I.**, $n \geq 4$. We shall show that $F_3 \rightarrow$ **P.I.** and $T_n \rightarrow F_n$, thus establishing $T_n \rightarrow$ **P.I.**, $n \geq 3$. The proof that $F_3 \rightarrow$ **P.I.** depends on results of Läuchli [4] which connect the coloring of infinite graphs and **P.I.**

A graph is said to be *n-colorable* if there is a function, f, mapping the vertices to $\{0, \ldots, n-1\}$ such that $f(v) \neq f(v')$ if v and v' are connected by an edge. P_n will denote the statement, proved by De Bruijn and Erdös [2], that a graph is n-colorable if every finite subgraph is n-colorable. Läuchli [4] shown that $P_n \rightarrow$ **P.I.**, $n \geq 3$.

Theorem 5 *$F_n \rightarrow P_n$, n a positive integer.*

Proof: Let $G = \langle V, R\rangle$ be a graph, where V is the set of vertices and R, the set of edges and assume every finite subgraph of G is n-colorable. If $W \subset V$, $G \upharpoonright W$ denotes the subgraph $\langle W, R \cap (W \times W)\rangle$. For each $v \in V$, let $A_v = \{\langle v, 0\rangle, \ldots, \langle v, n-1\rangle\}$. Define a relation S on $\bigcup_{v\in V} A_v$ by

$$\langle v, i\rangle\ S\langle v', j\rangle =_{df} vRv' \rightarrow i \neq j.$$

Given any finite $W \subset V$ and any n-coloring, f, of $G \upharpoonright W$, let $f^*(v) = \langle v, f(v)\rangle$, $v \in W$. Clearly f^* is an S-consistent choice function for $\{A_v\}_{v\in W}$. By F_n, there is an S-consistent choice function, h^*, for $\{A_v\}_{v\in V}$. Define h on V by $h^*(v) = \langle v, h(v)\rangle$, $v \in V$. Suppose vRv'; since h^* is S-consistent, $\langle v, h(v)\rangle\ S\langle v', h(v')\rangle$ and therefore, by the definition of S, $h(v) \neq h(v')$, that is, h n-colors G.

Theorem 6 *$F_n \rightarrow$ **P.I.**, $n \geq 3$.*

Proof: Follows immediately from Theorem 5 and the aforementioned theorem of Läuchli.

Theorem 7 *$T_n \rightarrow F_n$, n a positive integer.*

Proof: Suppose there exists S-consistent choice functions for all finite subcollections of $\{A_i\}_{i\in I}$, where $\overline{\overline{A}}_i \leq n$, $i \in I$, and for any finite $W \subset I$ let F_W be the set of S-consistent choice functions for $\{A_i\}_{i\in W}$. Assuming T_n, we shall show that there is an S-consistent choice function for $\{A_i\}_{i\in I}$.

Suppose W is a finite subset of I. A sequence of subsets of W, $W_1, \ldots, W_k$, is a *W-tower* if W_1 is a singleton, $W_k = W$ and $W_{i+1} = W_i \cup \{j\}$, $i = 1, \ldots, k-1$. For each such W-tower we form a tree as follows: the origin is $\emptyset$, level $i+1$ is F_{W_i} and any $f \in F_{W_i}$ is connected to $f \upharpoonright W_{i-1}$ which belongs to $F_{W_{i-1}}$, $2 \leq i \leq k$. Since $W_i - W_{i-1} = \{j\}$ and $f(j) \in A_j$, $\overline{\overline{A}}_j \leq n$, each vertex of the tree has at most n successors. Therefore if $\mathcal{T}$ is the set of all such trees, $o(\mathcal{T}) = n$.

If $\{F_{W_1}, \ldots, F_{W_m}\}$ is any finite set of levels of $\mathcal{T}$, let $V = \bigcup_{i=1}^{m} W_i$ and let

$f \in F_V$. Since f will dominate $f \upharpoonright W_i$ in all trees formed from V-towers containing W_i, $\{f \upharpoonright W_1, \ldots, f \upharpoonright W_m\}$ is a consistent set of vertices piercing each F_{W_i}, $i = 1, \ldots, m$. Therefore, by T_n, there is a consistent set F such that $\overline{\overline{F \cap F_W}} = 1$, for all finite $W \subset I$. Since any two functions in F are both restrictions of the same function, F uniquely determines an S-consistent choice function for $\{A_i\}_{i \in I}$.

Theorem 8 $T_n \rightarrow$ **P.I.**, $n \geqslant 3$.

Proof: Theorem 8 now follows from Theorem 6 and Theorem 7.

Finally we inquire whether or not T_2 or F_2 implies **P.I.**?

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