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THE Q-CONSISTENCY OF $\mathcal{F}_{22}$

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In his [CSC],[1] Curry proved the consistency of a system, which he there defines and calls $\mathcal{F}_{22}$, and which is closely related to the system $\mathcal{F}_{21}^{*}$ of [CLg.II] §15C.[2] This is essentially a type-free intuitionistic predicate calculus without conjunction, alternation, or negation but with quantification over propositions and propositional functions. However, Curry's consistency proof is rather weak, since it only proves that every theorem of the system belongs to a class of obs (terms) which are defined to be canonical (called *canobs*) and since the canobs are those obs which are to be interpreted as propositions this proof leaves open the possibility that every ob which is to be interpreted as a proposition is a theorem of the system.

In this paper,[3] a stronger consistency result is proved. This is done by proving the elimination theorem (Gentzen's *Hauptsatz*). From this it follows that if an atom (atomic ob, or atomic term) **Q** is introduced in a natural way to represent equality, then $\vdash \mathbf{Q}XY$ holds if and only if $X = Y$ in the underlying C-system (i.e., if and only if X is convertible to Y using the conversion rules of the underlying system of combinatory logic or λ-conversion). Since it can be shown that $\mathbf{S} \neq \mathbf{K}$, it will follow that $\vdash \mathbf{QSK}$ does not hold, and hence there is an ob, **QSK**, which is interpreted as a proposition ($\mathbf{S} = \mathbf{K}$) but which is not a theorem.

In addition, an error in the theory of canobs in [CLg.II] §15B3 will be corrected here.

1 *The theory of canobs* Since the theory of $\mathcal{F}_{22}$ depends heavily on the theory of canobs, I will begin with the latter.

The error in the theory of canobs of [CLg.II] §15B3 occurs in Lemma 3.3. In the proof of that lemma, it is established that

(1) $$Yx_1 \ldots x_n \succ Z,$$

and it is claimed that from this, by property (ξ), it follows that

(2) $$Y \succ \lambda x_1 \ldots x_n \centerdot Z.$$

But this holds only if the underlying system is synthetic.[4] If the underlying

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system is a system of λ-conversion, then this inference fails, as can be seen by taking $n = 1$, $Y \equiv \mathbf{E}$, and $Z \equiv \mathbf{E}x$, for we do have $\mathbf{E}x \succ \mathbf{E}x$ but not $\mathbf{E} \succ \lambda x.\mathbf{E}x$.

The rest of this section will consist of a corrected theory of canobs for the case in which the underlying C-system is $\lambda\eta$-conversion. The theory will be developed for the system $\mathcal{F}_{22}$, and hence will be based on Curry's theory [CSC] (which is suitable for $\mathcal{F}_{22}$ if the underlying C-system is synthetic). A theory of canobs for $\lambda\eta$-conversion for the system of [CLg.II] §15B can be obtained by dropping clause (d) of Definition 1 below and the corresponding references to $\mathbf{H}$ throughout the rest of the theory. The assumptions about canonical simplexes will be the same as those of [CLg.II] §15B3.[5]

Definition 1 The ob ξ is a *proper canob* of *rank m* and *degree n* if and only if

(a) ξ is a canonical simplex and $m = n = 0$;
(b1) $\xi \equiv \lambda x.\eta$ where η is a proper canob of rank m and degree $n - 1$;[6]
(b2) ξx, where x does not occur in ξ, is a proper canob of rank 0 and degree $n - 1$,[6] and $m = 0$;
(c) $\xi \equiv \Xi\zeta\eta$ where ζ and η are proper canobs of degree 1, $n = 0$, and $\text{rk}(\zeta) + \text{rk}(\eta) = m - 1$; or
(d) $\xi \equiv \mathbf{H}\eta$ where η is a proper canob of degree 0, $n = 0$, and $\text{rk}(\eta) = m - 1$.

Definition 2 The ob ξ is a *canob* of *rank m* and *degree n* if and only if there is a proper canob η of rank m and degree n such that $\xi \succ \eta$.

I shall now give an alternative proof of the analogue of [CLg.II] Theorem 15B3, which is the following:

Theorem 1 *If* $\text{Can}_k(X)$ *is interpreted to mean that* X *is a canob of degree* k, *then the following hold:*

N1. $\text{Can}_k(X) \;\&\; X = Y \to \text{Can}_k(Y)$,
N2. $\text{Can}_k(X) \rightleftarrows \text{Can}_{k+1}(\lambda x.X)$,
N3. $\text{Can}_k(X) \to \text{Can}_{k-1}(XU)$ *for any* ob U,
NΞ. $\text{Can}_1(X) \;\&\; \text{Can}_1(Y) \rightleftarrows \text{Can}_0(\Xi XY)$,
N$\mathbf{H}$. $\text{Can}_0(X) \rightleftarrows \text{Can}_0(\mathbf{H}X)$.

The theorem will follow from the following lemmas.

Lemma 1.1 *If* ξ *is a (proper) canob of rank m and degree n, and if x is any variable and U any ob, then* $[U/x]\xi$ *is also a (proper) canob of rank m and degree n.*

Proof: See [CLg.II] §15B3 Lemma 3.1. There is a minor complication in the induction step for clause (b2), where the assumption is that $m = 0$, ξy is a proper canob of rank 0 and degree $n - 1$, y does not occur in ξ and is distinct from x. If y does not occur in U, then we have by the induction hypothesis that $[U/x](\xi y) \equiv ([U/x]\xi)\, y$ is a proper canob of rank 0 and degree $n - 1$, and since y does not occur in $[U/x]\xi$, the latter is a proper canob of rank 0 and degree n. However, y may occur in U. In this case, let z be a variable which is distinct from x and does not occur in ξ or U. Then it can be shown that ξz is a proper canob of rank 0 and degree $n - 1$ by a proof isomorphic to the proof that ξy is such a proper canob; for the latter proof must have come from the statement that $\xi y u_1 \ldots u_{n-1}$ is a canonical simplex by repeated applications of Definition 1 (b2), and by assumption (b) for canonical simplexes, it follows that $\xi z u_1 \ldots u_{n-1}$ is also a canonical simplex, from which it can be shown that ξz is a proper canob by the same number of applications of Definition 1 (b2). Then, by the induction hypothesis, we have that $[U/x](\xi z)$ is a proper canob of rank 0 and degree $n - 1$, and we can proceed as in the case in which y does not occur in U. Q.E.D.

Lemma 1.2 *If ξ is a proper canob of rank m and degree n, then there is a proper canob ν of rank m and degree $k \leqslant n$, where, if $y_1, \ldots, y_k$ are distinct variables which do not occur in ν, then $\nu y_1 \ldots y_k$ is a canonical simplex if $m = 0$ and is of the form of clause* (c) *or* (d) *of* Definition 1 (*and hence $k = 0$*) *if $m > 0$, such that*

(3) $$\xi \equiv \lambda x_1 \ldots x_{n-k} \bullet \nu.$$

Proof: By induction on the proof that ξ is a proper canob according to Definition 1. For the basic step of the induction, it is sufficient to note that ξ itself is such a ν (with $n = k = 0$). For the induction step, there are the following four cases, depending on which clause of Definition 1 is used:

(b1) $\xi \equiv \nu x \bullet \eta$, where $n > 1$ and $\mathrm{dg}(\eta) = n - 1$. By the induction hypothesis, there is a proper canob μ of rank m and degree $k < n - 1$ which satisfies the conditions of the lemma for η such that

$$\eta \equiv \lambda x_1 \ldots x_{n-k-1} \bullet \mu.$$

It follows that

$$\xi \equiv \lambda x x_1 \ldots x_{n-k-1} \bullet \mu.$$

Let $\nu \equiv [x_1/x, x_2/x_1, \ldots, x_{n-k}/x_{n-k-1}]\mu$. Then (3) holds. By Lemma 1.1, ν is a proper canob of rank m and degree k. Moreover, it follows by the properties of substitution that ν satisfies the other conditions of the lemma.

(b2) ξx, where x does not occur in ξ, is a proper canob of rank m and degree $n - 1$. By the induction hypothesis, there is a proper canob μ of rank m and degree $k \leqslant n - 1$ which satisfies the conditions of the lemma such that

$$\xi x \equiv \lambda x_1 \ldots x_{n-k-1} \bullet \mu.$$

Since the underlying system is assumed to be a system of λ-calculus, this implies that $n - k - 1 = 0$, so that $\xi x \equiv \mu$, and ξ itself is a ν with the desired properties (for $k = n$).

(c) $\xi \equiv \Xi \zeta \eta$. Then $m > 0$, $n = 0$, and ξ itself is such a ν.

(d) $\xi \equiv \mathsf{H}\eta$. Then $m > 0$, $n = 0$, and ξ itself is such a ν. Q.E.D.

Remark: Since $\nu y_1 \ldots y_k$ can have the form of clause (c) or (d) of Definition 1 only if $k = 0$,[7] we have that $k > 0$ only if $m = 0$.

Lemma 1.3 *If ξ is a proper canob of rank m and degree n, and if $\xi \succ Y$, then Y is a proper canob of rank m and degree n.*

Proof: By induction on the proof that ξ is a proper canob of rank m and degree n using Definition 1. For the basic step, it is sufficient to note that if ξ is a canonical simplex, then so is Y by assumption (a) about canonical simplexes. For the induction step, there are the following four cases depending on the clause of Definition 1 used:

(b1) $\xi \equiv \lambda x_\bullet \eta$. Then $n \geqslant 1$. Now consider the reduction of $\lambda x_\bullet \eta$ to Y. Let $\lambda x_\bullet Z$ be the last step of that part of the reduction which proceeds by working entirely within η, so that $\eta \succ Z$. If there is no next step, then $Y \equiv \lambda x_\bullet Z$; by the induction hypothesis since $\eta \succ Z$, Z is a proper canob of rank m and degree $n - 1$, and hence Y is a proper canob of rank m and degree n by Definition 1 (b1). If there is a next step in the reduction of ξ to Y, then since it does not take place entirely within Z, it must be an η-contraction for which the redex is $\lambda x_\bullet Z$. Hence, $Z \equiv Wx$, and the result of the contraction is W (and x does not occur in W). The rest of the reduction is a reduction from W to Y. Furthermore, since x does not occur in W, it does not occur in Y. Now, since $W \succ Y$, it follows that $Wx \succ Yx$. Since $\eta \succ Z \equiv Wx$, it follows that $\eta \succ Yx$. Hence, by the induction hypothesis, Yx is a proper canob of rank m and degree $n - 1$. Furthermore, as in case (b2) of the proof of Lemma 1.2 we have that

$$Yx \equiv \lambda x_1 \ldots x_{n-k-1\bullet} \mu$$

where $n - k - 1 = 0$ so that $Yx \equiv \mu$ and hence has the form of clause (c) or (d) of Definition 1 or else is a simplex. If Yx has the form of clause (c) or (d) of Definition 1, then $Yx \equiv \Xi\zeta\eta$ or $Yx \equiv \mathsf{H}\eta$ where η is a proper canob of degree 0 or 1. But then $\eta \equiv x$, and x is a proper canob, which is impossible (see note 7). Hence Yx is a simplex, and $m = 0$. It follows by (b2) of Definition 1 that Y is a proper canob of rank m and degree n.

(b2) ξx is a proper canob of rank m and degree $n - 1$ and $m = 0$. Since $\xi \succ Y$, we have $\xi x \succ Yx$. Hence, by the induction hypothesis, Yx is a proper canob of rank 0 and degree $n - 1$. Furthermore, since x does not occur in ξ, it does not occur in Y. Hence, by Definition 1 (b2), Y is a proper canob of rank m and degree n.

(c) $\xi \equiv \Xi\zeta\eta$. Then $n = 0$ and ζ and η are proper canobs of degree 1 and $\mathrm{rk}(\zeta) + \mathrm{rk}(\eta) = m - 1$. By the property C-VC3 of [CLg.II] §11F3, since $\xi \succ Y$, it follows that $Y \equiv \Xi Z W$, where $\zeta \succ Z$ and $\eta \succ W$. By the induction hypothesis, it follows that Z and W are proper canobs of degree 1 and such that $\mathrm{rk}(Z) = \mathrm{rk}(\zeta)$ and $\mathrm{rk}(W) = \mathrm{rk}(\eta)$. Hence, $\mathrm{rk}(Z) + \mathrm{rk}(W) = m - 1$. Hence, by Definition 1 (c), Y is a proper canob of rank m and degree 0.

(d) $\xi \equiv \mathsf{H}\eta$. Then η is a proper canob of rank $m - 1$ and degree 0 and $n = 0$. By property C-VC3 of [CLg.II] §11F3, since $\xi \succ Y$ it follows that $Y \equiv \mathsf{H}Z$

where $\eta \succ Z$. By the induction hypothesis, it follows that Z is a proper canob of rank $m - 1$ and degree 0. By Definition 1 (d), it follows that Y is a proper canob of rank m and degree n.

Remark: In the case of the above proof for (b1), the following result was established for $\lambda\eta$-conversion: if $\lambda x.X \succ Y$, then either $Y \equiv \lambda x.Z$ where $X \succ Z$ or else $X \succ Yx$. This result fails in the synthetic theory (with strong reduction) since $[x](\mathsf{I}(Ux)) \equiv \mathsf{S}(\mathsf{KI})U \succ \mathsf{I}U$, where x does not occur in U, but $\mathsf{I}U \equiv [x]Z$ only if $Z \equiv \mathsf{I}Ux$ and $\mathsf{I}(Ux) \succ \mathsf{I}Ux$ (as can be seen by letting U be $y \not\equiv x$).

Lemma 1.4 *If ξ is a canob of rank m and degree n and if $\xi = Y$, then Y is a canob of the same rank and degree.*

Proof: By Definition 2, there is a proper canob η of rank m and degree n such that $\xi \succ \eta$. It follows that $Y = \eta$. Hence, by the Church-Rosser Theorem, there is a Z such that $Y \succ Z$ and $\eta \succ Z$. By Lemma 1.3, Z is a proper canob of rank m and degree n. Hence, by Definition 2, Y is a canob of rank m and degree n.

Lemma 1.5 *If ξ is a canob of rank m and degree $n > 0$, then for any ob U, ξU is a canob of rank m and degree $n - 1$.*

Proof: [CLg.II] §15B3 Lemma 3.4 (but replace the reference to Definition 1 (b) by a reference to Definition 1 (b1) and the reference to Lemma 3.3 by a reference to Lemma 1.4).

Lemma 1.6 *The ob ξ is a canob of rank m and degree n if and only if $\lambda x.\xi$ is a canob of rank m and degree $n + 1$.*

Proof: [CLg.II] §15B3 Lemma 3.5.

Lemma 1.7 *The ob ΞXY is a canob of rank m and degree* 0 *if and only if X and Y are canobs of degree* 1 *and* $m = \mathrm{rk}(X) + \mathrm{rk}(Y) + 1$.

Proof: [CLg.II] §15B3 Lemma 3.6.

Lemma 1.8 *The ob $\mathsf{H}X$ is a canob of rank m and degree* 0 *if and only if X is a canob of rank $m - 1$ and degree* 0.

Proof: Similar to that of Lemma 1.7.

Proof of Theorem 1: N1-3 are just Lemmas 1.4, 1.6, and 1.5; NΞ is Lemma 1.7; and NH is Lemma 1.8.

The rest of [CLg.II] §15B3 is correct as is. It might be worth adding to Theorem 15B5 of that section a new clause:

(d) *ξ is not an O_1 -ob.*

This clause corresponds to assumption (d) made about canonical simplexes at the beginning of the section.

It might be interesting to examine the theory of canobs from the point of view of $\lambda\beta$-conversion, but that is beyond the scope of this paper.

2 The system $\mathcal{J}_{22}$. By the definition of Curry [CSC], the system $\mathcal{J}_{22}$ is obtained from the $\mathcal{J}_{21}^*$ of [CLg.II] §15C2 by dropping H from the list of θ's (canonical atoms). Since, as Curry shows in [CSC] by proving that $\mathrm{Can}_0(\mathsf{L}X) \rightleftarrows \mathrm{Can}_1(X)$, this implies that H is not canonical of degree one, and since the canonicalness of H was used in [CLg.II] §15C2 to prove the equivalence of the A- and T-formulation of $\mathcal{J}_{21}^*$, it follows that this proof breaks down for $\mathcal{J}_{22}$. Hence, we do not consider an A-formulation of $\mathcal{J}_{22}$, but begin with a T-formulation (natural deduction system). This system has as axioms the following:

$(\mathsf{E}A)$ $\qquad \vdash \mathsf{E}A$, *if A is an atom*,[8]

(FE) $\qquad \vdash \mathsf{LE}$,

where we are taking $\Upsilon \equiv \mathsf{E}$, the universal category, and H as a C-indeterminate (atomic constant with no reduction rule), and hence

$$\mathsf{L} \equiv \mathsf{FEH},$$
$$\mathsf{F} \equiv \lambda xyz . \Xi x(\mathsf{B}yz).$$

The rules of the system are as follows:

Eq: $$\frac{X}{Y}, \text{ if } X = Y.$$

Ξe: $$\frac{\Xi XY \qquad XU}{YU}.$$

Ξi* $$\frac{\begin{matrix}[Xx] \\ Yx\end{matrix} \qquad LX}{\Xi XY},$$ *where x does not occur in X, Y, or in any other premise.*

H: $$\frac{X}{\mathsf{H}X}.$$

E: $$\frac{\mathsf{E}X \qquad \mathsf{E}Y}{\mathsf{E}(XY)}.$$

$\mathsf{H}\theta$: $$\frac{\mathsf{E}X_1 \quad \mathsf{E}X_2 \quad \dots \quad \mathsf{E}X_n}{\mathsf{H}(\theta X_1 \dots X_n)},$$ *where* $n = \mathrm{dg}(\theta)$.

$\mathsf{H}\Xi$: $$\frac{\mathsf{L}X \qquad \mathsf{L}Y}{\mathsf{H}(\Xi XY)}.$$

The derived rules **Fi***, **Pi***, and Πi can be derived as in [CLg.II] §15C2. The derivations of rules **Fe**, **Pe**, and Πe are relatively trivial.

In [CSC], Curry proves the following result:

Theorem 2 *If $\vdash X$ is derivable in $\mathcal{J}_{22}$, then* $\mathrm{Can}_0(X)$, i.e., *X is a basic canob of degree* 0.

This proof also goes through with the revised definition of canob for the case of λ-conversion. This same proof will also prove the following:

Corollary 2.1 *If $M \vdash X$ is derivable in $\mathcal{J}_{22}$ and if each constituent of M is a basic canob of degree* 0, *then so is X.*

As a further corollary to this theorem, Curry establishes the following:

Corollary 2.2 *The ob X is a basic canob of degree k if and only if* $\vdash \mathsf{H}_k X$ *is derivable in* $\mathcal{J}_{22}$.

3 An L-formulation of $\mathcal{J}_{22}$. In order to prove Gentzen's *Hauptsatz*, it is normal to develop an L-formulation (calculus of sequents) of a system. This will be done here for $\mathcal{J}_{22}$. I will follow the convention of [CLg.II] in writing

$$M \vdash^T X$$

to indicate that X can be derived from the premises M in the system of section 2. The system will be a restricted CL-system in the sense of [CLg.II] §12D1. Hence, its prime statements are the following:

(p1) $X \mid \mathfrak{a} \vdash X$, *if* $\mathrm{Can}_0(X)$,
($\mathsf{E}X$)* $\mid \mathfrak{a} \vdash \mathsf{E}X$, *where X is any ob*,
FE* $\mid \mathfrak{a} \vdash \mathsf{LE}$.[9]

The rules of the system are the following:

Structural rules: *C*, *W*, and *K* from [CLg.II] §12C2, but with the requirement that the principal constituent of *K* must be a basic canob of degree 0.

Expansion rules: *Exp* from [Clg.II] §12C2.

$*\Xi$ *If X and Y are canobs of degree* 1,

$$\frac{M \mid \mathfrak{a} \vdash XU, L \qquad M, YU \mid \mathfrak{a} \vdash N}{M, \Xi XY \mid \mathfrak{a} \vdash N, L}.$$

$\Xi*$
$$\frac{M, Xx \mid \mathfrak{a}, x \vdash Yx, L \qquad M \mid \mathfrak{a} \vdash \mathsf{L}X, L}{M \mid \mathfrak{a} \vdash \Xi XY, L}.$$

Irregular rules for H:

$\mathsf{H}*$
$$\frac{M \mid \mathfrak{a} \vdash X, L}{M \mid \mathfrak{a} \vdash \mathsf{H}X, L}.$$

$\mathsf{H}\theta*$ *If* $n = \mathrm{dg}(\theta)$,

$$\frac{M \mid \mathfrak{a} \vdash \mathsf{E}X_1, L; \quad \ldots; \quad M \mid \mathfrak{a} \vdash \mathsf{E}X_n, L}{M \mid \mathfrak{a} \vdash \mathsf{H}\,(\theta X_1 \ldots X_n), L}.$$

$\mathsf{H}\Xi*$
$$\frac{M \mid \mathfrak{a} \vdash \mathsf{L}X, L \qquad M \mid \mathfrak{a} \vdash \mathsf{L}Y, L}{M \mid \mathfrak{a} \vdash \mathsf{H}(\Xi XY), L}.$$

From these rules follow the L-rules $*\mathsf{F}$, $*\Pi*$, $*\mathsf{P}$ of [CLg.II] and, in addition,

$\mathsf{F}*$
$$\frac{M, Xx \mid \mathfrak{a}, x \vdash Y(Zx), L \qquad M \mid \mathfrak{a} \vdash \mathsf{L}X, L}{M \mid \mathfrak{a} \vdash \mathsf{F}XYZ, L}.$$

$\mathsf{P}*$[10]
$$\frac{M, X \mid \mathfrak{a} \vdash Y, L \qquad M \mid \mathfrak{a} \vdash \mathsf{H}X, L}{M \mid \mathfrak{a} \vdash \mathsf{P}XY, L}.$$

The system, which will be called $\mathcal{I}_{22}^{L}$, may be either singular or multiple. The conventions of [CLg.II] have been used to treat both cases simultaneously. It is easy to see that an ob X can occur as a constituent in a theorem of $\mathcal{I}_{22}^{L}$ only if it is a basic canob of degree 0.

4 The elimination theorem.

Theorem 3 (ET) *If*

(4) $$M, X \mid \mathfrak{a} \vdash Y, L$$

and

(5) $$M \mid \mathfrak{a} \vdash X, L,$$

then

(6) $$M \mid \mathfrak{a} \vdash Y, L.$$

The proof requires the following lemma:

Lemma 3.1 *If*

(7) $$M, U \mid \mathfrak{a} \vdash N,$$

where U consists of constituents of the form **LE** *or* **H(E**X**)**, *then*

(8) $$M \mid \mathfrak{a} \vdash N.$$

Proof: Let $A_1, \ldots, A_n$ be a proof of (7), and let A_k be

$$M_k, U_k \mid \mathfrak{a}_k \vdash N_k,$$

where U_k consists of the semiparametric ancestors in A_k of the constituents of U in (7). Let A_k' be

$$M_k \mid \mathfrak{a}_k \vdash N_k.$$

Since A_n' is (8), it is sufficient to prove A_k' for each k by induction on k. There are the following cases:

(α) U_k is void. Then $A_k' \equiv A_k$.

(β) U_k is not void and A_k is prime.

Then A_k must be an instance of (p1), U_k must be singular, and $N_k \equiv U_k$. The common constituent of U_k and N_k is either **LE** or **H(E**X**)** for some X. In the former case, A_k' is an instance of **FE***. In the latter case, we can proceed from (**E**X)* as follows:

$$\frac{\mid \mathfrak{a} \vdash \mathsf{E}X}{\mid \mathfrak{a} \vdash \mathsf{H}(\mathsf{E}X)}\,\mathsf{H}*$$

and the conclusion is A_k'.

(γ) U_k is not void and A_k is derived by a rule for which the constituents in U_k are parametric. Let the premise(s) be $A_i, A_j, \ldots$. By the induction hypothesis, $A_i', A_j', \ldots$ are derivable. Then A_k' follows by the same rule.

(δ) U_k is not void and A_k is derived by an irregular rule for which the principal constituent is in U_k. The only such rules are the structural or expansion rules (since there is no other irregular rule with principal constituent on the left). Let the premise be A_i. By the induction hypothesis, A_i' is provable, and $A_k' \equiv A_i'$.

(ε) U_k is not void and A_k is derived by a regular rule for which the principal constituent is in U_k. Then the rule is $*\Xi$ (really $*\mathsf{F}$), and the left premise, say A_i, is

$$M_k,\ U_j,\ \mathsf{H}(\mathsf{E}X)\ |\mathfrak{a}_k \vdash N_k,$$

where U_j is that part of U_k left over when the principal constituent, which must be LE, is removed. Then by the induction hypothesis, A_i', which is the same as A_k', is derivable. Q.E.D.

Proof of Theorem 3: If X has a semiparametric ancestor of the form LE in the proof of either (4) or (5), then $X \succ \mathsf{LE}$. In this case, by [CLg.II] Theorem 12C7, there is a derivation of

$$M,\ \mathsf{LE}\ |\mathfrak{a} \vdash Y,\ L.$$

Then (6) follows by Lemma 3.1. If there is no semiparametric ancestor of X of the form LE in the proof of either (5) or (6), then the proof follows the usual three-stage pattern of proofs of (ET) in [CLg.II], for example the proof of Theorem 15B7. Stage 1 goes through as usual. After Stage 1, it can be assumed that the head of X is Ξ, since all other cases have been taken care of in that stage. In Stage 2, the case of $\mathsf{FE}*$ can be excluded from case (β) by the assumption that no semiparametric ancestor of X in (6) has the form LE; furthermore, the cases of the irregular rules for H can be excluded from case (δ) by the assumption that X, and hence each of its semiparametric ancestors, has Ξ at its head. Stage 3 goes through as in the proof of [CLg.II] Theorem 15B7; the right premise of $\Xi*$ can be ignored. This completes the proof.

Theorem 4 *If the L-system is taken to be singular, then*

$$M\ |\mathfrak{a} \vdash X \rightleftarrows M \vdash^{T} X.$$

Proof: Similar to the proof of [CLg.II] Theorems 14E2-3 (see also Theorem 15B8).

5 Q-consistency. Let us now extend the system by introducing Q as a canonical atom of degree 2 and postulating, in the T-formulation, the additional axiom scheme

(ρ) $\vdash \mathsf{Q}XX$, *where X is any ob,*

and in the L-formulation the prime statement

(ρ)* $|\mathfrak{a} \vdash \mathsf{Q}XX$, *where X is any ob.*

Then Theorem 4 still holds. Furthermore, we have by Rule Eq,

(9) $X = Y \rightarrow\ \vdash \mathsf{Q}XY.$

The main consistency result of this paper is the converse of (9):

Theorem 9 *If*

(10) $$\vdash \mathbf{Q}XY$$

is provable in the T-formulation, then $X = Y$.

Proof: If (10) holds, then in the singular L-formulation, we have

(11) $$|\mathfrak{a} \vdash \mathbf{Q}XY.$$

Now the only rule of which (11) can be the conclusion is Exp*. Hence, the proof of (11) consists of a sequence of statements, each of which has the form

$$|\mathfrak{a} \vdash \mathbf{Q}UV$$

where $X \succ U$ and $Y \succ V$. The first of these statements must be an instance of $(\rho)*$, and hence in this statement $U \equiv V$. It follows by properties of equality that $X = Y$.

This system still does not include all of the desirable properties of equality. For although (9) holds, we do not have within the system

(12) $$\mathbf{Q}XY,\ ZX \vdash ZY.$$

In [EFT], a method of adjoining (12) to $\mathcal{J}_{21}$ of [CLg.II], provided that certain restrictions were fulfilled, was presented. A method of doing the same thing for $\mathcal{J}_{22}$ is presented in [EFTT].

NOTES

1. For an explanation of the letters in brackets, see the references.
2. Note that this is *not* the same system as the system also called $\mathcal{J}_{21}^{*}$ of [SIC].
3. Presented at the Congrès 1972 de Logique d'Orléans held at Orléans, France, September 4-13, 1972. This research was supported by Illinois State appropriated funds administered by the Mathematics Department of Southern Illinois University at Carbondale.
4. I.e., if the theory is based on a system of combinators. In all such systems considered in this part of [CLg.II] clause (c) of the bracket algorithm, which says that $[x]Ux \equiv U$ if x does not occur in U, holds. The result referred to in the text depends on clause (c).
5. The terminology and symbolism of [CLg.II] will be used freely throughout this paper.
6. Hence, we must have $n \geqslant 1$ in this clause.
7. Because no variable can be a canob of any degree, since otherwise by Lemma 1.1., U would be a canob for any U, and this is known to be false.

8. In case the system is based on λ-conversion, we must also have axiom (**E**A) in case A is a basic combinator; see [CLg.II] §15C1, Footnote 3. An alternative would be to postulate (**E**A) only when A is an atom and add the additional rule **FEE**$X \vdash$ **E**X. Then, to prove that $\vdash$**E**X holds for all obs X, it is sufficient to take the proof for the synthetic system and add the following case for the case $X \equiv \lambda x . Y$: by the induction hypothesis, we have $\vdash$**E**Y. Hence, using the (perhaps dummy) premise **E**x, Axiom (**FE**), and Rule **Fi***, we can conclude $\vdash$**FEE**$(\lambda x . Y)$, and then $\vdash$**E**$(\lambda x . Y)$ follows by the new rule. This alternative will not be used here.
9. The system is not quite a CL-system because of the presence of prime statement **FE***. However, the presence of this prime statement does not upset the proofs of [CLg.II] Theorems 12C3-7, which will be used in what follows.
10. Note that the rules Ξ*, **F***, and **P*** as given here differ from the rules usually called by these names in [CLg.II]. The rules given here are used in [CLg.II] §15C4.

REFERENCES

Each work is indicated by the author's name followed by an abbreviation in brackets. These are arranged alphabetically by author and then by abbreviation below. If no author is given, the author is Seldin or Seldin and others.

Curry, H.B.:

[CSC] "The consistency of a system of combinatory restricted generality," *The Journal of Symbolic Logic*, vol. 38 (1973), pp. 489-492.

Curry, H. B., J. R. Hindley, and J. P. Seldin:

[CLg.II] *Combinatory Logic*, vol. II, North-Holland, Amsterdam and London (1972).

Seldin, J. P.:

[EFT] "Equality in $\mathcal{F}_{21}$," *The Journal of Symbolic Logic*, vol. 38 (1973), pp. 521-525.

[EFTT] "Equality in $\mathcal{F}_{22}$," in H. E. Rose and J. C. Shepherdson, eds., *Logic Colloquium 1973, Proceedings of the Logic Colloquium, Bristol, July 1973,* North-Holland, Amsterdam (1975), pp. 433-444.

[SIC] *Studies in illative combinatory logic*. Dissertation, Amsterdam (1968).

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