Notre Dame Journal of Formal Logic
Volume XVI, Number 4, October 1975
NDJFAM

GENERALIZATION OF A RESULT OF HALLDÉN

ROBERT V. KOHN

We take M, $\vee$, $\neg$ as primitive connectives; let $\mathcal{L}$ be the set of all wffs in these connectives. We take the connectives $\wedge$, $\supset$, $\prec$, $\equiv$, and L to be defined in the usual ways. If $\alpha \in \mathcal{L}$, we write $\mathcal{L}[\alpha]$ for the smallest subset of $\mathcal{L}$ containing α and closed under the connectives $M, \vee, \neg$. A *modal logic* is a proper subset of $\mathcal{L}$ which is closed under the rules of uniform substitution and modus ponens, and contains all tautologies. If $\mathbf{L}_1$ and $\mathbf{L}_2$ are modal logics, then $\mathbf{L}_1$ is an *extension* of $\mathbf{L}_2$ iff $\mathbf{L}_2 \subseteq \mathbf{L}_1$. Let **PC** denote the classical propositional calculus. For any wff $\alpha \in \mathcal{L}$, let $\hat{\alpha}$ be the wff of **PC** obtained by erasing all occurrences of "M" in α.

Lemma *Let $\alpha \in \mathcal{L}[p]$, and suppose $\vdash_{\mathbf{PC}} \hat{\alpha} \supset p$. Then there is an $n \geq 1$ such that $\vdash_{S2} \alpha \prec M^n p$.*

Proof: First of all, notice that for any wffs γ, δ and any affirmative modality F, if $\vdash_{S2} \gamma \prec \delta$ then $\vdash_{S2} F\gamma \prec F\delta$; moreover, for each such F there is an n such that $\vdash_{S2} Fp \prec M^n p$. The proof now proceeds by induction, showing that the Lemma is true of both β and $\neg\beta$ for every $\beta \in \mathcal{L}[p]$. In the case $\beta = p$, the assertion of the Lemma is trivial for β and vacuous for $\neg\beta$. Suppose the Lemma has been verified for both γ and $\neg\gamma$. If β is $M\gamma$ and $\vdash_{\mathbf{PC}} \hat{\beta} \supset p$, then $\vdash_{\mathbf{PC}} \hat{\gamma} \supset p$, so by hypothesis there is an n such that $\vdash_{S2} \gamma \prec M^n p$. Then $\vdash_{S2} M\gamma \prec M^{n+1} p$. If β is $\neg M\gamma$ and $\vdash_{\mathbf{PC}} \hat{\beta} \supset p$, then $\vdash_{\mathbf{PC}} \widehat{\neg\gamma} \supset p$. So there is an n such that $\vdash_{S2} \neg\gamma \prec M^n p$. Then $\vdash_{S2} L\neg\gamma \prec LM^n p$, so $\vdash_{S2} \neg M\gamma \prec M^{n+1} p$. Now suppose the Lemma has been verified for γ_1, γ_2, $\neg\gamma_1$, and $\neg\gamma_2$. If β is $\gamma_1 \vee \gamma_2$ and $\vdash_{\mathbf{PC}} \hat{\beta} \supset p$, then $\vdash_{\mathbf{PC}} \hat{\gamma}_1 \supset p$ and $\vdash_{\mathbf{PC}} \hat{\gamma}_2 \supset p$. So there are n_1 and n_2 such that $\vdash_{S2} \gamma_1 \prec M^{n1} p$ and $\vdash_{S2} \gamma_2 \prec M^{n2} p$. Put $n = \max(n_1, n_2)$; then $\vdash_{S2} \gamma_1 \vee \gamma_2 \prec M^n p$. Now suppose β is $\neg(\gamma_1 \vee \gamma_2)$, and $\vdash_{\mathbf{PC}} \hat{\beta} \supset p$. Then $\vdash_{\mathbf{PC}} (\neg\hat{\gamma}_1 \wedge \neg\hat{\gamma}_2) \supset p$; since γ_1 and γ_2 are in $\mathcal{L}[p]$, it follows that $\vdash \neg\hat{\gamma}_i \supset p$ for either $i = 1$ or $i = 2$. Then by hypothesis, there is an n such that $\vdash_{S2} \neg\gamma_i \prec M^n p$, so $\vdash_{S2} \neg(\gamma_1 \vee \gamma_2) \prec M^n p$. The induction is now complete.

The modal logic **Tr** of [2] is that modal logic which contains all $\alpha \in \mathcal{L}$ such that $\vdash_{\mathbf{PC}} \hat{\alpha}$. McKinsey [3] has shown that **Tr** is the unique Post-complete extension of S4.

Received July 26, 1974

Theorem *Let* **L** *be any modal logic which extends* S2. *Then either* **L** $\subseteq$ **Tr** *or there is an* $n \geqslant 2$ *such that* $\vdash_{\mathsf{L}} M^n p$.

Proof: Suppose **L** is not a sublogic of **Tr**, and choose $\alpha \in \mathsf{L}$ such that $\overline{\overline{\mathsf{PC}}}\!\!\dashv \hat{\alpha}$. Then there is a substitution instance α^* of α such that $\alpha^* \in \mathcal{L}[p]$ and $\vdash_{\mathsf{PC}} \hat{\alpha}^* \supset p$. Hence by the Lemma there is an n such that $\vdash_{S2} \alpha^* \dashv M^n p$, and so $\vdash_{\mathsf{L}} M^n p$.

Corollary *Let* **L** *be a modal logic which extends* S3. *Then either* **L** $\subseteq$ **Tr** *or* S7 $\subseteq$ **L**.

Proof: This follows immediately from the Theorem, since $\vdash_{S3} M^{n_1} p \equiv M^{n_2} p$ for $n_1 \geqslant 2$ and $n_2 \geqslant 2$, and since S7 can be axiomatized by adding *MMp* to S3, with only uniform substitution and modus ponens as rules of inference.

REFERENCES

[1] Halldén, S., "Results concerning the decision problem of Lewis's calculi S3 and S6," *The Journal of Symbolic Logic*, vol. 14 (1949), pp. 230-236.

[2] Segerberg, K., "Post completeness in modal logic," *The Journal of Symbolic Logic*, vol. 37 (1972), pp. 711-715.

[3] McKinsey, J. C. C., "On the number of complete extensions of the Lewis systems of sentential calculus," *The Journal of Symbolic Logic*, vol. 9 (1944), pp. 42-45.

University of Warwick
Coventry, England