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CONCERNING SOME EXTENSIONS OF S4

BOLESŁAW SOBOCIŃSKI

In my papers [9], [11] and [12] several problems concerning some extensions of S4 are left open.* Namely:

(A) In [9], pp. 355-359, sections **2.6** and **2.7**, it has been proved about the modal formula

T1 $\mathfrak{C}\mathfrak{C}\mathfrak{C}pqqC\mathfrak{C}\mathfrak{C}Npqqq$

which was observed by Grzegorczyk in [1], p. 128: (i) that, in the field of S4, **T1** implies **J1**, i.e., the proper axiom of K1.1, *cf.* [10] and [9], p. 349; (ii) that **T1** is a consequence of K1.2, *cf.* [10] and [9], p. 349; (iii) and that **T1** is verified by characteristic matrix which, *cf.* [2], Makinson has constructed for his system D*, i.e., for my system K3.1, *cf.* [5].

But I was able neither to prove logically that **T1** is a consequence of K1.1 nor to establish that the systems K1.1.1 (= {S4; **T1**}) and K2.2 (= {K2; **T1**}), *cf.* [9], p. 367, are the proper extensions of K1.1 and K2.1 respectively.

(B) As Geach has observed, *cf.* [4], p. 139, and [11], p. 305, in the field of S4.2, the so-called Diodorean modal formulas **N1** and **M1** are inferentially equivalent. Although, clearly, in the field of S4, {**M1**} $\rightarrow$ {**N1**}, up to now it was unknown whether, in the field of the same system, {**N1**} $\rightarrow$ {**M1**}. Consequently, the problems whether S4.1 (= {S4; **N1**}) and S4.1.2 (= {S4; **L1**; **N1**}) are properly contained in S4.1.1 (= {S4; **M1**}) and in S4.1.3 (= {S4; **L1**; **M1**}) respectively remained open, *cf.* [11], p. 311, and [12].

(C) In [9], pp. 363-366, sections **3.4-3.6**, it has been proved that the system S4.7 which Schumm has established in [6], contains S4.6 (= {S4; **S1**}) which in its turn contains S4.5 (= {S4; **E1**} $\rightleftarrows$ {S4; **E2**}). Moreover, it has been

*An acquaintance with the papers cited in this note and especially, with the enumeration of the extensions of S4 and their proper axioms introduced in [9], pp. 347-350, and in [12], is presupposed.

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shown that S4.5 is a proper extension of S4.4 and that S4.6 is properly contained in V1. But, an open problem remained whether S4.5 and S4.6 are the proper subsystems of S4.6 and S4.7 respectively.

All the problems mentioned above are solved negatively now due to the research of Schumm [7] and of Zeman [14]. Namely:

(D) In [7] Schumm has proved metalogically that the following formulas

$\mathfrak{S}$**I** $CLCLCLCCpLqLCpLqCpLqCpLqCLCLCpLqLqCLCLCNpLqLqLq$

$\mathfrak{S}$**II** $CCLCLCCpLpLCpLpCpLpCMLCpLpCpLpCCLCLCCNpLNpLCNpLNp$
$CNpLNpCMLCNpLNpCNpLNpCLCLCpLpLpCMLpLp$

$\mathfrak{S}$**III** $CCMLpLMpCMCMCpMqMqCMCMCNpMqMqMq$

$\mathfrak{S}$**IV** $CCMCMCpMNCppMNCppCMCMCNpMNCppMNCppMNCppCMLpLMp$

are provable in S4. Therefore, since, in the field of S4, {**T1**} → {**J1**} and {**M1**} → {**N1**}, the formulas $\mathfrak{S}$**I** and $\mathfrak{S}$**II** show that, in the field of the same system, {**T1**} ⇄ {**J1**} and {**M1**} ⇄ {**N1**}. Hence, we have {K1.1.1} ⇄ {K1.1}, {K2.2} ⇄ {K2.1}, {S4.1.1} ⇄ {S4.1} and {S4.1.3} ⇄ {S4.1.2} immediately. Additionally, a provability of $\mathfrak{S}$**III** and $\mathfrak{S}$**IV** which has been established in [7] shows that the formula

G3 $\mathfrak{C}MCMCpMqMqCMCMCNpMqMqMq$

can serve as the proper axiom of S4.2.

(E) In [14], pp. 349-353, Zeman investigated Schumm's system S4.7 which he calls S4.9. Besides other results, he has proved that S4.9 (i.e., S4.7) is properly contained in V1. And the methods which he used in order to establish his axiomatizations of S4.9, *cf.* [14], p. 355, allow him to remark that the systems S4.5, S4.6 and S4.7 (i.e., his S4.9) are inferentially equivalent, *cf.* the last paragraph of [9], p. 367.

Since in [7] Schumm has obtained his results in a purely metalogical way which does not indicate entirely how in S4 the formulas $\mathfrak{S}$**I** - $\mathfrak{S}$**IV** can be obtained logically, in section **1** of this note such logical proof will be presented. Moreover, it will be shown that, in the field of S4, the following formula

N3 $\mathfrak{C}\mathfrak{C}\mathfrak{C}pqqC\mathfrak{C}\mathfrak{C}NpqqCMLpq$

is inferentially equivalent to **N1**, and, therefore, can serve as the proper axiom of S4.1, and that **G3** is inferentially equivalent to **G1** not only in the field of S4 but also in the field of S3. And, since in [14] Zeman did not elaborate his mentioned above remark, in section **2** a proof which is essentially based on Zeman's method will be given that {S4.5} ⇄ {S4.6} ⇄ {S4.7} ⇄ {S4.9}.

REMARK: In a letter of 9.23.1970 Dr. Krister Segerberg, Åbo Academy, Åbo, Finland, informed me that some results discussed in [7], [9], [12], [14] and the present note were known to him, but not yet published. *Cf.* his short abstract "On some extensions of S4" in *The Journal of Symbolic Logic*, vol. 35 (1970), p. 363.

1 Let us assume S4. Then:

1.1 In the field of S4 the proper axiom of K1.1, i.e., thesis **J1** is inferentially equivalent to Grzegorczyk's axiom **T1**, *cf.* Schumm [7]. Logical proof:

Z1	*ℭNpCpq*	[S1°]
Z2	*ℭℭpqℭℭvrCℭℭprsLℭℭqvCws*	[S4°]
Z3	*ℭℭℭpqqCℭℭNpqsLℭℭCpqℭpqCws*	[*Z2*, *p/Np*, *q/Cpq*, *v/ℭpq*, *r/q*; *Z1*]
Z4	*ℭLpp*	[S1]
Z5	*ℭℭpCqrℭℭpCvsℭℭsqℭpCvr*	[S3°]
Z6	*ℭℭLℭℭCpqℭpqCpqℭpqℭℭℭpqqCℭℭNpqqq*	[*Z5*, *p/ℭℭpqq*, *q/ℭpq*, *r/q*, *v/ℭℭNpqq*, *s/Lℭℭ Cpqℭ pqCpq*; *Z4*, *p/Cℭpqq*; *Z3*, *s/q*, *w/p*; S1°]
Z7	*ℭℭpqℭLpLq*	[S3°]
S1	*ℭℭℭℭCpqℭpqCpqCpqℭℭℭpqqCℭℭNpqqq*	[*Z7*, *p/ℭℭCpqℭpqCpq*, *q/Cpq*; *Z6*; S1]

It is self-evident that, in the field of S4, *S1* is inferentially equivalent to Schumm's formula **𝔖I**. Now, by S4 and

J1 *ℭℭℭpLppp*

we obtain

T1 *ℭℭℭpqqCℭℭNpqqq*

at once. Since, *cf.* [9], p. 355, in the field of S4, **T1** implies **J1**, we have

(i) {K1.1} ⇄ {S4; **J1**} ⇄ {S4; **T1**} ⇄ {K1.1.1}

(ii) {K2.1} ⇄ {S4.2; **J1**} ⇄ {S4.2; **T1**} ⇄ {K2.2}.

1.2 In the field of S4 the formulas **N1** and **M1**, i.e., the proper axioms of S4.1 and S4.1.1 respectively, are inferentially equivalent, *cf.* Schumm [7]. I shall present here two logical proofs of these facts. In the first proof it will be shown that, in the field of S4, **M1** implies a formula **N3**, as far as I know, previously unobserved, which in its turn gives **M1** and **N1** at once. In the second proof Schumm's formula **𝔖II** will be deduced logically.

1.2.1 Formula **N3**.

Z8	*ℭℭℭpqsCℭℭNpqqLℭℭCNpqℭNpqℭws*	[*Z3*, *p/Np*; S4°]
Z9	*ℭCMLCNpqrCMLpr*	[S2°]
Z10	*ℭℭpCqrℭℭrsCLpCqs*	[S3]
Z11	*ℭℭℭNpqqCℭMLCNpqℭNpqCMLpq*	[*Z10*, *p/CMLCNpqℭNpq*, *q/MLp*, *r/ℭNpq*, *s/q*; *Z9*, *r/ℭNpq*; S1°]
Z12	*ℭℭpCqrℭℭvCpsℭℭsqℭvCpr*	[S3°]
Z13	*ℭℭLℭℭCNpqℭNpqℭNpqℭMLCNpqℭNpqℭℭℭpqqCℭℭNpqqCMLpq*	[*Z12*, *p/ℭℭNpqq*, *q/ℭMLCNpqℭNpq*, *r/CMLpq*, *v/ℭℭpqq*, *s/Lℭℭ CNpqℭNpqℭNpq*; *Z11*; *Z8*, *w/Np*, *s/q*]
S2	*ℭℭℭℭCNpqℭNpqℭNpqCMLCNpqℭNpqℭℭℭpqqCℭℭNpqqCMLpq*	[*Z7*, *p/ℭℭCNpqℭNpqℭNpq*, *q/CMLCNpqℭNpq*; *Z13*; S1°]

Thus, *S2* is provable in the field of S4. Hence, let us assume *S2* and

M1 *ℭℭℭpLpLpCMLpLp.*

Then:

N3 *ℭℭℭpqqCℭℭNpqqCMLpq.* [*S2*; **M1**, *p/CNpq*; S1°]

Now, assume S4 and **N3**. Then:

Z14 *ℭℭNpLpLp* [S2°]
M1 *ℭℭℭpLpLpCMLpLp.* [**N3**, *a/Lp*; *Z14*; S1°]

Since, *cf.* [11], p. 308, section **2.3**, in the field of S4, **M1** implies **N1**, we have {S4.1} ⇄ {S4; **N1**} ⇄ {S4; **N3**} ⇄ {S4; **M1**} ⇄ {S4.1.1}, and, moreover, {S4.1.2} ⇄ {S4.1; **L1**} ⇄ {S4; **N1**; **L1**} ⇄ {S4; **N3**; **L1**} ⇄ {S4; **M1**; **L1**} ⇄ {S4.1.3}.

1.2.2 Proof of Schumm's formula **𝔖II**:

Z15 *ℭℭNppCpLp* [S2°]
Z16 *ℭℭℭpLpLpℭℭpLpp* [S2]
Z17 *ℭℭvCqrℭℭℭprsℭvℭℭpqs* [S4]
Z18 *ℭℭℭpLpLpℭℭNpℭpLpCpLp*
[*Z17*, *v/ℭℭpLpLp*, *q/ℭpLp*, *r/p*, *p/Np*, *s/CpLp*; *Z16*; *Z15*]
Z19 *ℭℭpqℭℭvℭℭprsℭvℭℭqrs* [S4]
Z20 *ℭℭℭpLpLpℭℭCpLpℭpLpCpLp*
[*Z19*, *p/Np*, *q/CpLp*, *v/ℭℭpLpLp*, *r/ℭpLp*, *s/CpLp*; *Z1*, *q/Lp*; *Z18*]
Z21 *ℭCMLCqLprCMLpr* [S4]
Z22 *ℭℭpqℭℭrsℭCqrCps* [S3°]
Z23 *ℭCℭℭCpLpℭpLpCpLpCMLCpLpCpLpCℭℭpLpLpCMLpCpLp*
[*Z22*, *p/ℭℭpLpLp*, *q/ℭℭCpLpℭpLpCpLp*, *r/CMLCpLpCpLp*, *s/CMLpCpLp*; *Z20*; *Z21*, *q/p*, *r/CpLp*]
Z24 *ℭℭℭpqrℭℭNrLNpLr* [S4]
Z25 *ℭℭℭpLpLpℭℭNpLNpLp* [*Z16*; *Z24*, *q/Lp*, *r/p*; S1°]
Z26 *ℭℭℭpLpLpCℭℭpLpLpℭℭpℭNpLNpLp*
[*Z17*, *v/ℭℭpLpLp*, *q/ℭNpLNp*, *r/Lp*, *s/Lp*; *Z25*]
Z27 *ℭℭℭpLpLpℭℭpℭNpLNpLp* [*Z26*; S1°]
Z28 *ℭpCNpq* [S1°]
Z29 *ℭLpCNpq* [*S28*; S2]
Z30 *ℭℭtsℭℭpqℭℭvℭℭprtℭvℭℭqrs* [S4]
Z31 *ℭℭℭpLpLpℭℭCNpLNpℭNpLNpCNpLNp*
[*Z30*, *t/Lp*, *s/CNpLNp*, *q/CpLNp*, *v/ℭℭpLpLp*, *r/ℭNpLNp*; *Z29*, *q/LNp*; *Z28*, *q/LNp*; *Z27*]
Z32 *ℭCMLCNpqrCMLpr* [S2°]
Z33 *ℭCMLCNpqCNpLNpCMLpCMpp* [*Z32*, *r/CNpLNp*; S1°]
Z34 *ℭCℭℭCNpLNpℭNpLNpCNpLNpCMLCNpLNpCNpLNp*
CℭℭpLpLpCMLpCMpp
[*Z22*, *p/ℭℭpLpLp*, *q/ℭℭCNpLNpℭNpLNpCNpLNp*, *r/CMLCNpLNpCNpLNp*, *s/CMLpCMpp*; *Z31*; *Z33*, *q/LNp*]
Z35 *ℭCqrCCpqCpr* [S1°]
Z36 *ℭCMpLqℭpLq* [S4°; *cf.* [8]]

Z37 *ℭℭpCqrℭℭrsℭpCqs* [S3°]
Z38 *ℭCpLpCCMppℭpLp* [*Z37, p/CpLp, q/CMpp, r/CMpLp, s/ℭpLp; Z35, q/p, r/Lp, p/Mp; Z36, q/p*]
Z39 *ℭCℭqrCsqCℭqrCsr* [S2]
Z40 *ℭℭvCwtℭℭCqCrtCqCrzℭℭpCqCrvℭℭuCqCrwℭpCuCqCrz* [S3°]
S3 *ℭCℭℭCpLpℭpLpCpLpCMLCpLpCpLpCCℭℭCNpLNpℭNpLNpCNpLNpCMLCNpLNpCNpLNpCℭℭpLpLpCMLpLp*
[*Z40, v/CpLp, w/CMpp, t/ℭpLp, q/ℭℭpLpLp, r/MLp, z/Lp, p/Cℭℭ CpLpℭpLpCpLpCMLCpLpCpLp, u/CℭℭCNpLNpℭNpLNpCNpLNpCMLCNpLNpCNpLNp; Z38; Z39, q/ℭpLp, r/Lp, s/MLp; Z23; Z34*]

S3 is Schumm's formula **𝔖II**. Thus, {S4; **N1**} ⇄ {S4; **M1**}.

1.3 Proof of the formulas **𝔖III** and **𝔖IV**.

Z41 *ℭLMpCrCMLNpq* [S1°]
Z42 *ℭMKpqMp* [S2°]
Z43 *ℭℭvsℭℭpCrCsqℭpCrCvq* [S3°]
Z44 *ℭpCqCrr* [S1°]
Z45 *ℭLMpCrCNLCLNpMqMq* [*Z43, v/MKLNpNMq, s/MLNp, p/LMp, q/Mq; Z42, p/LNp, q/NMq; Z41, q/Mq*; S1°]
Z46 *ℭLMpCrCCLCLNpMMqMMqMq* [*Z45; Z44, p/LMp, q/r, r/Mq*; S4]
Z47 *ℭCLpMqMCpq* [S2°; *cf.* [13], pp. 71-72]
Z48 *ℭLMpCrCMCMCNpMqMqMq*
[*Z46; Z47, p/Np, q/Mq; Z47, p/MCMCNpMqMq, q/Mq*; S1°]
Z49 *ℭNMLpCMCMCpMqMqCrMq* [*Z48, p/Np*; S1°]
S4 *ℭCMLpLMpCMCMCpMqMqCMCMCNpMqMqMq*
[*Z49, r/MCMCNpMqMq; Z48, r/MCMCpMqMq*; S2°]
Z50 *ℭLNqℭCpqNp* [S2°]
Z51 *LNMNCqq* [S4°]
Z52 *ℭCpMNCqqNp* [*Z50, q/MNCqq; Z51*]
Z53 *ℭMCpMNCqqMNp* [*Z52*; S2°]
Z54 *ℭMNpMCpq* [S2°]
Z55 *ℭMNpMCpMNCqq* [*Z54, q/MNCqq; Z53*; S1°]
Z56 *ℭMpMCNpMNCqq* [*Z55, p/Np*; S1°]
Z57 *ℭNpCpMNCqq* [*Z1, q/MNCqq; Z52*; S1°]
Z58 *ℭpp* [S1°]
Z59 *ℭCMNMNpNMNMpCMLpLMp* [*Z58, p/CMLpLMp*; S1°]
Z60 *ℭCMNMCpMNCqqNMNMCNpMNCqqCMLpLMp* [*Z59; Z55; Z56*; S1°]
Z61 *ℭCMCMCpMNCqqMNCqqNMCMCNpMNCqqMNCqqCMLpLMp*
[*Z60; Z55, p/MCpMNCqq; Z55, p/MCNpMNCqq*; S1°]
S5 *ℭCMCMCpMNCppMNCppCMCMCNpMNCppMNCppMNCppCMLpLMp*
[*Z61, q/p; Z57, p/MCMCNpMNCppMNCpp, q/p*; S1°]

S4 and *S5* are Schumm's formulas **𝔖III** and **𝔖IV**. Hence, in the field of S4, {**G1**} ⇄ {**G3**}.

1.3.1 In [13], pp. 75-76, section **2.6**, I have proved that each of the proper axioms of S4.2, i.e., **G1** and **G2** (formerly *L1*) possesses a property that its addition to S3 generates S4.2. It will be shown here that Schumm's axiom **G3** also has this property. For this end assume **G3** and let us use only the formulas provable in S3 in the following deductions. Then:

Z62 *ℭℭpCqrℭNrCpNq* [S2°]
Z63 *LNKpNp* [S1°]
Z64 *CMCMCpMKpNpMKpNpLNCMCNpMKpNpMKpNp*
[*Z62*, *p/MCMCpMKpNpMKpNp*, *q/MCMCNpMKpNpMKpNp*, *r/MKpNp*; **G3**, *q/KpNp*; S1°; *Z63*]
Z65 *ℭCMpLqℭpq* [S2°, *cf.* [8]]
Z66 *ℭCMCpMKpNpMKpNpNCMCNpMKpNpMKpNp*
[*Z65*, *p/CMCpMKpNpMKpNp*, *q/NCMCNpMKpNpMKpNp*; *Z64*]
Z67 *ℭℭpNCqrℭpNr* [S2°]
Z68 *ℭCMCpMKpNpMKpNpLNKpNp*
[*Z67*, *p/CMCpMKpNpMKpNp*, *q/MCNpMKpNp*, *r/MKpNp*; *Z66*; S1°]
Z69 *ℭℭCpqrℭqr* [S2°]
Z70 *ℭMKpNpLNKpNp* [*Z69*, *p/MCpMKpNp*, *q/MKpNp*, *r/LNKpNp*; *Z68*]
Z71 *ℭℭMpLqLLCpq* [*Z65*; S2°]
Z72 *LLCKpNpNKpNp* [*Z71*, *p/KpNp*, *q/NKpNp*; *Z70*]

Since Parry has proved in [3], p. 148, that an addition of any formula of the form $LL\alpha$ to S3 implies a system containing S4, we know, by *Z72*, that {S3; **G3**} → {S4}. Therefore, due to provability of *S4* and *S5* in S4, we have

{S4.2} ⇄ {S4; **G1**} ⇄ {S4; **G2**} ⇄ {S4; **G3**} ⇄ {S3; **G1**} ⇄ {S3; **G2**} ⇄ {S3; **G3**}.

2 Inferential equivalence of the systems S4.5, S4.6 and S4.7. As mentioned above, Zeman has remarked, see [9], pp. 363-366, sections **3.4-3.6**, that the systems S4.5 and S4.6 are inferentially equivalent to Schumm's system S4.7. Since in [9] it was proven that {S4.7} → {S4.6} → {S4.5}, it will be sufficient to show that {S4.5} → {S4.7}. Although, formally, the deductions given below differ in some respects from the proof which Zeman used in [14], pp. 349-353, in order to show that in the field of S4.4 system S4.7 implies his system S4.9, the idea of both these proofs is essentially the same and is due to Zeman.

Let us assume S4 and, *cf.* [9], section **3.5**, the proper axiom of S4.5

E2 *AℭMLpLpALpAℭpqℭpNq.*

Hence, we have at our disposal S4.4. Then:

Z1 *LAℭMLpLpALpAℭpqℭpNq* [**E2**; S4°]
Z2 *ℭLpℭqLp* [S4°]
Z3 *ℭℭpqCMLpMLq* [S3°]
Z4 *ℭℭpNqNMKpq* [S2°]
Z5 *ℭℭqpℭℭrvℭℭsNtℭLApAqArsℭNpCtv* [S3°]
Z6 *ℭNℭMLpLpCMKpqCMLpMLq* [*Z5*, *q/Lp*, *p/ℭMLpLp*, *r/ℭpq*, *v/CMLpMLq*, *s/ℭpNq*, *t/MKpq*; *Z2*, *q/MLp*; *Z3*; *Z4*; *Z1*]
Z7 *ℭNℭMpqMp* [S4°]

Z8 $\mathfrak{C}MLpCLMqMKpq$ [S4°]
Z9 $\mathfrak{C}\mathfrak{C}pq\mathfrak{C}\mathfrak{C}qCst\mathfrak{C}\mathfrak{C}pCtCqv\mathfrak{C}pCsv$ [S3°]
Z10 $\mathfrak{C}N\mathfrak{C}MLpLp\mathfrak{C}LMqMLq$ [*Z9*, $p/N\mathfrak{C}MLpLp$, q/MLp, s/LMq, $t/MKpq$, v/MLq; *Z7*, p/Lp, q/Lp; *Z8*; *Z6*]
Z11 $\mathfrak{C}N\mathfrak{C}MLpLp\mathfrak{C}MLMqMLq$ [*Z11*; S4°]
R1 $\mathfrak{C}MLpCpLp$ [**E2**; S4; *cf*. [9], p. 364, section **3.5.1**]
Q1 $A\mathfrak{C}MLpLp\mathfrak{C}MLMqCqLq$ [*Z11*; **R1**, p/q; S1°]

Thus, in the field of S4, **E2** implies **Q1**, i.e., the proper axiom of S4.7 (S4.9), *cf*. [9], pp. 361-362, section **3.1**. Since {S4.7} → {S4.6} → {S4.5}, the proof is complete.

3 Due to results which were discussed above the following rectifications in the enumeration of the extensions of S4 and their proper axioms, introduced in [9], pp. 347-350, should be made:

1. System S4.1 (= {S4; **N1**}). Besides

N1 $\mathfrak{C}\mathfrak{C}\mathfrak{C}pLppCMLpp$

each of the following formulas

N2 $\mathfrak{C}\mathfrak{C}\mathfrak{C}pLpLpCMLpLp$ [Formerly **M1**]
N3 $\mathfrak{C}\mathfrak{C}\mathfrak{C}pqqC\mathfrak{C}\mathfrak{C}NpqqCMLpq$ [*Cf*. section **1.2**]
N4 $\mathfrak{C}\mathfrak{C}\mathfrak{C}CpLp\mathfrak{C}pLpCpLpCMLCpLpCpLp$
[An inspection of Schumm's formula $\mathfrak{S}$II]

can serve as the proper axiom of this system.

2. System S4.2 (= {S4; **G1**}). Besides

G1 $\mathfrak{C}MLpLMp$
G2 $\mathfrak{C}MLpLMLp$

also

G3 $\mathfrak{C}MCMCpMqMqCMCMCNpMqMqMq$

can be adopted as the proper axiom of S4.2.

3. System S4.9 (= {S4; **Q1**}). I am accepting a suggestion of Zeman, *cf*. [14], p. 353, that Schumm's system S4.7 should be renamed. Besides

Q1 $A\mathfrak{C}MLpLp\mathfrak{C}MLMqCqLq$

each of the following formulas

Q2 $A\mathfrak{C}MLpLpALqA\mathfrak{C}qr\mathfrak{C}qNr$ [Formerly **S1**]
Q3 $A\mathfrak{C}MLpLpALqA\mathfrak{C}qp\mathfrak{C}qNp$ [Formerly **E1**]
Q4 $A\mathfrak{C}MLpLpALpA\mathfrak{C}pq\mathfrak{C}pNq$ [Formerly **E2**]

can serve as the proper axiom of this system. I omitted here the axiom-systems of S4.9 given in [7] and [14], p. 355, since instead of S4 they are based on S4.4.

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University of Notre Dame
Notre Dame, Indiana