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SUPERINDUCTIVE CLASSES IN CLASS-SET THEORY

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1. *Introduction*[1] Many definitions of the (von Neumann) ordinals have been given in set theory, but the one which seems most natural to us is that which parallels Frege's definition of the natural numbers, as the intersection of all inductive classes. This definition of ordinals as the intersection of all 'superinductive' classes has been proposed and its virtues discussed by Sion and Wilmot [3] and Smullyan [4]. In [4] a more general process of superinduction is discussed and the resulting minimally superinductive classes play a key role in particularly elegant proofs of Zorn's lemma, the Well-ordering theorem, and the Transfinite recursion theorem. Methods of establishing the existence of this minimally superinductive class in versions of Class-Set theory such as Gödel's [2], where we cannot assert the existence of classes defined by formulas containing bound class variables, have been briefly described in [4]. In Smullyan [5] a proof is given of the existence of the minimally superinductive class in Gödel's Class-Set theory which though proving a slightly more general theorem than the one we present here, requires both the axiom of substitution and the axiom of choice. In section 2, we present a new proof which avoids using the axiom of substitution and the axiom of choice. In addition as a by-product of our proof we obtain yet another definition of ordinal and a new definition of constructible set. In section 4, we present a proof that the minimally superinductive class under an arbitrary progressing function is well-ordered under inclusion. This theorem is given in [4] for slowly progressing functions. Again our proofs avoid using the axiom of substitution.

2. *Existence of Minimally Superinductive Classes*

Definition. A function g is called *progressing* if $x \subseteq g(x)$ for all x in the domain of g.

Definition. A function g is *slowly progressing* if g is progressing and $g(x)$ contains at most one more element than x.

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1. The results presented here are contained in the first chapter of the author's thesis [1], written under the supervision of Professor Raymond Smullyan at Yeshiva University.

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Definition. A set B is called a *chain* if $x, y \in B$ implies $x \subseteq y$ or $y \subseteq x$.

Definition. Let A, B be classes and g a function. Then the class S is *superinductive with respect to* (A, g, B) if

(i) $A \subseteq S \subseteq B$
(ii) $x \in S$ and $g(x) \in B$ implies $g(x) \in S$
(iii) D a chain of S and $\bigcup D \in B$ implies $\bigcup D \in S$, where $\bigcup D$ is the union of all sets in D.

Definition. S is *superinductive under* g if S is superinductive with respect to $(\{\Lambda\}, g, V)$, where Λ and V are the empty and universal class, respectively.

Definition. M is *minimally superinductive with respect to* (A, g, B) if M is the intersection of all classes which are superinductive with respect to (A, g, B).

Remark. Clearly M is superinductive with respect to (A, g, B).

Definition. M is *minimally superinductive under* g if M is minimally superinductive with respect to $(\{\Lambda\}, g, V)$.

Remark. In general it is not true that the minimally superinductive class with respect to (A, g, B) is the same as the union of all the minimally superinductive classes with respect to $(\{a\}, g, B)$, where $a \in A$. However for finite classes, A, it is true, for one can easily prove (by 'superinduction') that the minimally superinductive class with respect to $(\bigcup_{i=1}^{k} A_i, g, B)$ is the union of the minimally superinductive classes with respect to (A_i, g, B), $i = 1, \dots, k$.

For the remainder of this section, we shall only be concerned with progressing functions, so we shall assume that all functions, unless otherwise noted, are progressing (the proof of the existence of the minimally superinductive class in [5], though using the axioms of choice and substitution holds for arbitrary functions). This should not be thought of as being an arbitrary restriction; in fact condition (iii) in the definition of superinductive class seems unnatural unless we assume g is progressing. Assume, then, that g is a fixed progressing function, A and B fixed classes and that all superinductive classes mentioned are superinductive with respect to (A, g, B).

The definition of the minimally superinductive class involves quantifying over classes, so its existence is not immediately apparent if we restrict our comprehension axioms to formulas without bound class variables. The next definitions serve to define the minimally superinductive class without bound class variables; we then prove the definitions equivalent.

Definition. Given a set x, we call $S(x)$ (or equivalently, S_x) a *superinductive set for* x if

(i) $A \cap P(x) \subseteq S_x \subseteq B \cap P(x)$, where $P(x)$ is the power set of x
(ii) $y \in S_x$ and $g(y) \in B \cap P(x)$ implies $g(y) \in S_x$
(iii) D a chain of S_x and $\bigcup D \in B$ implies $\bigcup D \in S_x$.

Remark. If S is a superinductive class, then $S \cap B \cap \mathrm{P}(x)$ is a superinductive set for x; in particular $B \cap \mathrm{P}(x)$ is a superinductive set for x.

Definition. Given a set x, we define $M(x)$ (or equivalently, M_x), the *minimally superinductive set for x*, as the intersection of all superinductive sets for x.

Remark. Since the S_x are sets ($S_x \subseteq \mathrm{P}(x)$) this definition involves only bound set variables. Also it is clear that M_x is a superinductive set for x.

Definition. M' is the union of all the minimally superinductive sets for x, M_x.

Note. When the definition for M' is written out as a formula of set theory, the definition formula contains only set variables.

We now show M' is the minimally superinductive class. The proof depends on the following three lemmas.

LEMMA 1. *If $x \subseteq y$, then $M_x \subseteq M_y$.*

Proof. We will show $M_y \cap B \cap \mathrm{P}(x)$ is a superinductive set for x, by verifying conditions (i)-(iii) of the definition. This will imply $M_x \subseteq M_y \cap B \cap \mathrm{P}(x)$, *a fortiori*, $M_x \subseteq M_y$.

Suppose $a \in A$ and $a \in \mathrm{P}(x)$, then $a \in \mathrm{P}(y)$ and so $a \in M_y$ (since M_y is a superinductive set for y). Therefore $a \in M_y \cap B \cap \mathrm{P}(x)$ (since $A \subseteq B$), that is, $A \cap \mathrm{P}(x) \subseteq M_y \cap B \cap \mathrm{P}(x)$. Also $M_y \cap B \cap \mathrm{P}(x) \subseteq B \cap \mathrm{P}(x)$. Thus we have verified condition (i).

Suppose $w \in M_y \cap B \cap \mathrm{P}(x)$ and $g(w) \in B \cap \mathrm{P}(x)$. Then $g(w) \in \mathrm{P}(y)$. But $w \in M_y$ and $g(w) \in B \cap \mathrm{P}(y)$ implies $g(w) \in M_y$. Therefore $g(w) \in M_y \cap B \cap \mathrm{P}(x)$ verifying condition (ii).

Assume D is a chain of $M_y \cap B \cap \mathrm{P}(x)$ and $\bigcup D \in B$. Then D is a chain of both $\mathrm{P}(x)$ and M_y, and $\bigcup D \in M_y \cap B \cap \mathrm{P}(x)$, verifying condition (iii) and completing the proof.

LEMMA 2. *If $x \in M_y$, then $x \in M_x$.*

Proof. We can assume $y \supseteq x$; for if not let $y^* = y \cup x$ and then $y^* \supseteq y$ and so, by Lemma 1, $x \in M_{y^*}$, $y^* \supseteq x$. We shall show that there exists a superinductive set for y, S_y, such that $S_y \cap B \cap \mathrm{P}(x) = M_x$ and this will imply $x \in M_x$, since $x \in B \cap \mathrm{P}(x)$ and $x \in S_y$ because $x \in M_y$.

Let $S_y = B \cap \mathrm{P}(y) - (B \cap \mathrm{P}(x) - M_x)$. Then since $B \cap \mathrm{P}(x) \subseteq B \cap \mathrm{P}(y)$, $S_y \cap B \cap \mathrm{P}(x) = M_x$. We now show S_y is superinductive for y. Surely $S_y \subseteq B \cap \mathrm{P}(y)$. Let $a \in A \cap \mathrm{P}(y)$. If $a \in B \cap \mathrm{P}(y)$ and $a \in B \cap \mathrm{P}(x)$, then $a \in M_x$ (since M_x is superinductive for x), that is $a \in B \cap \mathrm{P}(y) - (B \cap \mathrm{P}(x) - M_x)$. Therefore $A \cap \mathrm{P}(y) \subseteq S_y$. Thus S_y satisfies condition (i).

For (ii) let $w \in S_y$ and $g(w) \in B \cap \mathrm{P}(y)$. If $g(w) \in B \cap \mathrm{P}(x)$, then $w \in B \cap \mathrm{P}(x)$, since g is progressing, and hence, by the definition of S_y, $w \in M_x$; thus $g(w) \in M_x$. Therefore $g(w) \in S_y$, as required.

Let D be a chain of S_y and $\bigcup D \in B \cap \mathrm{P}(y)$. Assume $\bigcup D \in B \cap \mathrm{P}(x)$; then $w \in D$ implies $w \in B \cap \mathrm{P}(x)$ which together with $w \in S_y$ implies $w \in M_x$. Thus D is a chain of M_x, and $\bigcup D \in M_x$. Hence $\bigcup D \in S_y$, verifying (iii) and completing the proof.

LEMMA 3. *If* $a \in A$, *then* $a \in M_a$.

Proof. Since M_a is a superinductive set for a and since $a \in A$ and $\{a\} \in \mathrm{P}(a)$, we have $a \in M_a$.

THEOREM 1. *M' is a superinductive class.*

Proof. Surely $M' \subseteq B$. Also, by Lemma 3, $A \subseteq M'$. Thus $A \subseteq M' \subseteq B$. Assume $x \in M'$ and $g(x) \in B$. Then $x \in M_y$, for some y. Let $z = y \cup g(x)$; then $x \in M_z$, by Lemma 1. Since M_z is a superinductive set for z and $g(x) \in B \cap \mathrm{P}(z)$, we have $g(x) \in M_z \subseteq M'$.

Let D be a chain of elements of M' with $\bigcup D \in B$. We shall show $\bigcup D \in M(\bigcup D)$. If $y \in D$, then $y \in M_y$, by Lemma 2, and hence $y \in M(\bigcup D)$, by Lemma 1. Therefore D is a chain of $M(\bigcup D)$ and hence $\bigcup D \in M(\bigcup D) \subseteq M'$.

THEOREM 2. *M' is the minimally superinductive class.*

Proof. Let S be superinductive. Then $x \in M'$ implies $x \in M_y$, for some y, which in turn implies $x \in S \cap B \cap \mathrm{P}(y)$ (since, as we have remarked $S \cap B \cap \mathrm{P}(y)$ is a superinductive set for y). Therefore $x \in S$ and the theorem now follows from Theorem 1.

Remark. The only place we really needed the hypothesis that g is progressing was in the verification of condition (iii) in Lemma 2. It remains an open question whether the hypothesis can be eliminated entirely.

3. *Examples* The class of ordinals can now be defined as follows:

Definition. The *class of ordinals* is the minimally superinductive class under σ, where $\sigma(x) = x \cup \{x\}$.

The various other definitions can now be proved equivalent to the above (see Sion and Wilmot [3]).

We can also define many important classes in Class-Set theory, directly, without resorting to the transfinite recursion theorem. For example, if $\mathrm{S}(x)$ is the set of subsets of x, which are first order definable over x, then the class of constructible sets, is the union of the minimally superinductive class under $g(x) = x \cup \mathrm{S}(x)$. If $g(x) = x \cup \mathrm{P}(x)$, where $\mathrm{P}(x)$ is the power set of x, then the union of the minimally superinductive class under g is just V, the universal class—if we assume the axiom of regularity.

Because of Lemma 2, we can also define the ordinals or the elements of any minimally superinductive class by the formula '$x \in M_x$'. In the case of the constructible sets this condition, when written out in primitive terms, is more easily proved absolute than the usual transfinite recursion condition.

4. *Well-ordering* The most important theorem about minimally superinductive classes under progressing functions, is that they are well-ordered by $\subseteq$. This result is proved in [4] for slowly progressing g. In this section we prove this result for arbitrary progressing functions and the proof does not require the axiom of substitution. Throughout this section let M be a fixed minimally superinductive class under g, progressing.

Definition. An element m of M is called *regular* if there is a chain $C(m)$ (or equivalently, C_m) of elements of M, satisfying:

(i) $m \in C_m$ and if $z \in C_m$ then $z \subseteq m$
(ii) if $x \in C_m$ and $x \neq m$ then $g(x) \in C_m$
(iii) if B is a chain of C_m then $\bigcup B \in C_m$
(iv) if $z \in M$ then either $z \in C_m$ or $z \supset m$.

THEOREM 3. *All elements of M are regular.*

The proof of the theorem consists of the following three lemmas.

LEMMA 1. *The empty set, Λ, is regular.*

Proof. It is easily checked that $C_\Lambda = \{\Lambda\}$ satisfies conditions (i) - (iv).

LEMMA 2. *If y is regular then $g(y)$ is regular.*

Proof. Suppose y is regular and let C_y be a chain for y satisfying (i)-(iv). We will show $C(g(y)) = C_y \cup \{g(y)\}$, which is a chain since g is progressing, satisfies (i)-(iv) also.

(i) Clearly $g(y) \in C(g(y))$. If $z \in C(g(y))$ and $z \neq g(y)$, then $z \in C_y$ and $z \subseteq y \subseteq g(y)$.

(ii) Suppose $x \in C(g(y))$ and $x \neq g(y)$; then $x \in C_y$. If $x \neq y$, then $g(x) \in C_y$, and so $g(y) \in C(g(y))$. If $x = y$, then $g(x) = g(y) \in C(g(y))$.

(iii) Suppose B is a chain of $C(g(y))$. If all elements of B are in C_y, then $\bigcup B \in C_y \subseteq C(g(y))$. If not, $g(y) \in B$ and $\bigcup B = g(y) \in C(g(y))$.

(iv) Suppose $z \in M$ and not $z \supset g(y)$. If $z \in C_y$, then $z \in C(g(y))$. Assume, then, $z \notin C_y$; we shall show $z = g(y) \in C(g(y))$. Let $S = C_y \cup \{w \in M \mid w \supset g(y)\}$. If we can show S is superinductive, we are done, since $z \in S$ (because $z \in M$, minimally superinductive) and $z \notin C_y$, hence $z \supseteq g(y)$ which together with not $z \supset g(y)$, implies $z = g(y)$.

Surely $\Lambda \in S$, since $\Lambda \in C_y$. Assume $w \in S$. If $w \in C_y, g(w) \in C_y$ or $w = y$ and $g(w) = g(y) \in S$. If $w \supseteq g(y)$, then $g(w) \supseteq g(y)$ and $g(w) \in S$.

If D is a chain of S and for some $d \in D, d \supseteq g(y)$ then $\bigcup D \supseteq g(y)$ and $\bigcup D \in S$. If not, all elements of D must be in C_y and $\bigcup D \in C_y \subseteq S$.

Hence S is superinductive and (iv) is proved.

LEMMA 3. *If D is a chain of regular elements of M, then $\bigcup D$ is regular.*

Proof. Let D be a chain of regular elements. For each $d \in D$, let C_d be a chain for d, satisfying (i)-(iv). We shall show $C(\bigcup D) = \bigcup_{d \in D} C_d / \cup \{\bigcup D\}$ is a chain for $\bigcup D$ satisfying (i)-(iv).

We must first show $C(\bigcup D)$ is a chain. We write x comp y for $x \subseteq y$ or $y \subseteq x$. If $x \in C_{d_1}$, $y \in C_{d_2}$, with $d_1 \subset d_2$, then $x \subseteq d_1 \subset d_2$, hence not $x \supset d_2$, so $x \in C_{d_2}$ and since C_{d_2} is a chain, x comp y. If $x \in C_d$ and $y = \bigcup D$, then $x \subseteq d \subseteq \bigcup D = y$. This proves $C(\bigcup D)$ is a chain.

(i) Clearly $\bigcup D \in C(\bigcup D)$. If $z \in C(\bigcup D)$ and $z \neq \bigcup D$, then $z \in C_d$, $d \in D$ and $z \subseteq d \subseteq \bigcup D$.

(ii) Suppose $x \in C(\bigcup D)$ and $x \neq \bigcup D$; then $x \in C_d$, for some $d \in D$. If $x \neq d$, $g(x) \in C_d \subseteq C(\bigcup D)$. If $x = d$, d is not the maximal element of D (for then $x = d = \bigcup D$), so there exists $d' \in D$, with $d' \supset x$. By condition (iv) this implies $x \in C_{d'}$, and $x \neq d'$. Therefore $g(x) \in C_{d'} \subseteq C(\bigcup D)$.

(iii) Let B be a chain of $C(\bigcup D)$. If there is a $b \in B$ with $b = \bigcup D$, then $\bigcup B = \bigcup D \in C(\bigcup D)$. Suppose, therefore, B is a chain of elements of $\bigcup_{d \in D} C_d$. If $\bigcup B \subseteq d$, for some $d \in D$, B is a chain of C_d, and $\bigcup B \in C_d \subseteq C(\bigcup D)$. So assume that for any $d \in D$, there is a $b \in B$, with not $b \subseteq d$. This implies that for any $d \in D$, there is a $b \in B$, with $b \supset d$. For if $b \in C_{d1}$, let $d_2 = \max(d_1, d)$, then $b \subseteq d_1 \subseteq d_2$, so not $b \supset d_2$ and thus $b \in C_{d_2}$; also $d \in C_{d_2}$ and since C_{d_2} is a chain b comp d; since not $b \subset d$, we have, finally, $b \supseteq d$. Therefore $\bigcup D \subseteq \bigcup B$; but $b \in B$ implies $b \in C_d$ and $b \subseteq d$ for some $d \in D$, that is $\bigcup B \subseteq \bigcup D$. Therefore $\bigcup B = \bigcup D \in C(\bigcup D)$.

(iv) Suppose $z \in M$ and not $z \supset \bigcup D$. If for some $d \in D$, not $z \supset d$, then $z \in C_d \subseteq C(\bigcup D)$. If $z \supseteq d$ for all $d \in D$, then $z \supseteq \bigcup D$. Since not $z \supset \bigcup D$, $z = \bigcup D$ and $z \in C(\bigcup D)$. This completes the proof of Lemma 3.

Proof of Theorem 3. Since the regular elements of M form a superinductive subclass S of M by the lemmas, and since M is minimally superinductive $S = M$, that is all elements of M are regular.

COROLLARY 1. *If $x \in M$ and $y \in M$, then x comp y, that is $x \subseteq y$ or $y \subseteq x$.*

Proof. If $x, y \in M$, x is regular, so either $y \in C_x$ and $y \subseteq x$ or $y \supset x$, that is x comp y.

COROLLARY 2. *If $x, y \in M$, then not $x \subset y \subset g(x)$.*

Proof. If $x, y \in M$ and $x \subset y$, then $x \in C_y$ and $x \neq y$. Therefore $g(x) \in C_y$ and hence $g(x) \subseteq y$.

THEOREM 4. *Let M be minimally superinductive under g, progressing. Then M is well-ordered by $\subseteq$.*

Proof. We have already shown that M is linearly ordered by $\subseteq$ (Corollary 1 of Theorem 3).

Let $M' \subseteq M$, $M' \neq \Lambda$. Let $L = \{x \in M \mid y \in M' \to x \subseteq y\}$. Since $M' \neq \Lambda$, $L \neq M$. Therefore L is not superinductive. However $\Lambda \in L$ and also it is easily verified that L is closed under chain unions. Therefore there is an element $x \in L$ such that $g(x) \notin L$. We assert this implies $x \in M'$ and hence x is the least element of M'. For any $y \in M'$ either $y \subset g(x)$ or $g(x) \subseteq y$, since M is linearly ordered by $\subseteq$. Now $g(x) \subseteq y$ cannot hold for all $y \in M'$, since $g(x) \notin L$, so there is an element $y \in M'$ with $y \subset g(x)$, that is, $x \subseteq y \subset g(x)$. But $x \subset y \subset g(x)$ contradicts Corollary 2 of Theorem 3, hence $x = y \in M'$.

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