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LEŚNIEWSKI AND FREGE ON COLLECTIVE CLASSES

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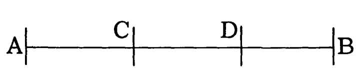
Between 1927 and 1931 Leśniewski published a series of articles on the foundations of mathematics in the Polish journal *Przegląd Filozoficzny*.[1] 65% of the work is devoted to various axiomatizations of Leśniewski's mereology (a theory of collective classes) while the remainder takes up various related issues. In the third part of this series Leśniewski informally sets forth his notion of a collective class, criticizes certain descriptions of distributive classes, and argues that there is no justification in Frege's statement that the conception of a class as consisting of individuals, so that the individual thing coincides with the unit class, cannot in any case be supported.[2]

Leśniewski's refutation of Frege's statement appears to be unknown to western logicians and philosophers. None of the recent books on Frege (e.g., Angelelli, Egidi, Sternfeld, Thiel, Walker) mentions it. Luschei, in his *The Logical Systems of Leśniewski*, mentions it but does not present it. My purpose here is to state and explain Leśniewski's refutation in the hope that it will help stimulate interest in his work. Since Leśniewski bases his refutation on his concept of a collective class, I shall first briefly and informally discuss this concept.

Leśniewski reports that in 1911 he became acquainted with symbolic logic and Russell's antinomy when he came upon Jan Łukasiewicz's *O zasadzie sprzeczności u Arystotelesa* (The Principle of Contradiction in Aristotle).[3] Initially Leśniewski was averse to symbolic logic but Russell's antinomy stimulated him to reflect on those cases in which he actually did consider an object to be (or not to be) a class of objects, and to analyze critically the assumptions of the antinomy from this point of view.[4] Taking the view that if some object is a class of objects a, then some object is an a, he dismissed empty classes as being mythological entities. He held that time and again it occurs that an object is a class of objects a and at the same time it is a class of completely different objects b, and he used the following example to illustrate this point. Consider the following segment:



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The segment AB is a class of segments which are either the segment AC or the segment CB, and at the same time AB is a class of segments which are either the segment AD or the segment DB. Leśniewski also maintained that if one and only one object is P, then P is the class of objects P. E.g., the segment AB is the class of segments AB.

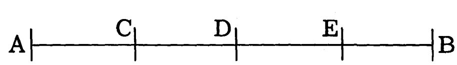
Leśniewski also reports that in analyzing Russell's antinomy he took as his starting point the conception of a class which permits one to assert of every class of objects that it consists of these objects, which are not necessarily discrete. For example, the segment AB, constituting the class of segments, which are segment AD or segment DB, consists of those segments which are the segment AD or the segment DB.

In 1915 Leśniewski formulated his conception of collective classes as a deductive theory, and in 1916 this theory was published in his *Podstawy ogôlnej teoryi mnogości*. I. (Foundations of the General Theory of Sets. I). He held that his theory, compared to Zermelo's, Russell's, and others avoids the antinomies of set theory without restricting the original, Cantorian extension of "set," and that his axiomatization does not lead to theorems which are in such glaring conflict with the intuitions of a "totality" as is the theorem of non-naive set theory which distinguishes an object from the set containing only this object as an element. He also held that he was concerned that his theorems should harmonize with "common sense." In this version of what came to be called "mereology" Leśniewski expressed his theory using Polish rather than a symbolic language, and he took the term "part" as the only primitive term of the theory. For our purposes only the first four axioms and first two definitions need be cited.

Axiom I. *If P is part of the object Q, then Q is not part of the object P.*
Axiom II. *If P is part of the object Q, and Q is part of the object R, then P is part of the object R.*
Definition I. *P is an ingredient of the object Q if and only if P is the same object as Q or is a part of the object Q.*
Definition II. *P is the class of objects a if and only if the following conditions are satisfied:*

a) *P is an object;*
b) *every a is an ingredient of the object P;*[5]
c) *for all Q, if Q is an ingredient of the object P, then some ingredient of the object Q is an ingredient of some a.*

Leśniewski used the following example to illustrate this definition. Consider the following segment:



The segment AB is the class of parts of the segment AB, since all three conditions of the definition are satisfied. The segment AB is *not* the class of parts of the segment AD because while conditions a) and b) are satisfied, c) is not, since the segment EB is an ingredient of the segment AB but it is not the case that some ingredient of the segment EB is an ingredient of some part of the segment AD. The segment AC is *not* the class of ingredients of the segment AB because while conditions a) and c) are satisfied, b) is not, since the segment AB is an ingredient of the segment AB but it is not an ingredient of the segment AC.

Axiom III. *If P is the class of objects a and Q is the class of objects a, then P is Q.*
Axiom IV. *If some object is a, then some object is the class of objects a.*

In 1918 Leśniewski created another axiomatization of mereology which is equivalent to his 1916 system. The 1918 system takes "part" as primitive but differs from the 1916 system in that no axiom contains any defined terms. In 1920 he discovered that another system equivalent to his 1916 system could be obtained by taking "ingredient" as the only primitive term, and in 1921 he discovered still another system equivalent to his previous system and which takes "external" as its only primitive term. For our purposes it is not necessary to consider these later systems. However, it is necessary to say something more about Leśniewski's 1916 system. He expressed the axioms, definitions, and theorems of this and the subsequent systems in Polish rather than in a symbolic language, and I have followed his practice by translating the cited axioms and definitions into English. Following Sobociński,[6] I shall formulate these using Peano-Russell notation. (In these formulations quantifiers are used in the familiar way; "ε", the formal counterpart of "is", is to be construed according to Leśniewski's Ontology; "$\mathsf{pt}(P)$" is a nominal expression meaning "part of P," while "$\sim(\mathsf{pt}(P))$" is to be taken as the negation of "$\mathsf{pt}(P)$," and to be read as "not-part of P"; "$P = Q$" expresses Ontological identity, and is equivalent to "$P \varepsilon Q . Q \varepsilon P$"; "$P \varepsilon \vee$" is to be understood as "P is an object"; "$\mathsf{ing}(Q)$" is a nominal expression to be read as "ingredient of Q"; "$\mathsf{Kl}(a)$" means "class of a's," and "$(EQ)(Q \varepsilon a)$" means "there exists at least one a".)

Axiom I. $(P,Q)(P \varepsilon \mathsf{pt}(Q) . \supset . Q \varepsilon \sim(\mathsf{pt}(P)))$.
Axiom II. $(P,Q,R)(P \varepsilon \mathsf{pt}(Q) . Q \varepsilon \mathsf{pt}(R) . \supset . P \varepsilon \mathsf{pt}(R))$.
Definition I. $(P,Q)(P \varepsilon \mathsf{ing}(Q) . \equiv : P \varepsilon \vee : P = Q .\vee. P \varepsilon \mathsf{pt}(Q))$.
Definition II. $(P,a)(P \varepsilon \mathsf{Kl}(a) . \equiv . P \varepsilon \vee . (EQ)(Q \varepsilon a) . (Q)(Q \varepsilon a . \supset . Q \varepsilon \mathsf{ing}(P)) . (Q)(Q \varepsilon \mathsf{ing}(P) . \supset . (EC)(ED)(C \varepsilon a . D \varepsilon \mathsf{ing}(C) . D \varepsilon \mathsf{ing}(Q))$.
Axiom III. $(P,Q,a)(P \varepsilon \mathsf{Kl}(a) . Q \varepsilon \mathsf{Kl}(a) . \supset . P = Q)$.
Axiom IV. $(A,a)(A \varepsilon a . \supset . (EB)(B \varepsilon \mathsf{Kl}(a)))$.

The expression "ε" occurring above needs some explanation. It is the only primitive term of Leśniewski's Ontology, a calculus of names (as it has been called) based upon his Protothetic, a propositional calculus admitting universal quantifiers binding propositional variables, and variables whose values are truth functions. "ε" is introduced into Ontology *via* a single axiom of Ontology. In the 1920 formulation of Ontology Leśniewski formulated this axiom as follows:

$$(x)(y)(x \varepsilon y . \equiv . (Ec)(c \varepsilon x) . (c)(d)(c \varepsilon x . d \varepsilon x . \supset . c \varepsilon d) . (c)(c \varepsilon x . \supset . c \varepsilon y).$$

In effect, this axiom states that x is y if and only if there is at least and at most one x, and whatever is x is y.[7]

Leśniewski's Mereology is a theory of *collective* classes, and is to be distinguished from the more familiar theories of Russell, Zermelo, *et al*, which formalize the notion of a *distributive* class. Collective classes differ from distributive classes in several respects. In the collective sense of "class" there is no empty class, while in the distributive sense there is. If we consider the class of the United States, then in the distributive sense of "class" California, Arizona, and Massachusetts are elements of this class, but San Francisco is not. In the collective sense, however, not only are the several states elements of this class but so are San Francisco, Long Island, and Cook County. In the collective sense of "class" elementhood is transitive, i.e., if x is an element of y, and y is an element of z, then x is an element of z. In the distributive sense, elementhood is not a transitive relation. In the collective sense of "class", but not in the distributive sense, if some class is a unit class, then it is the same object as its only element.

Leśniewski maintained that his conception of a class as being a collective totality which literally consists of its elements was consistent with the common usage of "class" in the ordinary language of those who have never been concerned with any theory of classes. And he further claimed that his conception was also consistent with Cantor's characterization of the relation holding between a class of objects and the objects constituting that class: Jede Menge wohlunterschiedener Dinge kann als ein einheitliches Ding für sich angesehen werden, in welchem jene Dinge Bestandteile oder constitutive Elemente sind.[8] If a class consists of those objects of which it is a class, then (according to Leśniewski) there cannot be an empty class; if some object is the class of objects a, then some object is a, i.e., $(P,a)(P \; \varepsilon \; \mathsf{Kl}(a) \supset (\mathrm{E}Q)(Q \; \varepsilon \; a))$, which is a consequence of Definition II above.[9] He objected vigorously against those (e.g., Fraenkel, Hausdorff, Sierpiński) who maintain both that a class of objects consists of these objects and that for some purposes it is necessary to introduce or invent the empty class, which consists of no objects, and he believed that this objection was supported by some of Frege's criticisms of Schröder and Dedekind. Frege maintained that according to both Schröder and Dedekind the elements of a class are the proper constituents of that class: ... nach Dedekind die Elemente den eigentlichen Bestand des Systemes ausmachen ... denn auch Schröder sieht im Grunde die Elemente als das an, was seine Klasse ausmacht.[10] Hence there is no empty class. Yet there is in both a felt need for an empty class, and (according to Frege) each invents such a class. But such an invention cannot be tolerated, since if the elements constitute a class, then where the elements are abolished so is the class.[11] Clearly both Frege and Leśniewski thought it inconsistent to assert that a class consists of its elements and that there is a class which consists of no elements. It is to be conjectured whether the source of this purported inconsistency is found in confusing collective classes with distributive classes. Collective classes are literally constituted by their elements, while distributive classes could not possibly be so constituted.

Frege's objections to those who both accept the existence of empty classes and also assert that classes consist of their elements, clearly, do not apply to Leśniewski's conception of a collective class, since he takes the position that if an object is a class of a's, then it consists of these a's. As indicated earlier, Leśniewski repudiated the existence of those (as he called them) "theoretical monsters" such as the class of square circles, since, as he maintained, he was well aware that nothing can consist of what does not exist. He believed that his repudiation of empty classes was supported by Frege's remark: If ... a class consists (besteht) of objects, is an aggregate (Sammlung) or collective unity of them, then it must vanish when these objects vanish. If we burn down all the trees of a forest, we thereby burn down the forest. Consequently, there cannot be an empty class.[12]

Leśniewski said that according to his conception of a class

(1) *If one and only one object is P, then P is the class of objects P.*

Leśniewski did not express (1) symbolically, but we can using some expressions from Ontology and some introduced above.[13] "$\mathsf{ex}\{P\}$" will be used to express "*at least one P exists*," and "$\rightharpoonup\{P\}$" to express "*at most one P exists*." (1) may then be expressed as

(1') $(P)(\mathsf{ex}\{P\} . \rightharpoonup \{P\} . \supset . P \; \varepsilon \; \mathsf{Kl}(P))$.

Now in the first formulation of mereology theorem VIII states

(a) *If P is an object, then P is the class of objects P,*

i.e.,

(a') $(P)(P \; \varepsilon \; \mathsf{V} . \supset . P \; \varepsilon \; \mathsf{Kl}(P))$.

In Ontology we have as a theorem that all and only individuals are objects, i.e.,

(b) $(x)(x \; \varepsilon \; \mathsf{V} . \equiv . \mathsf{ex}\{x\} . \rightharpoonup \{x\})$.

Thus, we can show that on the basis of (a) and (b), (1) is a theorem of the 1916 system of mereology. From (1) Leśniewski derived

(A) *If one and only one object is an element of the class K, then the element of the class K is the class of elements of the class K,*

i.e.,

(A') $\mathsf{ex}\{\mathsf{el}(\mathsf{Kl}(K))\} . \rightharpoonup \{\mathsf{el}(\mathsf{Kl}(K))\} . \supset . \mathsf{el}(\mathsf{Kl}(K)) \; \varepsilon \; \mathsf{Kl}(\mathsf{el}(\mathsf{Kl}(K)))$.

He maintained that

(B) *If X is the class of elements of the class K, then the class K is the same object as X,*

which we may express as

(B') $(X)(X \; \varepsilon \; \mathsf{Kl}(\mathsf{el}(\mathsf{Kl}(K))) . \supset . \mathsf{Kl}(K) = X)$.

From (A) and (B) he inferred

(C) *If one and only one object is an element of the class K, then the class K is the same object as the element of the class K,*

i.e.,

(C') $\mathsf{ex}\{\mathsf{el}(\mathsf{Kl}(K))\} . \multimap \{\mathsf{el}(\mathsf{Kl}(K))\} . \supset . \mathsf{Kl}(K) = \mathsf{el}(\mathsf{Kl}(K))$.

Using the expression "unit class" pursuant to the statement

(D) *K is a unit class if and only if one and only one object is an element of the class K,*

which we may express as

(D') $K \, \varepsilon \, \mathsf{UKl} . \equiv . \mathsf{ex}\{\mathsf{el}(\mathsf{Kl}(K))\} . \multimap \{\mathsf{el}(\mathsf{Kl}(K))\}$,

Leśniewski asserted, on the basis of (D) and (C),

(E) *If a class is a unit class, then it is the same object as its only element,*

i.e.,

(E') $K \, \varepsilon \, \mathsf{UKl} . \supset . \mathsf{Kl}(K) = \mathsf{el}(\mathsf{Kl}(K))$.

Leśniewski believed that the possibility of obtaining (E) from his conception of a class was completely consistent with Frege's assertion "Now our assumption that unit classes coincide with individuals is a necessary consequence of the conception that classes consist of individuals,"[14] provided that Frege's phrase "unit classes coincide with individuals" (which appears to be equivalent to Frege's phrase "a class which consists of only one object coincides with this object") may be expressed, as Leśniewski assumed it could, as (E) above.

While accepting (E), Leśniewski emphatically rejected the view according to which

(E*) *Every object is a class whose sole element is this object.*

Consider the segment AB of the first diagram above. According to Leśniewski's conception of a collective class, the segment AC is an element of the segment AB, which is the class of segments which are either the segment AC or the segment CB. This, he maintained, entitles him to assert that although one and only one object is the segment AB, and although pursuant to this the segment AB is also the class of segments AB (see (1) above), the segment AB is *not* a class whose only element is the segment AB. Leśniewski maintained that independently of determining whether there are unit classes, if some class is a unit class (e.g., a class whose only element is some "point" which is indivisible either spatially or temporally), then it is the same object as its only element; and he also held that (E*) is false, which, he said, seemed to put him in conflict with the position espoused by Frege. According to Frege:

(a) "The doubt whether each individual may be regarded as the class that consists of it alone is made stronger by the following consideration. In the discussion set forth above we may take P to be itself likewise a class comprising a number of individuals; for, as the author says (p. 148), such a class can be presented as an object of thought and consequently as an individual."

(b) "Now if Q, as before, is the class of objects that coincide with P, then Q is a unit class (eine singuläre Klasse) containing only P as an individual."

(c) "Now if it were right to hold that a unit class coincides with the only individual it contains, then Q would coincide with P. Let us suppose that a and b are different objects, contained within P as individuals; then they would also be contained within Q; i.e., both a and b would coincide with P. Consequently a would also coincide with b, contrary to our permissible supposition that they are different."[15]

Thus Frege argued against the supposition

(1) *Each individual may be regarded as the class that consists of it alone by appealing to the contradiction entailed by the supposition.*

(2) *A unit class coincides with the only element it contains.*

Leśniewski maintained that on the basis of his conception of classes these suppositions are distinct, and that Frege (in his article on Schröder) did not distinguish one from the other. In Leśniewski's terminology (1) may be expressed as

(1') *Every object may be regarded as a class whose sole element is this object.*

This is simply a variant of (E*), which is inconsistent with Leśniewski's conception of a class. The second supposition, in Leśniewski's terminology, is (E), which *is* consistent with his conception of a class. Frege's argument in (c) does not affect the second supposition (understood as Leśniewski's (E)), since (b), which permits Frege to say in (c) that in the case indicated "both a and b would coincide with P," does not hold in Leśniewski's conception of a class, and hence may be rejected by him. For consider the segment AB again. AB, being the only object which is the same object as segment AB, is (according to Leśniewski's thesis that if one and only one object is P, then P is the class of objects P) the class of objects which are the same object as segment AB. Yet AB is not a unit class (see (D) above), since (in virtue of Leśniewski's earlier assertions "*If P is a class, then P is an element of the class P*," and "*the segment* AC *is an element of the class* AB") AC and AB are elements of AB.

Thus, according to Leśniewski's conception of a collective class, there is no justification in Frege's peremptory remark: ... the conception of a class as consisting of individuals, so that the individual thing coincides with the unit class, cannot in any case be supported. On the basis of Leśniewski's conception of collective classes, to say that a class consists of individuals does not entail that the single thing coincides with a unit class.

NOTES

1. *Przegląd Filozoficzny,* XXX (1927), pp. 164-206; XXXI (1928), pp. 261-291; XXXII (1929), pp. 60-101; XXXIII (1930), pp. 77-105; XXXIV (1931), pp. 142-170.

2. "Kritische Beleuchtung einiger Punkte in E. Schröders *Vorlesungen über die Algebra der Logik,*" *Archiv für systematische Philosophie,* I (1895), p. 445.

3. *Przegląd Filozoficzny,* XXX (1927), p. 169.

4. Leśniewski presented a critical analysis of Russell's antinomy in his paper "Czy klasa klas, nie podporządkowanych sobie, jest podporządkowana sobie?" (Is the class of classes which are not elements of themselves an element of itself?), *Przegląd Filozoficzny,* XVII (1914), pp. 63-75. The second chapter of his series of articles on the foundations of mathematics is also devoted to the antinomy. His apparently final analysis of the antinomy is recounted in Bolesław Sobociński's "L'analyse de l'antinomie russellienne par Leśniewski," *Methodos,* I (1949), pp. 94-107, 220-228, 308-316, and II (1950), pp. 237-257.

5. According to Leśniewski, a sentence of the form "every a is b" is equivalent to "some object is a, and for all X, if X is a, then X is b," and it is *not* equivalent to "for all X, if X is a, then X is b." (Translation of Polish texts are by V.F.S.)

6. Bolesław Sobociński, "Studies in Leśniewski's Mereology," *Rocznik Polskiego Towarzystwa Naukowego na Obczyźnie* (Yearbook of the Polish Society of Arts and Sciences Abroad), rok 1954-55 (published 1955), pp. 34-43.

7. For accessible details on Ontology see Czesław Lejewski, "On Leśniewski's Ontology," *Ratio,* I (1958), pp. 150-176; Jerzy Słupecki, "St. Leśniewski's Calculus of Names," *Studia Logica,* III (1955), pp. 7-71; Bolesław Sobociński, "Successive Simplifications of the Axiom-System of Leśniewski's Ontology," in *Polish Logic 1920-1939* (edited by Storrs McCall), Oxford, 1967, pp. 188-200, and my paper "Nominalism and Common Names," *The Philosophical Review,* LXXI (1962), pp. 230-235.

8. Georg Cantor, "Mitteilungen zur Lehre vom Transfiniten," *Zeitschrift für Philosophie und philosophische Kritik,* N.F. vol. 91 (1887), p. 83.

9. It is of some interest to recall G. E. Moore's remark: With the ordinary meaning of "class" it is impossible that any class should have only *one* member or none. *The Commonplace Book* of *G. E. Moore, 1919-1953,* edited by Casimir Lewy, London, 1962, p. 14. Moore makes essentially the same comment in his *Lectures on Philosophy,* edited by Casimir Lewy, London, 1966, p. 126.

10. Gottlob Frege, *Grundgesetze der Arithmetik,* vol. 1, Jena, 1893, p. 2.

11. *Ibid.,* p. 3.

12. "Kritische Beleuchtung einiger Punkte . . . ", pp. 436-437.

13. The notation for Ontology is here based on that of Słupecki, *op. cit.*

14. "Kritische Beleuchtung einiger Punkte . . . ", p. 445.

15. *Ibid.,* pp. 444-445. This translation is by Peter Geach, *Translations from the Philosophical Writings of Gottlob Frege,* edited by Peter Geach and Max Black, Oxford, 1966, p. 96, except that I use "unit class" in place of his "singular class."

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