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ON THE CATEGORY OF COFINITE MODULES FOR PRINCIPAL IDEALS

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Abstract. In this paper, it is pointed out that $\mathcal{M}(A, I)_{cof}$ is an Abelian full subcategory of the category $\mathcal{M}(A)$ consisting of all A-modules for a principal ideal I over a noetherian ring A.

1. Introduction

We assume that all rings are commutative and noetherian with identity throughout this paper.

In the paper [3, §2, p. 147], the four questions were proposed over a regular ring R. In particular the following are given:

Question 1 (Second Question). Let $\mathcal{M}(R, J)_{cof}$ be the collection of all the R-modules N satisfying the condition

$$(*) \qquad \mathrm{Supp}_R(N) \subseteq V(J) \quad \text{and}$$
$$\mathrm{Ext}^j_R(R/J, N) \quad \text{is of finite type, for all } j,$$

where J is an ideal of R. Then does $\mathcal{M}(R, J)_{cof}$ form an Abelian subcategory of $\mathcal{M}(R)$? Here we denote by $\mathcal{M}(R)$ the category of all R-modules.

In this note, we call the object of $\mathcal{M}(R, J)_{cof}$ J-cofinite.

Question 2 (Fourth Question). Does there exist an Abelian category $\mathcal{M}_{cof}$ consisting of R-modules, such that elements $N^\bullet \in \mathcal{D}(R, J)_{cof}$ are characterized by the property "$H^i(N^\bullet) \in \mathcal{M}_{cof}$" for all i? Here we denote by $\mathcal{D}(R, J)_{cof}$ the essential image of $\mathcal{D}_{ft}(R)$ by the J-dualizing functor (See [3, p. 149, line 3] for the definition).

In [3, §3 An Example, p. 149], Question 1 and Question 2 are answered negatively for an ideal generated by two elements. The example is as follows: Let R be the formal power series ring $k[x, y][[u, v]]$ over a polynomial ring $k[x, y]$ and J the ideal

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(u, v) of R, where k is a field. Let M be the R-module $R/(xv+yu)$. Then it is proved that the local cohomology module $H^2_J(M)$ is not J-cofinite in [3, §3 An Example]. Even the socle $\mathrm{Hom}_R(k, H^2_J(M))$ is not of finite dimension as a k-vector space. The ideal J is generated by the two elements u, v, and there is an exact sequence:

$$0 \longrightarrow H^1_J(M) \longrightarrow H^2_J(R) \longrightarrow H^2_J(R) \longrightarrow H^2_J(M) \longrightarrow 0.$$

Since J is generated by a regular sequence u, v over R, the local cohomology module $H^2_J(R)$ is J-cofinite. If Question 1 is affirmatively answered for the ideal J, then the local cohomology module $H^2_J(M)$ must be J-cofinite, which is a contradiction. Further if Question 2 is affirmatively answered for the ideal J, then $\mathrm{Hom}_R(R/J, H^2_J(M))$ must be of finite type by the local duality theorem (cf. [2, Theorem 2.1, p. 148]) and the characterization of cofinite complexes (cf. [3, Thorem 5.1, p. 154]), which is also a contradiction.

In this paper, it is proved that $\mathcal{M}(A, I)_{cof}$ is Abelian, provided that I is principal up to radical over a noetherian (not necessarily local) ring. One can find the result in [3, Proposition 6.1, p. 158] for an ideal generated by a single *non-zero divisor* over a *regular* ring of finite Krull dimension. Further one can also find that in [6, Proposition 4, p. 605] for an ideal generated by a single element over a *local* ring (See [5, Thorem 1] also for a result for principal ideals). As related topics, several results have been obtained on $\mathcal{M}(A, I)_{cof}$ for an ideal I of dimension one of a local ring A (cf. [2, Theorem 2, p. 49] and [6, Theorem 1, Theorem 2]).

2. A result on principal ideals over rings

Now it is proved that $\mathcal{M}(A, I)_{cof}$ is an Abelian full subcategory of $\mathcal{M}(A)$, provided that I is principal up to radical. We state that as a theorem below.

Theorem 2.1. *Let A be a noetherian ring, and I an ideal of A. If I is an ideal generated by a single element of A up to radical, then $\mathcal{M}(A, I)_{cof}$ is an Abelian full subcategory of $\mathcal{M}(A)$.*

Proof. We may assume that I is a radical ideal by [4, Lemma 4·1, p. 426]. If I is a unit ideal, then we have nothing to prove. From now on, we suppose that I is an ideal generated by a non-unit element x of A by assumption, for it holds that $V(I) = V(\sqrt{I}) = V(\sqrt{x}) = V(x)$.

Let $\varphi : M \to N$ be an arbitrary A-module homomorphism between I-cofinite modules M and N, where $I = (x)$. Consider an ideal $(0 :_A x^m)$ of A for an integer $m \in \mathbb{N}$, so the series of ideals $(0 :_A x), (0 :_A x^2), (0 :_A x^3), \dots$ forms the ascending chain. Since A is noetherian, the chain is stationary. Let n be an integer such that $(0 :_A x^n) = (0 :_A x^{n+1}) = \cdots$. Consider submodules $(0 :_M x^n)$ and $(0 :_N x^n)$ of M and N respectively, so the two submodules are of finite type over A, since

$(0:_M x^n) = \mathrm{Hom}_R(A/(x^n), M) = \mathrm{Hom}_A(A/I^n, M)$ and $\mathrm{Ext}^j_A(A/I^n, M)$ is of finite type over A for all $j \geq 0$ by the assumption of M. And the same assertion also holds for the module N. Set $\bar{A} = A/(0:_A x^n)$, $\bar{M} = M/(0:_M x^n)$ and $\bar{N} = N/(0:_N x^n)$. Then it is easy to see that x is a regular element on $\bar{A}$. Consider the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & (0:_M x^n) & \to & M & \to & \bar{M} & \to & 0 \\ & & {\scriptstyle\varphi_0}\downarrow & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\bar{\varphi}} & & \\ 0 & \to & (0:_N x^n) & \to & N & \to & \bar{N} & \to & 0, \end{array}$$

so we find that the kernel and the cokernel of φ are I-cofinite if and only if those of $\bar{\varphi}$ are I-cofinite by Snake Lemma.

Replacing $\bar{A}$, $\bar{M}$, $\bar{N}$ and $\bar{\varphi}$ with A, M, N and φ respectively*, we may assume that x is a regular element of A. Denote the image of φ by $\mathrm{Im}\varphi$, so there are short exact sequences

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{Ker}\varphi & \to & M & \to & \mathrm{Im}\varphi & \to & 0, \\ 0 & \to & \mathrm{Im}\varphi & \to & N & \to & \mathrm{Coker}\varphi & \to & 0, \end{array}$$

where $\mathrm{Ker}\varphi$ and $\mathrm{Coker}\varphi$ are the kernel and cokernel of φ respectively. Since x is a non-zero divisor on A, we can consider the following short exact sequence:

$$0 \to A \stackrel{x}{\to} A \to A/I \to 0,$$

which gives the projective resolution of A/I. So we have long exact sequences

$$\begin{array}{llll} 0 & \to \mathrm{Hom}_A(A/I, \mathrm{Ker}\varphi) & \to \mathrm{Hom}_A(A/I, M) & \to \mathrm{Hom}_A(A/I, \mathrm{Im}\varphi) \\ & \to \mathrm{Ext}^1_A(A/I, \mathrm{Ker}\varphi) & \to \mathrm{Ext}^1_A(A/I, M) & \to \mathrm{Ext}^1_A(A/I, \mathrm{Im}\varphi) \to 0, \end{array}$$

and

$$\begin{array}{llll} 0 & \to \mathrm{Hom}_A(A/I, \mathrm{Im}\varphi) & \to \mathrm{Hom}_A(A/I, N) & \to \mathrm{Hom}_A(A/I, \mathrm{Coker}\varphi) \\ & \to \mathrm{Ext}^1_A(A/I, \mathrm{Im}\varphi) & \to \mathrm{Ext}^1_A(A/I, N) & \to \mathrm{Ext}^1_A(A/I, \mathrm{Coker}\varphi) \to 0. \end{array}$$

Since M and N are I-cofinite by assumption, $\mathrm{Ext}^j_A(A/I, M)$ and $\mathrm{Ext}^j_A(A/I, N)$ are of finite type for all j. It follows from the above exact sequence that $\mathrm{Ext}^j_A(A/I, \mathrm{Ker}\varphi)$ and $\mathrm{Ext}^j_A(A/I, \mathrm{Coker}\varphi)$ are of finite type for $j = 0, 1$. Now the projective dimension of A/I is one. So the modules $\mathrm{Ext}^j_A(A/I, \mathrm{Ker}\varphi)$ and $\mathrm{Ext}^j_A(A/I, \mathrm{Coker}\varphi)$ are of finite type for all j, namely $\mathrm{Ker}\varphi$ and $\mathrm{Coker}\varphi$ are I-cofinite, that is $\mathrm{Ker}\varphi$ and $\mathrm{Coker}\varphi$ are in $\mathcal{M}(A, I)_{cof}$. Therefore the category $\mathcal{M}(A, I)_{cof}$ is Abelian. □

Remark 2.2. Let M be a non-zero module in $\mathcal{M}(A, I)_{cof}$. If x is not a unit and $\sqrt{I} = \sqrt{(x)}$, then x^n is a zero divisor on M for some n, since $\mathrm{Supp}M$ is contained in $V(x)$. Further one can see that $\Gamma_I(M) = M$ for an arbitrary ideal I of a noetherian ring A.

Furthermore the following holds:

Proposition 2.3. *Let R be a unique factorization domain, and J an ideal of pure height one. Then $\mathcal{M}(R, J)_{cof}$ is an Abelian full subcategory of $\mathcal{M}(R)$.*

Proof. It is well-known that all the prime ideal of height one is principal in a unique factorization domain. So the ideal J is principal up to radical, since J is of pure height one, that is all the minimal prime ideals of J have the same height one. Therefore the category $\mathcal{M}(R, J)_{cof}$ is Abelian by Theorem 2.1. □

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***Note added in proof**. Consider the following spectral sequence (cf. [7, Theorem 11.65, p. 364]):

$$E_2^{p,q} = \mathrm{Ext}^p_{\bar{A}}(\mathrm{Tor}^A_q(\bar{A}, A/I), T) \underset{p}{\Longrightarrow} H^{p+q} = \mathrm{Ext}^{p+q}_A(A/I, T),$$

which is in the first quadrant. Here $A \to \bar{A}$ is the natural ring homomorphism, I is an ideal of A generated by x, and T is an $\bar{A}$-module $\mathrm{Ker}\,\bar{\varphi}$, which is recognized to be an A-module via the ring homomorphism $A \to \bar{A}$. If $E_2^{p,q}$ is finitely generated for all $p \geq 0, q \geq 0$, then H^n is finitely generated for all $n \geq 0$ (cf. [1, Lemma 3]).

If $E_2^{p,0} = \mathrm{Ext}^p_{\bar{A}}(\bar{A} \otimes_A A/I, T) = \mathrm{Ext}^p_{\bar{A}}(\bar{A}/I\bar{A}, T)$ is finitely generated for all $p \geq 0$, then $E_2^{p,q}$ is finitely generated for all $p \geq 0, q \geq 0$ by the lemma due to Huneke and Koh [4, Lemma 4·1, p. 426]. The element x is a regular element of $\bar{A}$. So we may assume that x is a regular element of A, replacing $\bar{A}$, $\bar{M}$, $\bar{N}$ and $\bar{\varphi}$ with A, M, N and φ, respectively.

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