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DIRICHLET SERIES IN THE THEORY OF SIEGEL MODULAR FORMS

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We are concerned with Dirichlet series which appear in the Fourier expansion of the non-analytic Eisenstein series on the Siegel upper half space H_m of degree m. In the case of $m = 2$ Kaufhold [1] evaluated them. Here we treat the general cases by a different method.

For a rational matrix R we denote the product of denominators of elementary divisiors of R by $\nu(R)$. For a half-integral symmetric matrix $T^{(n)}$ we put

$$b(s, T) = \sum \nu(R)^{-s} e(\sigma(TR)),$$

where R runs over $n \times n$ rational symmetric matrices modulo 1 and σ means the trace, and $e(z)$ is $\exp(2\pi iz)$. If $\operatorname{Re} s > n + 1$, then $b(s, T)$ is absolutely convergent. For a rational symmetric matrix R there is a unique decomposition $R \equiv \sum R_p \bmod 1$ where R_p is a rational symmetric matrix such that $\nu(R_p)$ is a power of prime p. Therefore we have a decomposition

$$b(s, T) = \prod b_p(s, T),$$
$$b_p(s, T) = \sum \nu(R)^{-s} e(\sigma(TR)).$$

where R runs over rational symmetric matrices modulo 1 such that $\nu(R)$ is a power of prime p. Our aim is to give $b_p(s, T)$ in a form easy to see. Shimura [7] also treats $b_p(s, T)$ in a more general situation. $b_p(s, T)$ here is a special case α_0, Case SP in [7]. His results about α_0 are weaker than ours.

Generalized confluent hypergeometric functions in the Fourier expansion of the non-analytic Eisenstein series are investigated by Shimura [6].

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THEOREM 1. *Let* $T_1^{(n-1)}$ *be a half-integral symmetric matrix and* $T^{(n)} = \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix}$. *Then we have*

$$b_p(s, T) = (1 - p^{-s})(1 + p^{1-s})(1 - p^{n+1-2s})^{-1} b_p(s - 1, T_1).$$

We prepare some lemmas to prove this theorem. Put $C(k; p) = \{C \in M_k(Z) \mid |C| \text{ is a power of } p\}$ and

$$\Lambda_k = \{S \in M_k(Z) \mid {}^tS = S\}.$$

The following lemma is known ([1], [5]).

LEMMA 1. $b_p(s, T) = \sum |C|^{-s} e(\sigma(TC^{-1}D))$,

where C, D *run over* $SL_n(Z)\backslash C(n; p)$, $\{D \in M_n(Z) \mid C^{-1}D = {}^t(C^{-1}D)$ *and* (C, D) *is primitive*$\}$ mod $C\Lambda_n$ *respectively.*

$$\prod_{k=0}^{n-1} (1 - p^{k-s})^{-1} b_p(s, T) = \sum |C|^{-s} e(\sigma(TC^{-1}D)),$$

where C, D *run over* $SL_n(Z)\backslash C(n; p)$, $\{D \in M_n(Z) \mid C^{-1}D = {}^t(C^{-1}D)\}$ mod $C\Lambda_n$ *respectively.*

The next lemma is easy.

LEMMA 2. *As representatives of* $SL_n(Z)\backslash C(n; p)$ *we can choose*

$$C = \begin{pmatrix} C_1^{(n-1)} & 0 \\ C_3 & C_4 \end{pmatrix},$$

where C_1, C_4 *and* C_3 *run oner* $SL_{n-1}(Z)\backslash C(n-1; p)$, $C(1; p)$ *and* $M_{1,n-1}(Z)$ mod $M_{1,n-1}(Z)C_1$ *respectively.*

LEMMA 3. *For* $C = \begin{pmatrix} C_1^{(n-1)} & 0 \\ C_3 & C_4 \end{pmatrix} \in C(n; p)$ *we can choose as representatives of* $\{D \in M_n(Z) \mid C^{-1}D = {}^t(C^{-1}D)\}$ mod $C\Lambda_n$

$$D = \begin{pmatrix} D_1^{(n-1)} & D_2 \\ D_3 & D_4 \end{pmatrix},$$

where D_1, D_2 *and* D_4 *run over* $\{D_1 \in M_{n-1}(Z) \mid C_1^{-1}D_1 = {}^t(C_1^{-1}D_1)\}$ mod $C_1\Lambda_{n-1}$, $\{D_2 \in M_{n-1,1}(Z) \mid C_4{}^tD_2 + C_3{}^tD_1 \in M_{1,n-1}(Z){}^tC_1\}$ mod $C_1M_{n-1,1}(Z)$ *and* Z mod C_4 *respectively and then* $D_3 = (C_4{}^tD_2 + C_3{}^tD_1){}^tC_1^{-1}$.

Proof. Since $C^{-1} = \begin{pmatrix} C_1^{-1} & 0 \\ -C_4^{-1}C_3C_1^{-1} & C_4^{-1} \end{pmatrix}$, we have

$$C^{-1}D = \begin{pmatrix} C_1^{-1}D_1 & C_1^{-1}D_2 \\ -C_4^{-1}(C_3C_1^{-1}D_1 - D_3) & -C_4^{-1}(C_3C_1^{-1}D_2 - D_4) \end{pmatrix}.$$

Since $C^{-1}D$ is symmetric, $C_1^{-1}D_1$ is symmetric and $D_3 = (C_4{}^tD_2 + C_3{}^tD_1){}^tC_1^{-1}$. For an integral symmetric matrix $S = \begin{pmatrix} S_1^{(n-1)} & S_2 \\ {}^tS_2 & S_4 \end{pmatrix}$,

$$CS = \begin{pmatrix} C_1S_1 & C_1S_2 \\ C_3S_1 + C_4{}^tS_2 & C_3S_2 + C_4S_4 \end{pmatrix} \quad \text{holds}.$$

From these follows easily our lemma.

The next lemma is an immediate corollary.

LEMMA 4. *Let $C_1 \in C(n-1; p)$, $D_1 \in M_{n-1}(Z)$ and $C_4 \in C(1; p)$. Denote by $x(C_1, D_1, C_4)$ the number of elements of the set*

$$\{D_2 \in M_{n-1,1}(Z) \bmod C_1M_{n-1,1}(Z),\ C_3 \in M_{1,n-1}(Z) \bmod M_{1,n-1}(Z)C_1$$
$$\text{such that } C_4{}^tD_2 + C_3{}^tD_1 \in M_{1,n-1}(Z){}^tC_1\}.$$

Then the number of $C = \begin{pmatrix} C_1 & 0 \\ C_3 & C_4 \end{pmatrix}$, $D = \begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix}$ where C_3, D run over $M_{1,n-1}(Z) \bmod M_{1,n-1}(Z)C_1$, $\{D \in M_n(Z) \bmod C\Lambda_n \mid C^{-1}D = {}^t(C^{-1}D)\}$ respectively is $C_4x(C_1, D_1, C_4)$.

LEMMA 5. *Let R be a rational symmetric matrix and $C_i^{-1}D_i = R$ for $C_i, D_i \in M_n(Z)$ $(i = 1, 2)$. If (C_1, D_1) is primitive then $(C_2, D_2) = W(C_1, D_1)$ for some $W \in M_n(Z)$.*

Proof. This is well known [5]).

LEMMA 6. *Let $W \in C(n-1; p)$, $C_4 \in C(1; p)$, $C_1 \in C(n-1; p)$ and $D_1 \in M_{n-1}(Z)$ such that $C_1^{-1}D_1$ is symmetric and (C_1, D_1) is primitive. Then we have*

$$x(WC_1, WD_1, C_4) = |WC_1| \prod_{i=1}^{n-1} (C_4, w_i),$$

where $\{w_i\}$ is the set of elementary divisors of W.

Proof. Let $A_1, B_1 \in M_{n-1}(Z)$ such that $\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \in Sp_{n-1}(Z)$. Suppose $z{}^tD_1 = w{}^tC_1$ for $z, w \in M_{1,n-1}(Z)$; then $z = z({}^tD_1A_1 - {}^tB_1C_1) = w{}^tC_1A_1 -$

$z^tB_1C_1 = (w^tA_1 - z^tB_1)C_1 \in M_{1,n-1}(Z)C_1$. Conversely, suppose $z = xC_1$ for $z, x \in M_{1,n-1}(Z)$; then $z^tD_1 = xC_1{}^tD_1 = xD_1{}^tC_1 \in M_{1,n-1}(Z)^tC_1$. Thus we have proved that for $z \in M_{1,n-1}(Z)$

$$z^tD_1 \in M_{1,n-1}(Z)^tC_1 \qquad \text{iff } z \in M_{1,n-1}(Z)C_1 .$$

Next we show that for $D_2 \in M_{n-1,1}(Z)$ there exists $C_3 \in M_{1,n-1}(Z)$ such that $C_4{}^tD_2 + C_3{}^t(WD_1) \in M_{1,n-1}(Z)^t(WC_1)$ iff $C_4{}^tD_2{}^tW^{-1} \in M_{1,n-1}(Z)$. The "only if" part is trivial. Suppose $C_4{}^tD_2{}^tW^{-1} = y \in M_{1,n-1}(Z)$; then $y - yA_1{}^tD_1 = -yB_1{}^tC_1$ implies $C_4{}^tD_2 + (-yA)^t(WD_1) = -yB_1{}^t(WC_1) \in M_{1,n-1}(Z)^t(WC_1)$. Hence we can take $-yA_1$ as C_3.

Lastly suppose that $D_2 \in M_{n-1,1}(Z)$, $C_{3,i} \in M_{1,n-1}(Z)$ satisfy

$$C_4{}^tD_2 + C_{3,i}{}^t(WD_1) \in M_{1,n-1}(Z)^t(WC_1) \qquad (i = 1, 2) ,$$

then $(C_{3,1} - C_{3,2})^tD_1 \in M_{1,n-1}(Z)^tC_1$ and then $C_{3,1} - C_{3,2} \in M_{1,n-1}(Z)C_1$. Therefore

$$\begin{aligned} &x(WC_1, WD_1, C_4) \\ &\quad = |W| \# \{D_2 \in M_{n-1,1}(Z) \bmod WC_1M_{n-1,1}(Z) \mid C_4{}^tD_2{}^tW^{-1} \in M_{1,n-1}(Z)\} . \end{aligned}$$

Let $W = UW_0V$ where

$$U,\ V \in GL_n(Z), \quad W_0 = \begin{bmatrix} w_1 & & \\ & \ddots & \\ & & w_{n-1} \end{bmatrix} \quad \text{and} \quad \text{put } {}^tD_2{}^tU^{-1} = (y_1, \cdots, y_{n-1}) .$$

${}^tD_2{}^tW^{-1} = (\cdots, y_i/w_i, \cdots)^tV^{-1}$ implies

$$\begin{aligned} &x(WC_1, WD_1, C_4) \\ &\quad = |W| \# \{(y_1, \cdots, y_{n-1}) \in M_{1,n-1}(Z) \bmod M_{1,n-1}(Z)^tC_1{}^tVW_0 \mid C_4y_i \equiv 0 \bmod w_i\} \\ &\quad = |W| [M_{1,n-1}(Z) \colon M_{1,n-1}(Z)^tC_1{}^tVW_0] / \\ &\qquad\qquad [M_{1,n-1}(Z) \colon \{(y_1, \cdots, y_{n-1}) \in M_{1,n-1}(Z) \mid y_i \equiv 0 \bmod w_i/(C_4, w_i)\}] \\ &\quad = |C_1W| \prod (C_4, w_i) . \end{aligned}$$

Proof of Theorem 1. From above lemmas follows that

$$\begin{aligned} &\prod_{k=0}^{n-1} (1 - p^{k-s})^{-1} b_p(s, T) \\ &\qquad = \sum |C_1|^{-s} C_4^{1-s} e(\sigma(T_1C_1^{-1}D_1)) x(C_1, D_1, C_4) , \end{aligned}$$

where C_1, D_1, C_4 run over $SL_{n-1}(Z)\backslash C(n-1; p)$, $\{D_1 \in M_{n-1}(Z) \bmod C_1\Lambda_{n-1} \mid C_1^{-1}D_1 = {}^t(C_1^{-1}D_1)\}$ and $C(1; p)$ respectively

$$= \sum |WC_1|^{-s} C_4^{1-s} e(\sigma(T_1C_1^{-1}D_1)) x(WC_1, WD_1, C_4) ,$$

where C_1, D_1, C_4 run over the same set as above with an additional condition that (C_1, D_1) is primitive, and W runs over $SL_{n-1}(Z)\backslash C(n-1; p)$

$$= \sum |C_1|^{1-s} e(\sigma(T_1 C_1^{-1} D_1)) \cdot \sum |W|^{1-s} C_4^{1-s} \prod (C_4, w_i),$$

where C_1, D_1, C_4, W run over the above set and $\{w_i\}$ is the set of elementary divisors of W.

Thus we have proved that $b_p(s, T) b_p(s-1, T_1)^{-1}$ is independent of T_1. Hence by the formula of $b_p(s, 0)$ ([7]) or evaluating $b_p(s, T)$, $b_p(s, T_1)$ for

$$T_1 = \frac{1}{2}\begin{bmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & 1 & 0 \end{bmatrix} \quad \text{or} \quad \frac{1}{2}\begin{bmatrix} 0 & 1 & & & & \\ 1 & 0 & & & & \\ & & \ddots & & & \\ & & & 0 & 1 & \\ & & & 1 & 0 & \\ & & & & & 2 \end{bmatrix}$$

similarly to the proof of the next theorem we have $b_p(s, T) b_p(s-1, T_1)^{-1} = (1-p^{-s})(1+p^{1-s})(1-p^{n+1-2s})^{-1}$.

Corollary 1. *Let* $T^{(n)} = \begin{pmatrix} T_1^{(n-r)} & 0 \\ 0 & 0 \end{pmatrix}$ $(1 \leqq r < n)$ *be a half-integral symmetric matrix. Then we have*

$$\begin{aligned} b_p(s, T) = (1-p^{-s})(1+p^{r-s}) &\prod_{0<i\leqq \min(r-1,[n/2])} (1-p^{2i-2s}) \\ \times &\prod_{\max(r,[n/2]+1)\leqq j\leqq [(n+r)/2]} (1-p^{2j-2s})^{-1} \\ \times &\prod_{\substack{n+1\leqq k\leqq n+r \\ 2\nmid k}} (1-p^{k-2s})^{-1} b_p(s-r, T_1), \end{aligned}$$

where [ ] *means the Gauss' symbol.*

Proof. By induction it is easy to see

$$\begin{aligned} b_p(s, T) = (1-p^{-s})(1+p^{r-s}) &\prod_{0<i<r} (1-p^{2i-2s}) \\ \cdot &\prod_{n+1\leqq j\leqq n+r} (1-p^{j-2s})^{-1} b_p(s-r, T_1). \end{aligned}$$

From this follows our formula.

Corollary 2. *Let* $O^{(n)}$ *be the* $n \times n$ *zero matrix. Then we have*

$$\begin{aligned} b_p(s, O^{(n)}) = (1-p^{-s}) &\prod_{0<k\leqq [n/2]} (1-p^{2k-2s}) \\ \cdot &\Big\{(1-p^{n-s}) \prod_{\substack{n+1\leqq j<2n \\ 2\nmid j}} (1-p^{j-2s})\Big\}^{-1}. \end{aligned}$$

Proof. $b_p(s, O^{(1)}) = (1-p^{-s})(1-p^{1-s})^{-1}$ is easy to see. Applying Corollary 1 to $r = n-1$, $T = O^{(n)}$ we get

$$b_p(s, O^{(n)}) = (1-p^{-s})(1-p^{n-s})^{-1} \prod_{1 \leqq k \leqq n-1} \{(1-p^{2k-2s})(1-p^{n+k-2s})^{-1}\}.$$

From this follows our formula.

Remark. In Corollary 1 there is a cancellation

$$(1+p^{r-s})(1-p^{2j-2s})^{-1} = (1-p^{r-s})^{-1} \quad (j = r) \quad \text{if } r \geqq [n/2]+1.$$

In the rest of this paper we will show that $b_p(s, T)$ is a polynomial in p^{-s} for regular half-integral symmetric matrices T.

Put $E_n = \{S = (s_{ij}) \in M_n(Z_p) \mid S = {}^tS,\ s_{ii} \in 2Z_p\ (1 \leqq i \leqq n)\}$ and

$$H_s = \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & 1 & 0 \end{pmatrix} \in E_{2s}.$$

For $N \in E_n$ we put

$$\begin{aligned} \alpha(N, H_s; p^t) &= \{T \in M_{2s,n}(Z_p/(p^t)) \mid H_s[T] - N \in p^tE_n\}, \\ B(N, H_s; p^t) &= \{T \in \alpha(N, H_s; p^t) \mid T\text{: primitive}\}. \end{aligned}$$

LEMMA 7. *Let $N \in E_n$ with $|N| \neq 0$ and $G \in GL_n(\boldsymbol{Q}_p) \cap M_n(Z_p)$. If $t > \operatorname{ord}_p |N|$, we have*

$$\begin{aligned} &(p^t)^{n(n+1)/2-2sn} \#\{T \in \alpha(N, H_s; p^t) \mid M_{2s,n}(Z_p) \ni TG^{-1}\text{: primitive}\} \\ &\quad = (p^{\operatorname{ord}_p |G|})^{n-2s+1} p^{n(n+1)/2-2sn} \# B(N[G^{-1}], H_s; p). \end{aligned}$$

Proof. Let $T \in M_{2s,n}(Z_p)$ and suppose that $H_s[T] - N \in p^tE_n$ and $T_1 = TG^{-1}$ is primitive. Then $H_s[T] = N + p^tC$ holds for some $C \in E_n$ and $H_s[T] \equiv N \bmod p^t$. Hence $|H_s[T_1]||G|^2 \equiv |N| \bmod p^t$ holds and $2\operatorname{ord}_p|G| \leqq \operatorname{ord}_p|N| < t$ follows from $\operatorname{ord}_p|N| < t$. Denote by $C_1, \cdots, C_a$ the representatives of the set $\{p^t\bar{C}[G^{-1}] \mid \bar{C} \in E_n\} \bmod p^tE_n$, then we have $H_s[T_1] = N[G^{-1}] + p^tC[G^{-1}] \equiv N[G^{-1}] + C_k \bmod p^tE_n$. Conversely suppose that $T_1 \in M_{2s,n}(Z_p)$ and T_1 is primitive and $H_s[T_1] \equiv N[G^{-1}] + C_k \bmod p^tE_n$, then we have $H_s[T_1G] \equiv N \bmod p^tE_n$. Therefore we get

$$\begin{aligned} &\#\{T \in M_{2s,n}(Z_p) \bmod p^tM_{2s,n}(Z_p)G \mid H_s[T] - N \in p^tE_n,\ TG^{-1}\text{: primitive}\} \\ &\quad = \sum_{k=1}^{a} \#B(N[G^{-1}] + C_k, H_s; p^t). \end{aligned}$$

Since C_k is in pE_n, by virtue of 2.2 in [2] we have

$$\begin{aligned}(p^t)^{n(n+1)/2-2sn}\#B(N[G^{-1}] + C_k, H_s; p^t) \\ = p^{n(n+1)/2-2sn}\#B(N[G^{-1}] + C_k, H_s; p) \\ = p^{n(n+1)/2-2sn}\#B(N[G^{-1}], H_s; p) .\end{aligned}$$

Let $p^{a_1}, \cdots, p^{a_n}$ be elementary divisors of G, then from the definition of a follows immediately

$$\begin{aligned}a &= \#[\{p^t(c_{ij}p^{-a_i-a_j}) \mid (c_{ij}) \in E_n\} \bmod p^t E_n] \\ &= (p^{\mathrm{ord}_p|G|})^{n+1} .\end{aligned}$$

Thus we have

$$\begin{aligned}(p^{\mathrm{ord}_p|G|})^{2s}\#\{T \in \alpha(N, H_s; p^t) \mid M_{2s,n}(Z_p) \ni TG^{-1}: \text{primitive}\} \\ = (p^{\mathrm{ord}_p|G|})^{n+1}(p^{-t})^{n(n+1)/2-2sn}p^{n(n+1)/2-2sn}\#B(N[G^{-1}], H_s; p) .\end{aligned}$$

As a corollary we get

LEMMA 8. *Let $N \in E_n$ with $|N| \neq 0$ and $t > \mathrm{ord}_p|N|$. Then we have*

$$\begin{aligned}(p^t)^{n(n+1)/2-2sn}\#\alpha(N, H_s; p^t) \\ = \sum (p^{\mathrm{ord}_p|G|})^{n+1-2s}p^{n(n+1)/2-2sn}\#B(N[G^{-1}], H_s; p)\end{aligned}$$

where G runs over $GL_n(Z_p)\backslash\{GL_n(\boldsymbol{Q}_p) \cap M_n(Z_p)\}$.

Proof. Let $T \in \alpha(N, H_s; p^t)$ and suppose that TG^{-1} is primitive for $G \in GL_n(\boldsymbol{Q}_p) \cap M_n(Z_p)$. For any matrix $T_1 \equiv T \bmod p^t$ T_1G^{-1} is also primitive since $2\,\mathrm{ord}_p|G| < t$ as in the proof of the previous lemma. If TG_1^{-1}, TG_2^{-1} are primitive for $G_i \in GL_n(\boldsymbol{Q}_p) \cap M_n(Z_p)$, then $G_1G_2^{-1} \in GL_n(Z_p)$ since $TG_1^{-1}(G_1G_2^{-1}) = TG_2^{-1}$. Now Lemma 7 completes the proof of Lemma 8.

Let $H = Z/(p)[e, f]$ be a quadratic space over $Z/(p)$ such that $q(e) = q(f) = 0$, $b(e, f) = 1(q(x + y) - q(x) - q(y) = b(x, y))$, and $\overline{H}_s = \perp_s H$. For a quadratic space N over $Z/(p)$ we put

$$B(N, \overline{H}_s) = \#\{\text{isometries form } N \text{ to } \overline{H}_s\} .$$

If $N \in E_n$, then

$$q(x_1, \cdots, x_n) = \frac{1}{2}N\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

makes a quadratic space N' over $Z/(p)$ corresponding to N and $\#B(N, H_s; p) = B(N', \overline{H}_s)$ holds.

LEMMA 9. *Let N be a quadratic space over $Z/(p)$ and* $\dim N = n$. *Let $N = N_1 \perp N_2$ where N_2 is a maximal totally singular subspace, that is, N_2 has a maximal dimension among the subspaces in N such that $q(N_2) = 0$. Put* $\dim N_1 = d$ *and $\varepsilon = 1$ if N_1 is isometric to $\overline{H}_k$ for some k or $d = 0$, otherwise $\varepsilon = -1$. Then for a sufficiently large s we have*

$$p^{n(n+1)/2-2sn}B(N, \overline{H}_s) = \begin{cases} (1-p^{-s})(1+\varepsilon p^{n-d/2-s}) \prod_{1 \leqq i \leqq n-d/2-1} (1-p^{2i-2s}) & 2 \mid d\,, \\ (1-p^{-s}) \prod_{1 \leqq i \leqq n-(d+1)/2} (1-p^{2i-2s}) & 2 \nmid d\,. \end{cases}$$

Proof. Let p be an odd prime. This follows from the proof of Lemma 1 in [2]. For a sufficiently large s there is an isometry u from N into $\overline{H}_s$. Let M be the orthogonal complement of $u(N_1)$ in $\overline{H}_s$. By the theorem of Witt the isometry class of M is independent of the choice of u. Then we have

$$B(N, \overline{H}_s) = B(N_1, \overline{H}_s)B(N_2, M)\,,$$

where $B(N_2, M)$ is the number of isometries from N_2 into M. Then it is known ([8], [2]).

$$p^{d(d+1)/2-2sd}B(N_1, \overline{H}_s) = \begin{cases} (1-p^{-s})(1+\varepsilon p^{d/2-s}) \prod_{1 \leqq k \leqq d/2-1} (1-p^{2k-2s}) & 2 \mid d > 0\,, \\ (1-p^{-s}) \prod_{1 \leqq k \leqq (d-1)/2} (1-p^{2k-2s}) & 2 \nmid d\,, \end{cases}$$

$$p^{-(2s-d)(n-d)+(n-d)(n-d+1)/2}B(N_2, M) = \begin{cases} \prod_{0 \leqq k \leqq n-d-1} \{(1-\varepsilon p^{n-s-d/2-k-1})(1+\varepsilon p^{n-s-d/2-k})\} & 2 \mid d\,, \\ \prod_{0 \leqq k \leqq n-d-1} (1-p^{2n-2s-d-1-2k}) & 2 \nmid d\,. \end{cases}$$

From this follows our formula. Similarly we get the same formulas for $p = 2$. There is nothing to change in the above proof for an odd prime p.

Let T be a half-integral symmetric matrix with $|T| \neq 0$. Put

$$b_p(s, T; p^t) = \sum_{R \bmod p^t} \nu(p^{-t}R)^{-s} e(\sigma(T(p^{-t}R)))\,,$$

where R runs over integral symmetric matrices $\bmod p^t$. Then it is known ([4]) that for a natural number s

$$\begin{aligned} b_p(s, T; p^t) &= (p^t)^{n(n+1)/2-2ns} \#\{K^{(n,2s)} \bmod p^t \mid p^{-t}(\tfrac{1}{2}H_s[{}^tK] + T) \in 2^{-1}E_n\} \\ &= (p^t)^{n(n+1)/2-2ns} \#\alpha(-2T, H^s; p^t)\,. \end{aligned}$$

By definition $b_p(s, T; p^t)$ is a polynomial in p^{-s}. On the other hand by virtue of Lemma 8,9 there exists a polynomial $f(x, T)$ which depends only on T such that $b_p(s, T; p^t) = f(p^{-s}, T)$ if s, t are sufficiently large integers. Hence we have $b_p(s, T; p^t) = f(p^{-s}, T)$ for any $s \in C$, and $b_p(s, T) = f(p^{-s}, T)$ as $t \to \infty$.

Thus we have proved

THEOREM 2. *Let $T^{(n)}$ be a half-integral symmetric matrix with $|T| \neq 0$. Then we have*

$$b_p(s, T) = \sum_G (p^{\mathrm{ord}_p|G|})^{n+1-2s} a(-T[G^{-1}], s),$$

where G runs over $GL_n(Z_p)\backslash\{GL_n(Q_p) \cap M_n(Z_p)\}$ and $a(T, s)$ is defined as follows. If T is not half-integral, $a(T, s) = 0$. If T is half-integral, we define a quadratic space N over $Z/(p)$ with $\dim N = n$ by

$$q(x_1, \cdots, x_n) = T\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \bmod p, \quad \textit{and} \quad N = N_1 \perp N_2$$

where N_2 is a maximal totally singular subspace. Put $d = \dim N_1$ and $\varepsilon = 1$ if N_1 is a hyperboilc space or $d = 0$, otherwise $\varepsilon = -1$. Then we set

$$a(T, s) = \begin{cases} (1 - p^{-s})(1 + \varepsilon p^{n-d/2-s}) \prod_{1 \leq i \leq n-d/2-1} (1 - p^{2i-2s}) & 2|d, \\ (1 - p^{-s}) \prod_{1 \leq i \leq n-(d+1)/2} (1 - p^{2i-2s}) & 2\nmid d. \end{cases}$$

In the above formula for $b_p(s, T)$ G runs over a finite set.

COROLLARY. (i) *Let $O^{(n)}$ be the $n \times n$ zero matrix. Then*

$$b_p(s, O^{(n)}) = (1 - p^{-s}) \prod_{0 < k \leq [n/2]} (1 - p^{2k-2s}) \{(1 - p^{n-s}) \prod_{\substack{n+1 \leq j < 2n \\ 2\nmid j}} (1 - p^{j-2s})\}^{-1}.$$

Let $T^{(n)} = \begin{pmatrix} T_1^{(n-r)} & 0 \\ 0 & 0 \end{pmatrix}$ be a half-integral symmetric matrix with $|T_1| \neq 0$ $(0 \leq r < n)$.

(ii) *If p does not divide $|2T_1|$, then*

$$b_p(s, T) = (1 - p^{-s}) \prod_{1 \leq j \leq [n/2]} (1 - p^{2j-2s}) \prod_{\substack{n+1 \leq k \leq n+r \\ 2\nmid k}} (1 - p^{k-2s})^{-1}$$
$$\times \begin{cases} (1 - \varepsilon(T_1) p^{(n+r)/2-s})^{-1} & 2|n - r, \\ 1 & 2\nmid n - r \end{cases}$$

where $\varepsilon(T_1)=1$ *if* T_1 *corresponds to a hyperbolic space over* $Z/(p)$, *and* $\varepsilon(T_1)=-1$ *otherwise*, i.e., $\varepsilon(T_1)=(((-1)^{(n-r)/2}|2T_1|)/p)$ (*Kronecker symbol*).

(iii) *If* $n-r$ *is odd, then*

$$b_p(s, T) = (\textit{polynomial in } p^{-s})(1-p^{-s})\prod_{1\leqq j\leqq[n/2]}(1-p^{2j-2s}) \times \prod_{\substack{n+1\leqq k\leqq n+r\\ 2\nmid k}}(1-p^{k-2s})^{-1}\,.$$

(iv) *If* $n-r$ *is even, then*

$$b_p(s, T) = (\textit{polynomial in } p^{-s}) \times (1-\eta p^{(n+r)/2-s})^{-1}(1-p^{-s}) \times \prod_{1\leqq j\leqq[n/2]}(1-p^{2j-2s})\prod_{\substack{n+1\leqq k\leqq n+r\\ 2\nmid k}}(1-p^{k-2s})^{-1}\,,$$

where η *is defined as follows*:

If there is an integral matrix $G^{(n-r)}$ *such that* $T_1[G^{-1}]$ *is half-integral and* $|2T_1[G^{-1}]|$ *is not divided by* p, *then*

$$\eta = \varepsilon(T_1[G^{-1}]) \qquad (\textit{in } (\mathrm{ii}))\,.$$

(η *is uniquely determined by* T_1).

$$\textit{Otherwise } \eta = 0\,.$$

Especially $\eta=0$ *if* $\mathrm{ord}_p|2T_1|$ *is odd.*

Proof. (i) is already proved. (ii) follows from Corollary 1 and Theorem 2. Let $T_2^{(n-r)}$ be a half-integral matrix with $|T_2|\neq 0$. If $n-r$ is odd or $p\,||2T_2|$, then $a(T_2, s)$ is divided by

$$(1-p^{-s})\prod_{1\leqq i\leqq[(n-r)/2]}(1-p^{2i-2s})\,.$$

(iii) and (iv) for $\eta=0$ follow from this and Corollary 1 and Theorem 2. Suppose that there is an integral matrix $G^{(n-r)}$ such that $T_1[G^{-1}]$ is half-integral and $|2T_1[G^{-1}]|$ is not divided by p. Then

$$a(T_1[G^{-1}], s) = (1-p^{-s})(1+\varepsilon(T_1[G^{-1}])p^{(n-r)/2-s})\prod_{1\leqq i\leqq(n-r)/2-1}(1-p^{2i-2s})\,.$$

The coset $G_{n-r}(Z_p)G$ is not necessarily unique, but $\varepsilon(T_1[G^{-1}])$ depends only on T_1. Taking these terms into account, we complete the proof of the case $\eta\neq 0$.

Remark 1. Let $n=2k$ be an even integer and $T^{(n)}$ a half-integral symmetric regular matrix. Let $L=Z_p[e_1,\cdots,e_n]$ be a free module over and define a bilinear form $B(e_i, e_j)$ on it by $(B(e_i, e_j)) = 2T$. Then there an integral matrix G such that $T[G^{-1}]$ is half-integral and $p \nmid |2T[G^{-1}]|$ and only if there is a unimodular lattice M such that $M \supset L$ and the m of M is $2Z_p$. A corresponding matrix to M is diag $[1, \cdots, 1, \delta]$ $\in Z_p^\times$) if $p \neq 2$,

$$\begin{cases} \operatorname{diag}\left[\begin{pmatrix}0&1\\1&0\end{pmatrix}, \cdots, \begin{pmatrix}0&1\\1&0\end{pmatrix}\right] \\ \text{or} & \text{if } p = 2\,. \\ \operatorname{diag}\left[\begin{pmatrix}0&1\\1&0\end{pmatrix}, \cdots, \begin{pmatrix}0&1\\1&0\end{pmatrix}, \begin{pmatrix}2&1\\1&2\end{pmatrix}\right] \end{cases}$$

t $|2T| = p^a \cdot u$ $(p \nmid u)$. Then there is an integral matrix G such that $G^{-1}]$ is half-integral and $p \nmid |2T[G^{-1}]|$ if and only if the following condins hold:

(i) a is even,

(ii) if $p \neq 2$, then the Hasse invariant is 1,

(iii) if $p = 2$, then $(-1)^k u \equiv 1 \bmod 4$ and the Hasse invariant is $(-1)^{k(k+1)/2}$ if $(-1)^k u \equiv 1 \bmod 8$, $(-1)^{k(k+1)/2+1}$ if $(-1)^k u \equiv 5 \bmod 8$.

re the Hasse invariant S is defined as follows: Taking a regular matrix such that $2T[H] = \operatorname{diag}\ [d_1, \cdots, d_n]$, we put

$$S = \prod_{1 \leqq i \leqq n} (d_i, \prod_{1 \leqq j \leqq i} d_j)\,,$$

iere $(\ ,\ )$ is the Hilbert symbol of degree 2 on $Q_p^\times$. S is uniquely deterned by T.

Remark 2. Let K be a finite extension field over the p-adic rational umber field Q_p, O the maximal order of K and (δ) the different of K over , $(\delta \in K)$. For $x \in K$ we denote by $|x|_K$ the normalized valuation of x. r a prime element π of K we have $|\pi|_K^{-1} = \sharp(O/(\pi))$. Let R be a symetric matrix in $M_n(K)$. Then R is decomposed as $R = C^{-1}D$ such that $\begin{pmatrix} * & * \\ C & D \end{pmatrix} \in Sp_n(O)$ and we put $\nu(R) = |\det C|_K^{-1}$. This is well-defined. For $\in Q_p$ we put $e(x) = \exp(2\pi i$ (the fractional part of x)). Let T be a halftegral matrix, that is, $2T \in M_n(O)$, $T = {}^tT$ and all diagonal entries of T e in O. Then we put

$$b(s, T) = \sum \nu(R)^{-s} e(\operatorname{tr}_{K/Q_p}(\sigma(TR)\delta^{-1}))\,,$$

here R runs over $\{R \in M_n(K) \mid R = {}^tR\} \bmod O$. Then all theorems and corollaries hold for $b(s, T)$ instead of $b_p(s, T)$ with the following minor changes:

(i) p should be $|\pi|_K^{-1}$.

(ii) In Theorem 2 G runs over $GL_n(O)\backslash\{GL_n(K)\cap M_n(O)\}$ and $p^{\mathrm{ord}_p|G|}$ should be $|\det G|_K^{-1}$ and a quadratic form q should be defined over $O/(\pi)$ (also in Corollary).

Conjecture 6.3 for $\lambda = 0$, Case SP in [7] where the denominator can be solved therein does not necessarily refer to the reduced denominator.

Remark 3. Let T be a half-integral symmetric binary regular matrix. Denote by t^* the discriminant of $\boldsymbol{Q}(\sqrt{-|T|})$ and let α be the integer such that $p^{2\alpha}\|\,|2T|/t^*$. Then from the explicit formula of $b_p(s, T)$ ([1], [3]) follows that $b_p(s, pT) - p^{2-s}b_p(s, T)$ does not depend on T itself but only on α, (t^*/p) (Kronecker symbol). A weaker assertion holds for the function α_1 (Case SP) defined in [7] from [3].

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