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# STRICTLY LOCALIZABLE MEASURES

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## Introduction

In this paper it is proved that every locally strictly localizable Radon measure of type ($\mathscr{H}$), is strictly localizable, from where it follows immediately the existence of lifting for these measures.

R. Ryan states in [9] that a complete measure has a lifting if and only if it is strictly localizable. The existence of lifting for the Lebesgue measure in $\boldsymbol{R}^n$ has been proved by von Neumann [4] and for general $\sigma$-finite measures by D. Maharam [3]. A. and C. Ionescu Tulcea [2] have proved the existence of lifting for positive Radon measures in locally compact spaces, and L. Schwartz [10] has solved the problem for locally finite Radon measures (of type ($\mathscr{K}$)) in arbitrary topological Hausdorff spaces.

B. Rodríguez-Salinas and P. Jiménez Guerra [7] and [8] have proved that every locally $\sigma$-finite Radon measure of type ($\mathscr{H}$) is strictly localizable, result which is an immediate consequence of the Maharam's theorem and of the theorem 2 in this paper (see Corollary 3).

Proposition 4 allows to extend, for locally strictly localizable Radon measures of type ($\mathscr{H}$), many results which are known for finite Radon measures of type ($\mathscr{H}$).

The results concerning the existence of different types of liftings for locally $\sigma$-finite Radon measures of type ($\mathscr{H}$), that were obtained by Rodríguez-Salinas in [6], can be easily extended for locally strictly localizable Radon measures of type ($\mathscr{H}$), using Theorem 2 and Proposition 4 of this work.

## Notations and fundamentals

We will denote by E an arbitrary topological space (Hausdorff or not) and by $\mathscr{H}$ a class of closed subsets of $E$. If $\mu$ is a Radon measure of

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type $(\mathscr{H})$ on $E$ and $A \subset E$ we will denote by $\mu_A$ the Radon measure of type $(\mathscr{H}_A)$ on $A$, induced by the measure $\mu$ (see Theorem 78 of [7]).

By $\mu$-compact set and Radon measure of type $(\mathscr{H})$ we will understand the same as in [5].

A Radon measure $\mu$ of type $(\mathscr{H})$ on a topological space $E$ is strictly localizable (Definition 8, p. 16 and 17 of [2]) if and only if there exists a family $\mathscr{C}$ of $\mu$-measurable disjoint subsets of $E$, with positive and finite measure which verify one of the two following equivalent conditions:

$M_1$. $\sup\{\tilde{K}\colon K \in \mathscr{C}\} = E$ (where $\tilde{K}$ is the equivalence class of the set $K$ with respect to the equivalence relation:

$$A \equiv B \Leftrightarrow \mu^{\cdot}(A \triangle B) = 0\ ,$$

being $A$ and $B$ $\mu^{\cdot}$-measurable subsets of $E$).

$M_2$. For every set $A \subset E$ with $\mu^{\cdot}(A) < +\infty$, there is a countable subset $\mathscr{C}_A \subset \mathscr{C}$ such that $A - \bigcup_{K \in \mathscr{C}_A} K$ is $\mu^{\cdot}$-negligible.

From now on we will say that $\mathscr{C}$ is a family of strict localizability for $\mu$ and we will denote by $\overline{\mathscr{C}}$ the set $\bigcup_{K \in \mathscr{C}} K$.

LEMMA 1. *If $\mu$ is a locally strictly localizable Radon measure of type $(\mathscr{H})$ on $E$, $G$ is an open subset of $E$ such that $\mu(E - G) > 0$ and $\mathscr{C}$ is a family of strict localizability for $\mu_G$, then there exists an open subset $G'$ of $E$ and a family $\mathscr{C}'$ of strict localizability for $\mu_{G'}$, such that $\mathscr{C} \subset \mathscr{C}'$ and $G$ is strictly contained in $G'$.*

*Proof.* We have that $\mu(E - G) > 0$, then there exists a set $H \in \mathscr{H}$ of measure $\mu(H) > 0$, such that $H \subset E - G$. Since $H$ is $\mu$-comact, $\mu$ is locally strictly localizable and $\mu(H) > 0$, it is easily deduced the existence of an open subset $U$ of $E$ such that $\mu_U$ is strictly localizable and $\mu(U \cap H) > 0$. Evidently, $G$ is strictly contained in $G' = G \cup U$.

Let $\mathscr{D}$ be a family of strict localizability for $\mu_U$. For every subset $\mathscr{S} \subset \mathscr{D}$ we set

$$\mathscr{S}' = \{K - G\colon K \in \mathscr{S}\}$$

and

$$\mathscr{S}'' = \{K' \in \mathscr{S}'\colon \mu(K') > 0\}\ .$$

We will prove now that $\mathscr{C}^* = \mathscr{C} \cup \mathscr{D}''$ is a family of strict localizability for $\mu_{G'}$ for which it is enough to verify that $\mathscr{C}^*$ satisfies $M_2$.

If $A \subset G'$ and $\mu^{\cdot}_{G'}(A) < +\infty$ then $\mu^{\cdot}_G(A \cap G)$ and $\mu^{\cdot}_U(A \cap U)$ are finite and there exist two countable subsets $\mathscr{C}_A \subset \mathscr{C}$ and $\mathscr{D}_A \subset \mathscr{D}$ such that

$$\mu_G(A \cap G - \bar{\mathscr{C}}_A) = 0$$

and

$$\mu_U(A \cap U - \bar{\mathscr{D}}_A) = 0 .$$

So, $\mathscr{C}^*_A = \mathscr{C}_A \cup \mathscr{D}''_A$ is a countable subfamily of $\mathscr{C}^*$ which verifies:

$$\begin{aligned}\dot{\mu}_{G'}(A - \bar{\mathscr{C}}^*_A) &\leq \dot{\mu}_{G'}(A \cap G - \bar{\mathscr{C}}^*_A) + \dot{\mu}_{G'}[A \cap (U - G) - \bar{\mathscr{C}}^*_A] \\ &\leq \dot{\mu}_{G'}(A \cap G - \bar{\mathscr{C}}_A) + \dot{\mu}_U[A \cap (U - G) - \bar{\mathscr{D}}''_A] \\ &\leq \dot{\mu}_{G'}(A \cap G - \bar{\mathscr{C}}_A) + \dot{\mu}_U[A \cap (U - G) - \bar{\mathscr{D}}_A] \\ &\leq \dot{\mu}_G(A \cap G - \bar{\mathscr{C}}_A) + \dot{\mu}_U(A \cap U - \bar{\mathscr{D}}_A) \\ &= 0\end{aligned}$$

and, consequently, $\mathscr{C}^*$ verifies $\mathrm{M}_2$ and the lemma is proved because $\mathscr{C} \subset \mathscr{C}^*$ by construction.

It should be notice that it follows from $\mathrm{M}_2$ that for every $H \in \mathscr{H}$ there exists a family $\mathscr{S}_A \subset \mathscr{D}_A$ such that

$$\mu(A \cap U \cap H - \bar{\mathscr{S}}_A) = 0$$

and

$$\begin{aligned}\dot{\mu}_U[A \cap (U - G) \cap H \cap \bar{\mathscr{S}}_A] &= \dot{\mu}_U[A \cap (U - G) \cap H \cap \bar{\mathscr{S}}'_A] \\ &= \sum_{K \in \mathscr{S}'_A} \dot{\mu}_U[A \cap (U - G) \cap H \cap K] \\ &= \sum_{K \in \mathscr{S}''_A} \dot{\mu}_U[A \cap (U - G) \cap H \cap K] \\ &= \dot{\mu}_U[A \cap (U - G) \cap H \cap \bar{\mathscr{S}}''_A] ,\end{aligned}$$

therefore the inequality

$$\dot{\mu}_U[(A \cap (U - G) - \bar{\mathscr{D}}''_A) \cap H] \leq \dot{\mu}_U[(A \cap (U - G) - \bar{\mathscr{D}}_A) \cap H]$$

holds, and it follows from Theorem 74.2 of [7] that

$$\dot{\mu}_U[A \cap (U - G) - \bar{\mathscr{D}}''_A] \leq \dot{\mu}_U[A \cap (U - G) - \bar{\mathscr{D}}_A] .$$

THEOREM 2. *Every locally strictly localizable Radon measure of type* ($\mathscr{H}$) *on* $E$, *is strictly localizable.*

*Proof.* Let $\mu$ be a locally strictly localizable Radon measure of type ($\mathscr{H}$) on $E$ and let us consider the set $\mathscr{A}$ of all pairs $(G, \mathscr{C})$ where $G$ is an open subset of $E$, such that $\mu_G$ is strictly localizable and $\mathscr{C}$ is a family of strict localizability for $\mu_G$. We consider in $\mathscr{A}$ the following order:

$$(G_1, \mathscr{C}_1) \leq (G_2, \mathscr{C}_2) \Leftrightarrow G_1 \subset G_2 \quad \text{and} \quad \mathscr{C}_1 \subset \mathscr{C}_2 .$$

We will see that if $\{(G_i, \mathscr{C}_i)\}_{i \in I}$ is a chain in $(\mathscr{A}, \leq)$ then $\mathscr{C} = \bigcup_{i \in I} \mathscr{C}_i$ is a family of strict localizability for $\mu_G$, being $G = \bigcup_{i \in I} G_i$, and therefore $(\mathscr{A}, \leq)$ is inductive.

If $A \subset G$ and $\dot{\mu_G}(A) < +\infty$ then $A$ is $\dot{\mu_G}$-compact and there is a countable subset $I'$ of $I$ such that

$$\mu_G(A - \bigcup_{i \in I'} G_i) = 0 .$$

For every $i \in I'$ we have that $\dot{\mu}_{G_i}(A \cap G_i) < +\infty$ and there exists a countable subfamily $\mathscr{C}_i^*$ of $\mathscr{C}_i$ such that

$$\mu_{G_i}(A \cap G_i - \overline{\mathscr{C}}_i^*) = 0$$

holds. Consequently $\mathscr{C}^* = \bigcup_{i \in I'} \mathscr{C}_i^*$ is a countable subset of $\mathscr{C}$ such that

$$\begin{aligned} \dot{\mu_G}(A - \overline{\mathscr{C}}^*) &= \dot{\mu_G}[(A \cap \bigcup_{i \in I'} G_i) - \overline{\mathscr{C}}^*] \\ &\leq \sum_{i \in I'} \mu_{G_i}(A \cap G_i - \overline{\mathscr{C}}_i^*) \\ &= 0 \end{aligned}$$

and $M_2$ holds. Therefore $\mathscr{C}$ is a family of strict localizability for $\mu_G$ and $(G, \mathscr{C}) \in \mathscr{A}$.

From Zorn's axiom it is deduced the existence of a maximal element $(G, \mathscr{C}) \in \mathscr{A}$ and it follows from Lemma 1 that $E - G$ is $\mu$-negligible.

COROLLARY 3. *Every locally $\sigma$-finite Radon measure of type $(\mathscr{H})$ on $E$ is strictly localizable.*

*Proof.* It is an immediate consequence of Theorem 2, because every $\sigma$-finite measure is strictly localizable.

PROPOSITION 4. *Let $\mu$ be a Radon measure of type $(\mathscr{H})$ on $E$ and $\mathscr{C}$ a family of strict localizability for $\mu$, then we have:*

4.1. *If $A \subset E$ is such that $A \cap K$ is $\mu$-negligible for all $K \in \mathscr{C}$, then $A$ is $\mu$-negligible.*

4.2. *If $A \subset E$ is such that $A \cap K$ is $\dot{\mu}_K$-measurable for all $K \in \mathscr{C}$, then $A$ is $\dot{\mu}$-measurable.*

*Proof.* 4.1. For every $H \in \mathscr{H}$ there exists a countable subclass $\mathscr{C}_H$ of $\mathscr{C}$ such that $\mu(H - \overline{\mathscr{C}}_H) = 0$ and

$$\mu^{\cdot}(A \cap \overline{\mathscr{C}} \cap H) \leq \sum_{K \in \mathscr{C}_H} \mu(A \cap K \cap H) \\ = 0$$

holds. Therefore it follows from Theorem 74.2 of [7] that

$$\mu^{\cdot}(A \cap \overline{\mathscr{C}}) = \sup \{\mu^{\cdot}(A \cap \overline{\mathscr{C}} \cap H) \colon H \in \mathscr{H}\} \\ = 0$$

and $\mu^{\cdot}(A) = 0$.

4.2. For every $H \in \mathscr{H}$ there exists a countable subclass $\mathscr{C}_H$ of $\mathscr{C}$ such that $\mu(H - \overline{\mathscr{C}}_H) = 0$. Consequently,

$$\begin{aligned} \mu(H) &= \mu(H \cap \overline{\mathscr{C}}_H) \\ &= \sum_{K \in \mathscr{C}_H} \mu(H \cap K) \\ &= \sum_{K \in \mathscr{C}_H} [\mu^{\cdot}(H \cap K \cap A) + \mu^{\cdot}((H - A) \cap K) \\ &= \mu^{\cdot}(H \cap A) + \mu^{\cdot}(H - A) \end{aligned}$$

and it follows from Theorem 75.2 of [7] that $A$ is $\mu^{\cdot}$-measurable.

*Remark* 5. If $\mu$ is a Radon measure of type ($\mathscr{H}$) and $\mathscr{C}$ is a family of strict localizability for $\mu$, then there exists a family $\mathscr{C}'$, of strict localizability for $\mu$, such that $\mathscr{C}' \subset \mathscr{H}$ and every $K' \in \mathscr{C}'$ is contained in some $K \in \mathscr{C}$,

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