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PERTURBED BILLIARD SYSTEMS, I. THE ERGODICITY OF THE MOTION OF A PARTICLE IN A COMPOUND CENTRAL FIELD

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§ 1. Introduction

The ergodicity of classical dynamical systems which appear really in the statistical mechanics was discussed by Ya. G. Sinai [9]. He announced that the dynamical system of particles with central potential of special type in a rectangular box is ergodic. However no proofs have been given yet. Sinai [11] has given a proof of the ergodicity of a simple one-particle model which is called a Sinai billiard system.

In this article, the author will show the ergodicity of the dynamical system of a particle in a compound central field in 2-dimensional torus (see. § 10). For such a purpose, a new class of transformations, which are called perturbed billiard transformations will be introduced. Let T_* be a perturbed billiard transformation which satisfies the assumptions (H–1), (H–2) and (H–3) (see § 3). Then T_* is expressed in the form

$$T_* = T_1 T \tag{1.1}$$

where T is a Sinai billiard transformation and T_1 is a rotation such that

$$T_1(\iota, r, \varphi) = (\iota, r + H_\iota(\varphi), \varphi) \ . \tag{1.2}$$

In Theorem 1, 2 and 3, the following assertions will be shown.

(a) There exists a generator $\alpha^{(c)}$ with finite entropy.

(b) Every element of the partition $\zeta^{(c)} = \bigvee_{i=0}^{\infty} T_*^i \alpha^{(c)}$ (resp. $\zeta^{(e)} = \bigvee_{i=-1}^{-\infty} T_*^i \alpha^{(c)}$) is a connected decreasing (resp. increasing) curve.

(c) $T_*^{-1}\zeta^{(c)} > \zeta^{(c)}$, $T_*\zeta^{(e)} > \zeta^{(e)}$,

$$\bigvee_{i=-\infty}^{\infty} T_*^i \zeta^{(c)} = \bigvee_{i=-\infty}^{\infty} T_*^i \zeta^{(e)} = \varepsilon \ ,$$

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$$\bigwedge_{i=-\infty}^{\infty} T_*^i \zeta^{(c)} = \bigwedge_{i=-\infty}^{\infty} T_*^i \zeta^{(e)} = \text{the trivial partition.}$$

A potential field is called a compound central field, if the potential function is expressed in the form

$$U(q) = \sum_{\iota=1}^{I} U_\iota(|q - \bar{q}(\iota)|) , \tag{1.3}$$

where U_ι is a central potential with range R_ι and $\bar{q}(\iota)$ is a fixed point for each ι, $1 \leq \iota \leq I$. The ergodicity of the motion of a particle in a compound central field can be reduced to the ergodicity of a perturbed billiard transformation (see § 2 and § 10). Hence by applying Theorem 3, the following theorem will be shown.

THEOREM. *If U_ι, $\iota = 1, 2, \cdots, I$, are bell-shaped and if the energy E satisfies the inequality*

$$0 < E < \frac{1}{4} \min_\iota \frac{-R_\iota L_{\min}}{R_\iota + L_{\min}} U_\iota'(R_\iota - 0) , \tag{1.4}$$

then the dynamical system is ergodic, where $L_{\min}$ is the minimum distance between different potential ranges.

The K-property of this system is not proved yet. However a partial result will be presented in the forthcoming article [7]. Moreover, in the article, the following theorems will be shown.

THEOREM. *Under the assumptions* (H–1), (H–2) *and* (H–3), *a perturbed billiard transformation T_* is Bernoullian. In particular, $\alpha^{(c)}$ is a weak Bernoullian generator. Further, every finite partition whose elements have smooth boundaries is weakly Bernoullian.*

THEOREM. *If the dynamical system of a particle in a compound central field with bell-shaped potentials satisfying* (1.4) *has not point spectrum, then the dynamical system is a Bernoulli flow.*

§ 2. Observations

Consider a potential field on a 2-dimensional torus $\boldsymbol{T}$ which is governed by several potential functions $U_\iota(q)$, $\iota = 1, 2, \cdots, I$, with finite ranges. Suppose that the potential ranges do not overlap and that the boundary ∂Q_ι of the range of U_ι is a closed curve of C^3-class and ∂Q_ι encloses a strictly convex open domain $\overline{Q}_\iota$ for every ι. Assume that $U_\iota(q)$ is continuous in the torus T and is continuously differentiable in $\overline{Q}_\iota$ Observe the motion of a particle with mass m and energy E in the potential field. Then the motion of the particle is described by the Hamilton canonical equations

$$\begin{cases} \dfrac{dq^{(i)}}{dt} = \dfrac{\partial H}{\partial p^{(i)}} \\ \dfrac{\partial p^{(i)}}{dt} = -\dfrac{\partial H}{\partial q^{(i)}} \end{cases} \qquad i = 1, 2$$

with *the Hamiltonian*

$$H(p, q) = \frac{1}{2m}\{(p^{(1)})^2 + (p^{(2)})^2\} + \sum_{\iota=1}^{I} U_\iota(q^{(1)}, q^{(2)}) ,$$

where $q = (q^{(1)}, q^{(2)})$ means the position of the particle and $p = (p^{(1)}, p^{(2)})$ means the momentum. Denote by $\{S_t\}$ the flow induced from the dynamical system; that is, for each (q, p), $S_t(q, p)$ means the state of the particle at time t whose initial state is (q, p). Then the Liouville theorem tells that

$$(2.1) \qquad dqdp = dq^{(1)}dq^{(2)}dp^{(1)}dp^{(2)}$$

is a measure invariant under $\{S_t\}$. As usual one can restrict $\{S_t\}$ to the energy surface M_E. *The energy surface* is represented in the form

$$M_E = \{(q, p)\,;\, (p^{(1)})^2 + (q^{(2)})^2 = 2m(E - U(q)), q \in Q_E\}$$

with $Q_E \equiv \{q\,;\, U(q) \leq E\}$, moreover the measure

$$(2.2) \qquad d\mu_E = \text{const.}\ d\omega dq^{(1)}dq^{(2)}$$

on M_E is invariant under $\{S_t\}$, where $(p^{(1)}, p^{(2)}) = (\{2m(E - U(q))\}^{1/2} \cos \omega, \{2m(E - U(q))\}^{1/2} \sin \omega)$.

Let π be *the natural projection* from M_E to the configuration space Q_E; $\pi(q, p) = q$. Put $Q \equiv T - \bigcup_{\iota=1}^{I} \overline{Q}_\iota$ and $M_0 \equiv \pi^{-1}(Q)$. Then the boundary ∂Q of Q coincides with $\bigcup_\iota \partial Q_\iota$. Assume that Q_E is connected, then almost every motion of the particle crosses the curves ∂Q. Put for $x = (q, p)$

$$(2.3) \qquad \begin{aligned} \tilde{\tau}(x) &\equiv \sup\{t < 0\,;\, S_t x \in \pi^{-1}(\partial Q)\} , \\ \tilde{v}(x) &\equiv \inf\{t \geq 0\,;\, S_t x \in \pi^{-1}(\partial Q)\} . \end{aligned}$$

Then a transformation $\tilde{T}$ of $\pi^{-1}(\partial Q)$ is defined by

$$\tilde{T}x = S_{\tau(x)}x \qquad \text{for } x = (q, p) \text{ in } \pi^{-1}(\partial Q) . \tag{2.4}$$

It can be seen that $\{S_t\}$ is a Kakutani-Ambrose flow built by the basic space $\pi^{-1}(\partial Q)$, the basic transformation $\tilde{T}$ and the ceiling function $-\tilde{\tau}(x)$ (see [1]). In order to clarify this, it is convenient to introduce notation: A point q in ∂Q can be parametrized by (ι, r), where ι shows the number of the curve ∂Q_ι which contains q and r is the arclength between the point q and a fixed origin of ∂Q_ι measured along the curve ∂Q_ι clockwise. Let $n(q) = n(\iota, r)$ be the inward normal at $q = (\iota, r)$ in ∂Q_ι, and let $k(q) = k(\iota, r)$ be the curvature of ∂Q_ι at $q = (\iota, r)$. A point $x = (q, p)$ in $\pi^{-1}(\partial Q)$ is represented by the coordinates (ι, r, φ), where $q = (\iota, r)$ shows the position of q and φ is the angle between $n(\iota, r)$ and p.

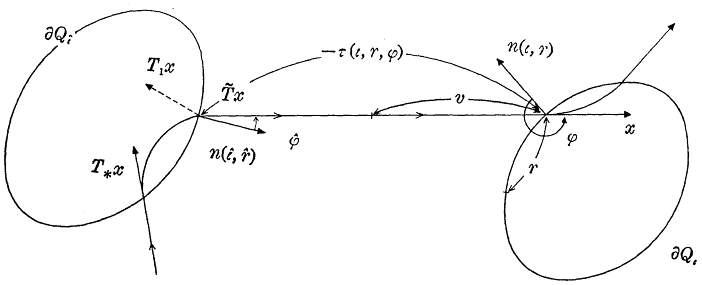

Fig. 2-1

One can introduce new coordinates of M_E; a point $x = (q, p)$ is represented by (ι, r, φ, v), where $v = \tilde{v}(x)$ and (ι, r, φ) shows the point $S_v x$ in ∂Q. Then M_E is naturally identified with the set $\{(\iota, r, \varphi, v)\,;\, 0 \leq v < -\tilde{\tau}(\iota, r, \varphi, 0), r \in \partial Q_\iota, 0 \leq \varphi < 2\pi, \iota = 1, 2, \cdots, I\}$. Then the invariant measure is expressed in the form

$$d\mu_E(\iota, r, \varphi, v) = \text{const.} \cos \varphi dv d\varphi dr d\iota \ , \tag{2.5}$$

where $d\iota$ means unit masses distributed on the set $\{\iota;\ \iota = 1, 2, \cdots, I\}$. Moreover, the measure ν on $\pi^{-1}(\partial Q)$ defined by

$$d\nu = \text{const.} \cos \varphi d\varphi dr d\iota \tag{2.6}$$

is invariant under $\tilde{T}$. Since the restriction of the measure μ_E to $M_0 = \pi^{-1}(Q)$ is expressed in the form (2.5) (see [6]), (2.5) and (2.6) are easily seen by results about induced flows and about Kakutani-Ambrose flows (see [1] and [2]). Put $\tilde{\tau}(\iota, r, \varphi) \equiv \tilde{\tau}(\iota, r, \varphi, 0)$. Then the action of $\{S_t\}$ is expressed in the form.

$$
(2.7) \qquad S_t x = \begin{cases} \left(\tilde{T}^{-k}x_0,\ v - t - \sum_{j=1}^{k} \tilde{\tau}(\tilde{T}^{-j}x_0)\right) \\ \qquad \text{if } 0 \le v - t - \sum_{j=1}^{k} \tilde{\tau}(\tilde{T}^{-j}x_0) < -\tilde{\tau}(\tilde{T}^{-k}x_0),\ k \ge 1\ , \\ (x_0, v - t) \\ \qquad \text{if } 0 \le v - t < -\tilde{\tau}(x_0)\ , \qquad k = 0\ , \\ \left(\tilde{T}^{-k}x_0,\ v - t + \sum_{j=0}^{k+1} \tilde{\tau}(\tilde{T}^{-j}x_0)\right) \\ \qquad \text{if } 0 \le v - t + \sum_{j=0}^{k+1} \tilde{\tau}(\tilde{T}^{-j}x_0) < -\tilde{\tau}(\tilde{T}^{-k}x_0),\ k \le -1\ , \end{cases}
$$

with $x = (\iota, r, \varphi, v)$ and $x_0 \equiv (\iota, r, \varphi)$ in $\pi^{-1}(\partial Q)$.

It is well known that $\{S_t\}$ is ergodic, if and only if $\tilde{T}$ is ergodic. Thus the ergodicity of $\{S_t\}$ can be reduced to the ergodicity of $\tilde{T}$. Now continue reduction. Put

$$
M \equiv \left\{(\iota, r, \varphi) \in \pi^{-1}(\partial Q)\,;\ \frac{\pi}{2} \le \varphi \le \frac{3\pi}{2}\right\}\ ,
$$

namely M is the set of all incident vectors at ∂Q. Introduce an involution Inv on $\pi^{-1}(\partial Q)$ by

$$
(2.8) \qquad \mathrm{Inv}\,(\iota, r, \varphi) \equiv (\iota, r, \pi - \varphi) \quad \bmod 2\pi\ .
$$

Since $\nu(\tilde{T}M \cap M) = 0$ and $\tilde{T}^2 M = M$, $\{S_t\}$ is a Kakutani-Ambrose flow built by the basic space M, the basic transformation $\tilde{T}^2$ and the ceiling function $-\tilde{\tau}(\iota, r, \varphi) - \tilde{\tau}(\tilde{T}(\iota, r, \varphi))$. Therefore $\{S_t\}$ is ergodic if and only if $\tilde{T}^2$ is ergodic. Put

$$
(2.9) \qquad S = \left\{(\iota, r, \varphi)\,;\ \varphi = \frac{\pi}{2} \text{ or } \frac{3\pi}{2}\right\}\ .
$$

Since $\pi^{-1}(\partial Q) - M$ is the set of vectors at ∂Q going out from $\bigcup_{\iota=1}^{I} \bar{Q}_\iota$, the restriction of $\tilde{T}$ to $\pi^{-1}(\partial Q_\iota) - M$ is a differentiable mapping from $\pi^{-1}(\partial Q_\iota) - M$ to $\pi^{-1}(\partial Q_\iota) \cap M$. Since Inv maps $M - S$ onto $\pi^{-1}(\partial Q) - M$ and Inv is identical on S, one can define a transformation T_1 of M by

$$
T_1 x = \begin{cases} \tilde{T}\,\mathrm{Inv}\,x & x \in M - S \\ x & x \in S\ . \end{cases}
$$

Then each $M^{(\iota)} \equiv \pi^{-1}(\partial Q_\iota) \cap M$ is invariant under T_1, and T_1 is differentiable. Since the particle moves along straight lines in Q, during the particle is staying in the interior of Q, the transformation T of M defined by

$$T = \mathrm{Inv}\cdot\tilde{T}$$

is the transformation which appears in the Sinai billiard system given in the domain Q with elastic collision at ∂Q (see [6] and [11]). The transformation T is called *a Sinai billiard transformation* (or *automorphism*). Thus the restriction of $\tilde{T}^2$ to M is resolved into the product of two transformations;

$$\tilde{T}^2 x = T_1 T x \qquad \text{for } x \in M - S .$$

LEMMA 2.1. *The flow $\{S_t\}$ is ergodic if and only if the product T_1T is ergodic.*

Generally, a transformation T_* of M is called *a perturbed billiard transformation* (or *automorphism*), if T_* is expressed in the form

$$T_* \equiv T_1 T . \tag{2.10}$$

where T_1 is a differentiable transformation of M which preserves the measure ν and T is a Sinai billiard transformation given in M with elastic collision at ∂Q.

If one obtains a condition of T_1 under which $T_* = T_1 T$ is ergodic, then one can solve the problem of the ergodic hypothesis for the case of one particle in a potential field (moreover for the case of two particles with interaction potential on a torus).

In the following sections, a special class of perturbed billiard transformations, which has some connection with the dynamical system of a particle in a compound central field, will be discussed, and a sufficient condition for the ergodicity will be given.

§ 3. Fundamental properties

In what follows, a special class of perturbed billiard transformations are discussed. Assume the assumption

(H–1) *the transformation T_1 is given by*

$$T_1(\iota, r, \varphi) = (\iota, r - H(\iota, \varphi), \varphi)$$

with functions $H(\iota,\varphi)$ of C^2-class satisfying $H(\iota, \pi/2) = H(\iota, (3/2)\pi) = 0$ for $\iota = 1, 2, \cdots, I$.

Obviously, T_1 preserves the measure ν. It is convenient to assume that ν is normalized;

$$d\nu = -\nu_0 \cos\varphi d\varphi dr d\iota$$

with $\nu_0 = (2|\partial Q|)^{-1}$, where $|\partial Q|$ is the total arclength of the curves $\partial Q = \bigcup_{\iota=1}^{I} \partial Q_\iota$. For (ι, r, φ) in M, put

$$\tau(\iota, r, \varphi) \equiv (2E/m)^{1/2}\tilde{\tau}(\iota, r, \varphi) .$$

Since the particle moves with speed $(2E/m)^{1/2}$, $-\tau(\iota, r, \varphi)$ is the distance between the point in ∂Q described by (ι, r) and the last point crossing ∂Q measured in Q.

It is convenient to use the following notations for a given $x = (\iota, r, \varphi)$ in M; $\iota(x) \equiv \iota$, $r(x) \equiv r$, $\varphi(x) \equiv \varphi$, $k(x) \equiv k(\iota, r)$, $k'(x) \equiv k(\iota, r + H(\iota, \varphi))$, $h(x) \equiv h(\iota, \varphi)$, $\tau(x) \equiv \tau(\iota, r, \varphi)$ and $\tau_1(x) \equiv \tau(T_*^{-1}x)$, with $h(\iota, \varphi) \equiv dH(\iota, \varphi)/d\varphi$. More simply, put $x_i = (\iota_i, r_i, \varphi_i) \equiv T_*^{-i}x$, $\iota_i \equiv \iota(x_i)$, $r_i \equiv r(x_i)$, $\varphi_i \equiv \varphi(x_i)$, $k_i \equiv k(x_i)$, $k'_i \equiv k'(x_i)$, $h_i \equiv h(x_i)$ and $\tau_i \equiv \tau(x_i)$.

LEMMA 3.1. *The Jacobian matrix of the transformation $T_*^{-1} = T^{-1}T_1^{-1}$ is given by*

$$\begin{bmatrix} \dfrac{\partial r_1}{\partial r}, & \dfrac{\partial r_1}{\partial \varphi} \\ \dfrac{\partial \varphi_1}{\partial r}, & \dfrac{\partial \varphi_1}{\partial \varphi} \end{bmatrix} \tag{3.1}$$
$$= \begin{bmatrix} -\dfrac{\cos\varphi + k'\tau_1}{\cos\varphi_1}, & -\dfrac{(\cos\varphi + k'\tau_1)h + \tau_1}{\cos\varphi_1} \\ -\dfrac{k_1\cos\varphi + k'\cos\varphi_1 + k_1k'\tau_1}{\cos\varphi_1}, & -\dfrac{(k_1\cos\varphi + k'\cos\varphi_1 + k_1k'\tau_1)h + \tau_1k_1}{\cos\varphi_1} - 1 \end{bmatrix}$$

or by

$$\begin{bmatrix} \dfrac{\partial r}{\partial r_1}, & \dfrac{\partial \varphi}{\partial \varphi_1} \\ \dfrac{\partial \varphi}{\partial r_1}, & \dfrac{\partial \varphi}{\partial \varphi_1} \end{bmatrix} \tag{3.1'}$$
$$= \begin{bmatrix} -\dfrac{(k_1\cos\varphi + k'\cos\varphi_1 + k_1k'\tau_1)h + \tau_1k_1 + \cos\varphi}{\cos\varphi}, & \dfrac{(\cos\varphi + k'\tau_1)h + \tau_1}{\cos\varphi} \\ \dfrac{k_1\cos\varphi + k'\cos\varphi_1 + k_1k'\tau_1}{\cos\varphi}, & -\dfrac{k'\tau_1}{\cos\varphi} - 1 \end{bmatrix} .$$

Proof. Put $(\iota', r', \varphi) \equiv T_1^{-1}(\iota, r, \varphi)$ and $(\iota_1, r_1, \varphi_1) \equiv T^{-1}(\iota', r', \varphi')$. Since $\iota' = \iota$, $r' = r + H(\iota, \varphi)$ and $\varphi' = \varphi$,

$$\begin{bmatrix} \dfrac{\partial r'}{\partial r'} & \dfrac{\partial r'}{\partial \varphi} \\ \dfrac{\partial \varphi'}{\partial r} & \dfrac{\partial \varphi'}{\partial \varphi} \end{bmatrix} = \begin{pmatrix} 1 & h(\iota, \varphi) \\ 0 & 1 \end{pmatrix}$$

is obviously true. On the other hand,

$$\begin{bmatrix} \dfrac{\partial r_1}{\partial r'}, & \dfrac{\partial r_1}{\partial \varphi'} \\ \dfrac{\partial \varphi_1}{\partial r'}, & \dfrac{\partial \varphi_1}{\partial \varphi'} \end{bmatrix} = \begin{bmatrix} -\dfrac{\cos\varphi + k'\tau_1}{\cos\varphi_1}, & -\dfrac{\tau_1}{\cos\varphi_1} \\ -\dfrac{k_1\cos\varphi + k'\cos\varphi_1 + k_1k'\tau_1}{\cos\varphi_1}, & -\dfrac{k_1\tau_1}{\cos\varphi_1} - 1 \end{bmatrix}$$

holds (see [5] § 4). Therefore the assertion is true. Q.E.D.

Since T is differentiable on the domain on which T is continuous, T_* is so. More precise statement of the properties concerning with the continuity and the discontinuity will be presented later.

LEMMA 3.2. *Let γ be a curve of C^1-class in $M^{(\iota)} \equiv \pi^{-1}(\partial Q_\iota) \cap M$ on which T_*^{-1} is continuous, and suppose that γ is given by the equation $= \psi(r)$. Put $\gamma_1 \equiv T_*^{-1}\gamma$ and suppose that γ_1 is given by $\varphi_1 = \psi_1(r_1)$ in $M^{(\iota_1)}$. Then it holds that*

$$\frac{d\psi_1}{dr_1} = \frac{(k_1\cos\psi + k'\cos\psi_1 + k_1k'\tau_1)(h + dr/d\psi) + k_1\tau_1 + \cos\psi_1}{(\cos\psi + k'\tau_1)(h + dr/d\psi) + \tau_1},$$

$$\frac{d\psi}{dr} = -\frac{k_1\cos\psi + k'\cos\psi_1 + k_1k'\tau_1 - (\cos\psi + k'\tau_1)d\psi_1/dr_1}{(k_1\cos\psi + k'\cos\psi_1 + k_1k'\tau_1)h + k_1\tau_1 + \cos\psi_1 - \{(\cos\psi + k'\tau_1)h + \tau_1\}d\psi_1/dr_1}$$

$$\frac{d\psi_1}{d\psi} = -\frac{k_1\cos\psi + k'\cos\psi_1 + k_1k'\tau_1}{\cos\psi_1}\left\{h + \frac{dr}{d\psi}\right\} - \frac{k_1\tau_1}{\cos\psi_1} - 1,$$

$$\frac{d\psi}{d\psi_1} = \frac{k_1\cos\psi + k'\cos\psi_1 + k_1k'\tau_1}{\cos\psi}\frac{dr_1}{d\psi_1} - \frac{k'\tau_1}{\cos\psi} - 1,$$

$$\frac{dr_1}{dr} = -\frac{\cos\psi + k'\tau_1}{\cos\psi_1} - \frac{(\cos\psi + k'\tau_1)h + \tau_1}{\cos\psi_1}\frac{d\psi}{dr},$$

$$\frac{dr}{dr_1} = -\frac{(k_1\cos\psi + k'\cos\psi_1 + k_1k'\tau_1)h + k_1\tau_1 + \cos\psi_1}{\cos\psi} + \frac{(\cos\psi + k'\tau_1)h + \tau_1}{\cos\psi}\frac{d\psi_1}{dr_1},$$

$$\frac{d\tau_1}{dr} = -\sin\psi_1\left\{\frac{\cos\psi + k'\tau_1}{\cos\psi_1}\left(1 + h\frac{d\psi}{dr}\right) + \frac{\tau_1}{\cos\psi_1}\right\} - \sin\psi\left(1 + h\frac{d\psi}{dr}\right),$$

$$\frac{d\tau_1}{dr_1} = \sin\psi_1 + \sin\psi\left\{\frac{\cos\psi_1}{\cos\psi} + \frac{\tau_1}{\cos\psi}\left(k_1 - \frac{d\psi_1}{dr_1}\right)\right\}.$$

Proof. Since

$$\frac{\partial\tau_1}{\partial r_1} = \tan\varphi(\cos\varphi_1 + k_1\tau_1) + \sin\varphi_1 \quad \text{and} \quad \frac{\partial\tau_1}{\partial\varphi_1} = -\tau_1\tan\varphi$$

hold, the last equality of the lemma is true. The other equalities follow from Lemma 3.1. Q.E.D.

Assume the following two additional assumptions throughout this article;

(H–2) every $\overline{Q}_\iota$ is a strictly convex domain such that the boundary ∂Q_ι is a curve of C^3-class, and $\{\overline{Q}_\iota \cup \partial Q_\iota;\ \iota = 1, 2, \cdots, I\}$ are disjoint.

(H–3) $\min_{\iota,\varphi}\left\{h(\iota,\varphi) + \left[\max_r k(\iota,r) + \left(\min_{\iota,r,\varphi'}|\tau(\iota,r,\varphi')|\right)^{-1}\right]^{-1}\right\} > 0\ .$

It is useful to introduce the following constants;

$$k_{\min} \equiv \min_{\iota,r} k(\iota,r),\ |\tau|_{\min} \equiv \min_{\iota,r,\varphi}|\tau(\iota,r,\varphi)|,\ \eta \equiv k_{\min}|\tau|_{\min}\ ,$$

$$K_{\max}(\iota) \equiv \max_r k(\iota,r) + \left(\min_{r,\varphi}|\tau_1(\iota,r,\varphi)|\right)^{-1},$$

$$K_{\max} \equiv \max_\iota\left[\min_\varphi h(\iota,\varphi) + 1/K_{\max}(\iota)\right]^{-1},$$

$$K_{\min} \equiv \left[\max_{\iota,\varphi} h(\iota,\varphi) + 1/k_{\min}\right]^{-1} \text{ and } \eta_1 \equiv \min\{\eta, (1+\eta)^2 K_{\min}/K_{\max}\}\ .$$

Then $0 < K_{\min} \le k_{\min} < K_{\max}(\iota) \le K_{\max} < \infty$ holds. Further constants $c_1 \equiv (1 + K_{\min}^{-2})^{1/2}$, $c_2 \equiv K_{\max}/K_{\min}$, $c_3 \equiv \log 16c_2^4$ and $c_4 \equiv 1 + c_2$ will be used.

For a subset F of M, define $\varphi_{\max}(F)$, $\varphi_{\min}(F)$, $\max\cos(F)$ and $\min\cos(F)$ by

$$\varphi_{\max}(F) \equiv \sup_{(\iota,r,\varphi)\in F}\varphi\ , \qquad \varphi_{\min}(F) \equiv \inf_{(\iota,r,\varphi)\in F}\varphi\ ,$$

$$\max\cos(F) \equiv \sup_{(\iota,r,\varphi)\in F}|\cos\varphi| \quad \text{and} \quad \min\cos(F) \equiv \inf_{(\iota,r,\varphi)\in F}|\cos\varphi|\ .$$

For a monotone connected curve γ in $M^{(\iota)}$, define $\theta(\gamma)$ and $\rho(\gamma)$ by

$$\theta(\gamma) \equiv \int_\gamma d\varphi = \varphi_{\max}(\gamma) - \varphi_{\min}(\gamma) \quad \text{and} \quad \rho(\gamma) \equiv \int_\gamma dr\ .$$

For a fixed point x in γ, define $\bar{\theta}(\gamma, x)$ and $\underline{\theta}(\gamma, x)$ by

$$\bar{\theta}(\gamma, x) \equiv \varphi_{\max}(\gamma) - \varphi(x) \quad \text{and} \quad \underline{\theta}(\gamma, x) \equiv \varphi(x) - \varphi_{\min}(\gamma) .$$

For a countable union γ of monotone connected curves $\gamma^{(j)}$, $j = 1, 2, 3, \cdots$, define $\theta(\gamma)$ and $\rho(\gamma)$ by

$$\theta(\gamma) \equiv \sum_{j=1}^{\infty} \theta(\gamma^{(j)}) \quad \text{and} \quad \rho(\gamma) \equiv \sum_{j=1}^{\infty} \rho(\gamma^{(j)}) .$$

LEMMA 3.3. *Let γ be a curve of C^1-class as in Lemma 3.2. Then the following assertions hold.*

(i) *If $0 \leq d\psi/dr \leq K_{\max}(\iota)$, then*

$$k_{\min} \leq \frac{d\psi_1}{dr_1} \leq K_{\max}(\iota_1) ,$$

$$-\frac{d\psi_1}{d\psi} \geq 1 + \eta, \quad -\frac{dr_1}{dr} \geq \frac{\cos \psi_1}{\cos \psi} \quad \textit{and} \quad \theta(\gamma_1) \geq (1 + \eta)\theta(\gamma) .$$

(ii) *If $d\psi_1/dr_1 \leq 0$, then*

$$K_{\min} \leq -\frac{d\psi}{dr} \leq K_{\max} ,$$

$$-\frac{d\psi_1}{d\psi} \geq 1 + \eta \quad \textit{and} \quad \theta(\gamma) \geq (1 + \eta)\theta(\gamma_1) .$$

Proof. If $0 \leq d\psi/dr \leq K_{\max}(\iota)$, then it follows from the assumption (H–3) that $h(\iota, \psi) + dr/d\psi \geq 0$. Hence by Lemma 3.2, the estimate

$$\frac{k_1 \cos \psi + k' \cos \psi_1 + k_1 k' \tau_1}{\cos \psi + k' \psi_1} \leq \frac{d\psi_1}{dr_1} \leq k_1 + \frac{\cos \psi}{\tau_1}$$

is given. Therefore one can prove (i). The assertion (ii) is obvious from the estimate

$$h + \frac{\tau_1}{\cos \psi + k' \tau_1} \leq -\frac{dr}{d\psi} \leq h + \frac{\cos \psi_1 + k_1 \tau_1}{k_1 \cos \psi + k' \cos \psi_1 + k_1 k' \tau_1}$$

which is true under the assumption (H–3) and the condition $d\psi_1/dr_1 \leq 0$.

Q.E.D.

In order to investigate the ergodicity of T_*, it is useful to see properties of *the curves of discontinuity of* T_* and T_*^{-1}. Here the curves of discontinuity of T_* (resp. T_*^{-1}) is defined by

$$T_*^{-1} S \qquad (\text{resp. } T_* S) ,$$

with $S = \{(\iota, r, \varphi) \in M ;\ \cos \varphi = 0\}$. By assumption (H–1), $T_1 S = S$ holds, hence

$$T_*^{-1}S = T^{-1}S \qquad (\text{resp. } T_*S = T_1TS) .$$

Therefore the curves of discontinuity of T_* coincides with those of T, and the curves of discontinuity of T_*^{-1} are merely a deformation of those of T^{-1} in the r-direction, that is,

$$T_*^{-1}S = \{(\iota, r - H_\iota(\varphi), \varphi)\,;\,(\iota, r, \varphi) \in TS\} .$$

Hence almost all properties of the curves of discontinuity are preserved under a small perturbation. The image $T_*^{-1}S$ (or T_*S) consists of countably many curves of C^2-class. A maximal connected component of such a curve in C^2 is called *a branch* of the curves of discontinuity.

(1°) Let γ be a branch of the curves of discontinuity of T_* (resp. T_*^{-1}). Then γ is an increasing curve (resp. a decreasing curve) which satisfies the equation

$$\frac{d\varphi}{dr} = k + \frac{\cos\varphi}{\tau}$$

$$\left(\text{resp. } \frac{d\varphi}{dr} = -\frac{\cos\varphi + k'\tau_1}{(\cos\varphi + k'\tau_1)h + \tau_1}\right),$$

though the solution of the equation are not unique.

Proof. By Lemma 3.2, the equations are easily obtained and the 2-times differentiability is obvious. The non uniqueness is checked by observing the curve $\tilde{\gamma}: r = r_0 - H_\iota(\varphi)$ and $T_*^{-1}\tilde{\gamma}$ (resp. $\tilde{\gamma}': r = r_0$ and $T_*\tilde{\gamma}$) with a constant r_0. Q.E.D.

(2°) Put $S(+) = \{(\iota, r, \varphi)\,;\,\varphi = \pi/2\}$ and $S(-) = \{(\iota, r, \varphi)\,;\,\varphi = 3\pi/2\}$. Give *a sign* to each branch γ of $T_*^{-1}S$ (resp. T_*S) as follows: sign$(\gamma) = (+)$ if γ is included in the image of $S(+)$, and sign$(\gamma) = (-)$ if γ is included in the image of $S(-)$. Then, only the following types of branching of the curves of discontinuity appear:

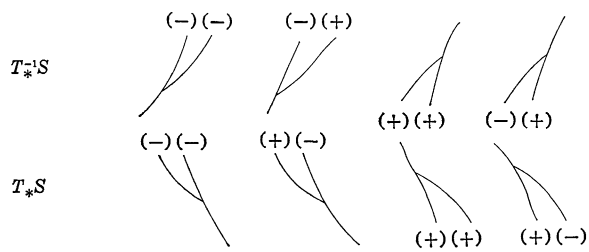

Fig. 3-1

In general for given connected curves γ, γ' and γ'', let us say that γ *joins* γ' *and* γ'' if one of ends of γ lies on γ' and the other end lies on γ''.

For any x in $T_*^{-1}S$ (or T_*S), there exists a monotone curve γ in $T_*^{-1}S$ (resp. T_*S) with x on γ such that γ joins $S(+)$ and $S(-)$.

(3°) The situation of the mapping T_* near the curves of discontinuity is shown in Fig. 3–2.

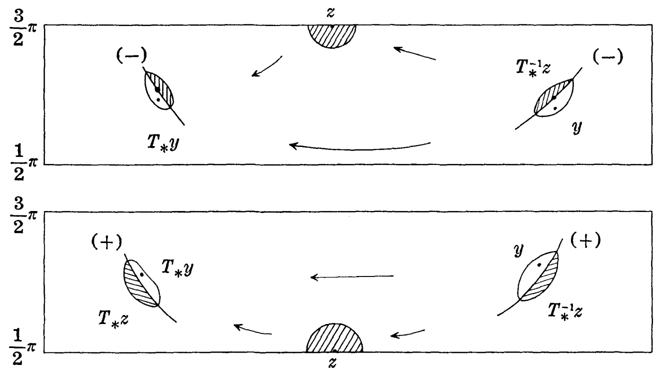

Fig. 3-2

Let γ be a branch of $T_*^{-1}S$ (resp. T_*S) and let W be a small closed neighbourhood of z in γ. If sign $(\gamma) = (+)$, then T^*(resp. T_*^{-1}) is continuous on the closed half part of W below γ and the image intersects with $S(+)$. While if sign $(\gamma) = (-)$, then T^* (resp. T_*^{-1}) is continuous on the closed half part of W above γ and the image intersects with $S(-)$.

(4°) Let $\alpha^{(e)}$ be a partition of M such that each element $X_j^{(e)}$ of $\alpha^{(e)}$ is a maximal connected set on which T_* is continuous. Then $\alpha^{(e)}$ is the partition separated by the curves $T_*^{-1}S$. Let γ be a segment of a branch such that γ is a part of the boundary of $X_j^{(e)}$. Then, γ is included in $X_j^{(e)}$, either if sign $(\gamma) = (+)$ and γ lies above $X_j^{(e)}$ or if sign $(\gamma) = (-)$ and γ lies below $X_j^{(e)}$.

Let $\alpha^{(c)}$ be a partition of M such that each element $X_j^{(c)}$ of $\alpha^{(c)}$ is a maximal connected set on which T_*^{-1} is continuous. Then $\alpha^{(c)}$ is the partition separated by the curves T_*S. Let γ be a segment of branch such that γ is a part of boundary of $X_j^{(c)}$. Then, γ is included in $X_j^{(c)}$, either if sign $(\gamma) = (-)$ and γ lies below $X_j^{(c)}$ or if sign $(\gamma) = (+)$ and γ lies above $X_j^{(c)}$.

Further one can choose the numbering of $\{X_j^{(e)}\}$ and $\{X_j^{(c)}\}$ such that

$T_*X_j^{(e)} = X_j^{(c)}$. Then T_* is a C^2-diffeomorphism from the interior of $X_j^{(e)}$ onto the interior of $X_j^{(c)}$.

(5°) One can see that $\bigcap_i T_*^i S$ consists of at most a finite number of points, say $z(1), z(2), \cdots, z(I_1)$. There exists branches Σ_i^+ of T_*S and Σ_i^- of $T_*^{-1}S$ which contain z(i) as a common end point. There exist an at most countable branches $\Sigma_{i,j}^+$ of T_*S (resp. $\Sigma_{i,j}^-$ of $T_*^{-1}S$), $j = 1, 2, \cdots$, such that one end lies on Σ_i^+ (resp. Σ_i^-) and the other end lies on S. Put $z^+(i, j) \equiv S \cap \Sigma_{i,j}^+$, $z_*^+(i, j) \equiv \Sigma_i^+ \cap \Sigma_{i,j}^+$, $z^-(i, j) \equiv S \cap \Sigma_{i,j}^-$, $z_*^-(i, j) \equiv \Sigma_i^- \cap \Sigma_{i,j}^-$. Then one can choose suffices j's such that distance between $z(i)$ and $z^+(i, j)$ (resp. $z^-(i, j)$) are decreasing with increasing j. The remaining branches $T_*S - \bigcup_{i=1}^{I_1} \Sigma_i^+ - \bigcup_{i=1}^{I_1} \bigcup_j \Sigma_{i,j}^+$ (resp. $T_*^{-1}S - \bigcup_{i=1}^{I_1} \Sigma_i^- - \bigcup_{i=1}^{I_1} \bigcup_j \Sigma_{i,j}^-$) are finite in number, say

$$\Sigma_i^+, \ I_1 + 1 \le i \le I_2 \qquad (\text{resp. } \Sigma_i^-, \ I_1 + 1 \le i \le I_2) .$$

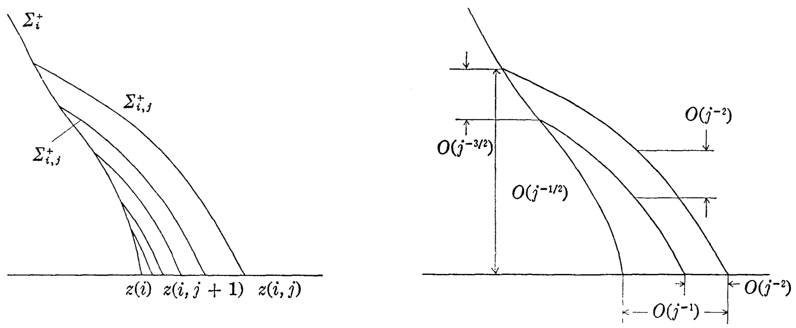

Fig. 3-3

Generally, a decreasing curve γ, $\varphi = \psi(r)$, is said to be *K-decreasing*, if

$$K_{\min} \le -\frac{\psi(r) - \psi(r')}{r - r'} \le K_{\max} \qquad \text{for } r \ne r' .$$

For an increasing curve γ in $M^{(\iota)}$, $\varphi = \psi(r)$, is said to be *K-increasing*, if

$$k_{\min} \le \frac{\psi(r) - \psi(r')}{r - r'} \le K_{\max}(\iota) \qquad \text{for } r \ne r' .$$

LEMMA 3.4. *There exist constants $c_{10} \sim c_{17}$ which admit the following estimates:*

(i) $c_{11}j^{-1/2} \le \theta(\Sigma_{i,j}^+) \le c_{12}j^{-1/2}$, $c_{11}j^{-1/2} \le \theta(\Sigma_{i,j}^-) \le c_{12}j^{-1/2}$,
$c_{11}j^{-3/2} \le \theta(\Sigma_{i,j}^+) - \theta(\Sigma_{i,j+1}^+) \le c_{12}j^{-3/2}$ *and*
$c_{11}j^{-3/2} \le \theta(\Sigma_{i,j}^-) - \theta(\Sigma_{i,j+1}^-) \le c_{12}j^{-3/2}$.

(ii) *Let γ be a K-increasing (resp. K-decreasing) curve which joins $\Sigma^+_{i,j}$ and $\Sigma^+_{i,j+1}$ (resp. $\Sigma^-_{i,j}$ and $\Sigma^-_{i,j+1}$). Then*

$$c_{13}j^{-2} \le \theta(\gamma) \le c_{14}j^{-2} .$$

(iii) *Let $X^+_{i,j}$ (resp. $X^-_{i,j}$) be the element of $\alpha^{(e)}$ (resp. $\alpha^{(c)}$) enclosed by $\Sigma^+_i, \Sigma^+_{i,j}, \Sigma^+_{i,j+1}$ and S (resp. by $\Sigma^-_i, \Sigma^-_{i,j}, \Sigma^-_{i,j+1}$ and S). Then*

$$c_{15}j \le \inf_{x\in X^+_{i,j}} |\tau(T_*^{-1}x)| \le \sup_{x\in X^+_{i,j}} |\tau(T_*^{-}x)| \le c_{15}\, j,$$

$$c_{15}j \le \inf_{x\in X^-_{i,j}} |\tau(x)| \le \sup_{x\in X^-_{i,j}} |\tau(x)| \le c_{16}j ,$$

$$\sup_{x\in X^+_{i,j}, y\in X^+_{i,j+1}} |\tau(T_*^{-1}x) - \tau(T_*^{-1}y)| \le c_{17} ,$$

$$\sup_{x\in X^-_{i,j}, y\in X^-_{i,j+1}} |\tau(x) - \tau(y)| \le c_{17} .$$

(iv) *Let Σ and Σ' be two branches of T_*S (resp. $T_*^{-1}S$) such that Σ lies below (resp. above) Σ' and that* $\operatorname{sign}(\Sigma) = (-)$ *and* $\operatorname{sign}(\Sigma') = (+)$. *Let γ be a K-increasing (resp. K-decreasing) curve which joins Σ and Σ'. Then*

$$\theta(\gamma) \ge c_{10} .$$

(6°) One can choose a suitable numbering of $\{X^{(c)}_j\}$ and $\{X^{(e)}_j\}$ which admits the following lemma for suitablly rechosen constants $c_{11} \sim c_{16}$.

LEMMA 3.5.

(i)
$$c_{11}j^{-1/2} \le \max\cos(X^{(c)}_j) \le c_{12}j^{-1/2} ,$$
$$c_{11}j^{-1/2} \le \max\cos(X^{(e)}_j) \le c_{12}j^{-1/2} .$$

(ii) *Except for a finite number of j's, $X^{(c)}_j$ (resp. $X^{(e)}_j$) is enclosed by three K-decreasing (resp. K-increasing) branches and a segment of S. Let γ be a K-increasing (resp. K-decreasing) curve which joins two sides of $X^{(c)}_j$ (resp. $X^{(e)}_j$) with the same sign. Then*

$$c_{13}j^{-2} \le \theta(\gamma) \le c_{14}j^{-2} .$$

(iii)
$$c_{15}j \le \inf_{x\in X^{(c)}_j} |\tau(T_*^{-1}x)| \le \sup_{x\in X^{(c)}_j} |\tau(T_*^{-1}x)| \le c_{16}j ,$$
$$c_{15}j \le \inf_{x\in X^{(e)}_j} |\tau(x)| \le \sup_{x\in X^{(e)}_j} |\tau(x)| \le c_{16}j .$$

§ 4. Construction of transversal fibres

The purpose of this section is to construct transversal fibres, and to show that $\alpha^{(c)}$ and $\alpha^{(e)}$ are generators and that almost every element of $\zeta^{(c)} \equiv \bigvee_{i=0}^{\infty} T_*^i \alpha^{(c)}$ (and $\zeta^{(e)} \equiv \bigvee_{i=0}^{-\infty} T_*^i \alpha^{(e)}$) is a local fibre. The method of the construction of the transversal fibres is similar to Sinai billiard systems (see [6], [11]).

LEMMA 4.1. *Let C be an element of $\bigvee_{i=0}^{n-1} T_*^i \alpha^{(c)}$ (resp. $\bigvee_{i=0}^{n-1} T_*^{-i} \alpha^{(e)}$), and fix x, y in C.*

(i) *C is a maximal connected set on which T_*^{-n} (resp. T_*^n) is continuous.*

(ii) *The boundary of C consists of several K-decreasing (resp. K-increasing) curves of C^2-class and segments of S.*

(iii) *If x and y are joined by a connected increasing (resp. decreasing) curve, then the curve is included in C.*

(iv) *If x and y are not joined by connected increasing (resp. decreasing) curve, then there exists a decreasing (resp. increasing) curve, which joins x, y and is included in C.*

Proof. The assertion (i) is obvious by (4°) in § 3. (ii) is a consequence of (1°) in § 3 and Lemma 3.2. (iii) and (iv) are obvious by (i), (ii) and the property (2°) in § 3. Q.E.D.

Let $\operatorname{dist}(x, y)$ be the Euclidean distance between x and y in the same $M^{(\iota)}$. Put for $\ell = 0, \pm 1, \pm 2, \cdots$,

$$d^{(\ell)}(x) \equiv \operatorname{dist}\left(x, \bigcup_{i=0}^{\ell} T_*^{-i} S\right). \tag{4.1}$$

LEMMA 4.2.

(i) $$\nu(\{x ; d^{(\ell)}(x) < u\}) \leq p_1(\ell) u^{p(\ell)}$$

with some constant $p_1(\ell)$ and $p(\ell) \equiv (2^{|\ell|+1} - 1)^{-1}$.

(ii) *Put $c_1 \equiv (1 + K_{\min}^{-2})^{1/2}$ and*

$$\Delta^{(\ell)}(x) \equiv \begin{cases} \inf\limits_{0 \leq i < \infty} \dfrac{(1+\eta)^i}{2c_1} d^{(\ell)}(T_*^{-i} x) & \text{if } \ell \geq 1 \\ \inf\limits_{0 \leq i < \infty} \dfrac{(1+\eta)^i}{2c_1} d^{(\ell)}(T_*^{i} x) & \text{if } \ell \leq -1 . \end{cases} \tag{4.2}$$

Then $\Delta^{(\ell)}(x) > 0$ for almost every x and for $\ell \neq 0$.

Proof. From the properties (5°) and (6°) in § 3, it follows that for any $j \geq 1$ and $j' \geq c_{12}^2 c_{13}^{-2} j^2$

$$X_{j'}^{(e)} \cap X_j^{(c)} = \emptyset$$

holds. Hence the intersection $T_*^{-1} S \cap X_j^{(c)}$ consists of K-increasing curves whose number is less than $c_{12}^2 c_{13}^{-2} j^2$. Since $X_j^{(e)} = T_*^{-1} X_j^{(c)}$, $T_*^{-2} S \cap X_j^{(e)}$ consists of K-increasing curves whose number is less than $c_{12}^2 c_{13}^{-2} j^2$. Since by the above discussion $T_*^{-1} S \cap T_*(X_{j'}^{(e)} \cap T_* X_j^{(e)})$ consists of K-decreasing curves whose number is less than $c_{12}^2 c_{13}^{-2} j'^2$, $T_*^{-3} S \cap X_j^{(e)}$ consists of K-decreasing curves whose number is less than

$$\sum_{j'=1}^{[c_{12}^2 c_{13}^{-2} j^2]} c_{12}^2 c_{13}^{-2} j'^2 \leq (c_{12}^2 c_{13}^{-2})^4 j^6 .$$

Recursively, it can be proved that the intersection $T_*^{-\ell} S \cap X_j^{(e)}$ consists of K-increasing curves whose number is less than const. $j^{2^{\ell+1}-2}$. Hence $(\bigcup_{k=1}^{\ell} T_*^{-k} S) \cap X_{i,j}^-$ consists of K-increasing curves whose number is less than const. $j^{2^{\ell+1}-1}$. Therefore, for $\ell \geq 1$ and $\ell_1 = 2^{\ell+1}$

$$\nu(\{x ; d^{(\ell)}(x) < u\}) < \pi u^{p(\ell)} + \text{const.}\, u \sum_{j=1}^{\text{const.}\, u^{-1/\ell_1}} j^{\ell_1 - 2} \leq \text{const.}\, u^{p(\ell)} .$$

holds. For $\ell \leq -1$, one can see similarly. The second assertion is obtained from (i) using the Borel-Cantelli lemma. Q.E.D.

Put

$$\zeta^{(c)} \equiv \bigvee_{i=0}^{\infty} T_*^i \alpha^{(c)} \quad \text{and} \quad \zeta^{(e)} \equiv \bigvee_{i=0}^{\infty} T_*^{-i} \alpha^{(e)} = \bigvee_{i=1}^{\infty} T_*^{-i} \alpha^{(c)} .$$

It will be shown that almost every element of $\zeta^{(c)}$ is a connected curves of C^1-class. Let $\bar{x} = (\bar{t}, \bar{r}, \bar{\varphi})$ be a fixed point with $\Delta^{(1)}(\bar{x}) > 0$, and let C be the element of $\zeta^{(c)}$ which contains $\bar{x}$. Since $\zeta^{(c)} \geq \bigvee_{i=0}^{n-1} T_*^i \alpha^{(c)}$, there exists the element Y_n of $\bigvee_{i=0}^{n-1} T_*^i \alpha^{(c)}$ which includes C. Therefore T_*^{-n} is continuous on C (of course on Y_n) by Lemma 4.1. Note that $T_*^{-n} Y_n$ is an element of $\bigvee_{i=1}^{n} T_*^{-i} \alpha^{(c)}$.

Let $\gamma_n^{(n)}$ be a K-decreasing curve of C^1-class passing through $\bar{x}_n \equiv T_*^{-n} \bar{x}$ such that

$$\bar{\theta}(\gamma_n^{(n)}, \bar{x}_n) = \underline{\theta}(\gamma_n^{(n)}, \bar{x}_n) = (1+\eta)^{-n} \Delta^{(1)}(\bar{x}) .$$

By definition, $(1+\eta)^{-n} \Delta^{(1)}(\bar{x}) \leq d^{(1)}(\bar{x}_n)/2c_1$. Hence for any y in $\gamma_n^{(n)}$, the inequality $d(y) \geq \frac{1}{2} d^{(1)}(\bar{x}_n)$ holds, since $\text{dist}(\bar{x}_n, y) \leq d^{(1)}(\bar{x}_n)/2$. Therefore T_* is continuous on $\gamma_n^{(n)}$. By Lemma 3.3, $T_* \gamma_n^{(n)}$ is a connected K-decreasing curve and satisfies the inequality

$$\min\{\bar{\theta}(T_*\gamma_n^{(n)}, \bar{x}_{n-1}), \underline{\theta}(T_*\gamma_n^{(n)}, \bar{x}_{n-1})\} \geq (1+\eta)^{-n+1}\Delta^{(1)}(\bar{x}) .$$

Therefore one can choose a connected segment $\gamma_{n-1}^{(n)}$ of $T_*\gamma_n^{(n)}$ such that

$$\bar{\theta}(\gamma_{n-1}^{(n)}, \bar{x}_{n-1}) = \underline{\theta}(\gamma_{n-1}^{(n)}, \bar{x}_{n-1}) = (1+\eta)^{-n+1}\Delta^{(1)}(\bar{x}) .$$

By the same reason in above, one can choose a sequence of connected K-decreasing curves of C^1-class such that

$$\bar{x}_i \in \gamma_i^{(n)} \subset T_*\gamma_{i+1}^{(n)} , \qquad i = 0, 1, 2, \cdots, n-1 .$$
$$\bar{\theta}(\gamma_i^{(n)}, \bar{x}_i) = \underline{\theta}(\gamma_i^{(n)}, \bar{x}_i) = (1+\eta)^i\Delta^{(1)}(\bar{x}) \qquad i = 0, 1, \cdots, n .$$

And T_* is continuous on $\gamma_i^{(n)}$, $1 \leq i \leq n$. In particular,

$$\bar{\theta}(\gamma_0^{(n)}, \bar{x}) = \underline{\theta}(\gamma_0^{(n)}, \bar{x}) = \Delta^{(1)}(\bar{x})$$

and T_*^{-n} is continuous on $\gamma_0^{(n)}$. Furthermore,

$$\text{(4.4)} \qquad \operatorname{dist}(T_*^{-i}\gamma_0^{(n)}, S \cup T_*^{-1}S) \geq \tfrac{1}{2}d^{(1)}(\bar{x}_i) \qquad 0 \leq i \leq n .$$

Hence $\gamma_0^{(n)}$ is included in Y_n. Thus for any $n \geq 1$, there exists a connected K-decreasing curve $\gamma_0^{(n)}$ of C^1-class which is defined on the interval $[\bar{\varphi} - \Delta^{(1)}(\bar{x}), \bar{\varphi} + \Delta^{(1)}(\bar{x})]$ and is included in Y_n. Let $\hat{\gamma}^{(n)}$ be a segment of the line given by the equation $\varphi = \hat{\varphi}$ for a fixed $\hat{\varphi}$ in the interval such that the segment $\hat{\gamma}^{(n)}$ joins $\gamma_0^{(n)}$ and $\gamma_0^{(n+1)}$. By Lemma 4.1, $\hat{\gamma}^{(n)}$ is included in Y_n, and hence $\rho(\hat{\gamma}^{(n)}) \leq (1+\eta)^{-n}\rho(T_*^{-n}\hat{\gamma}^{(n)})/|\cos\hat{\varphi}| \leq (1+\eta)^{-n}\pi/|\cos\hat{\varphi}|$ by Lemma 3.3 (i). Therefore $\sum_{n=1}^{\infty}\rho(\hat{\gamma}_n) < \infty$ and hence $\gamma_0^{(n)}$ converges uniformly in $[\bar{\varphi} - \Delta^{(1)}(\bar{x}), \bar{\varphi} + \Delta^{(1)}(\bar{x})]$ as $n \to \infty$. Let γ_0 be the limit curve of $\{\gamma_0^{(n)}\}$. Then by (4.4)

$$\operatorname{dist}(T_*^{-i}\gamma_0, S \cup T_*^{-1}S) \geq \tfrac{1}{2}d^{(1)}(\bar{x}_i) \qquad \text{for } i \geq 0$$

holds, and of course $\gamma_0 \subset Y_n$ for all $n \geq 0$. Therefore C includes the curve γ_0. Now it will be proved that C is a curve. Let y be a point in C which is different from $\bar{x}$. Then $\bar{x}$ and y are joined by a decreasing curve. In fact, suppose the contrary, then there exists a point z in C such that $r(z) = r(y)$, $\varphi(z) = \varphi(\bar{x})$, $z \neq y$ and $z \neq \bar{x}$. Let γ be the horizontal line which joins $\bar{x}$ and z. Then for any $n \geq 1$

$$\rho(\gamma) \leq \frac{(1+\eta)^{-n}}{|\cos\varphi(\bar{x})|}\rho(T_*^{-n}\gamma) \leq \frac{\pi(1+\eta)^{-n}}{K_{\min}|\cos\varphi(\bar{x})|} .$$

Hence $\rho(\gamma) = 0$; that is, $r(y) = r(\bar{x})$. Thus the above assertion was proved. Since $T_*^{-n}\zeta^{(c)} = \bigvee_{i=-n}^{\infty} T_*^i\alpha^{(c)} \geq \zeta^{(c)}$, $T_*^{-n}C$ is included in an element

C' of $\zeta^{(c)}$. Hence $T_*^{-n}x$ and $T_*^{-n}y$ are joined by a decreasing curve $\bar{\gamma}_n^{(n)}$ in $T_*^{-n}Y_n$. By the same reason in the above, $T_*^n\bar{\gamma}_n^{(n)}$ converges to a curve $\bar{\gamma}_0$ which contains $\bar{x}$ and y. Furthermore T_*^n is continuous on $\bar{\gamma}_0$ for all $n \geq 0$. Therefore C is a curve.

Denote by $\gamma^{(c)}(\bar{x})$ the element of $\zeta^{(c)}$ which is a K-decreasing curve passing through $\bar{x}$. Then $T_*^n\gamma^{(c)}(T_*^{-n}\bar{x})$ is the element of $T_*^n\zeta^{(c)}$ which contains x, and is an at most countable union of curves which are elements of $\zeta^{(c)}$. Put $\Gamma^{(c)}(\bar{x}) \equiv \bigcup_{n\geq 0} T_*^n\gamma^{(c)}(T_*^{-n}\bar{x})$. Then $\Gamma^{(c)}(x)$ is a countable union of curves which are elements of $\zeta^{(c)}$. The connected component of $\bar{x}$ in $\Gamma^{(c)}(\bar{x})$ coincides with $\gamma^{(c)}(\bar{x})$. By the Borel-Cantelli Lemma, for almost every $\bar{x}$ the inequality $d^{(1)}(T_*^{-j}\bar{x}) \geq 2\pi(1+\eta)^{-j}$ holds for all sufficiently large j's. Hence the estimate

$$\theta(T_*^{-j}\gamma^{(c)}(\bar{x})) \leq \pi(1+\eta)^{-j} \leq \tfrac{1}{2}d^{(1)}(T_*^{-j}\bar{x})$$

is obtained. Therefore for z in $\gamma^{(c)}(\bar{x})$

$$d^{(1)}(T^{-j-i}z) \geq \tfrac{1}{2}d^{(1)}(T_*^{-i-j}\bar{x})$$

and hence

$$\inf_{j\geq 0} \frac{(1+\eta)^{-j-i}}{2c_1} d^{(1)}(T^{-j-i}z) \geq \frac{(1+\eta)^{-i}}{2} \Delta^{(1)}(T_*^{-i}\bar{x}) \geq \frac{1}{2}\Delta^{(1)}(\bar{x}) > 0 .$$

Since z is not in $\bigcup_{j=0}^{i} T_*^{-j}S$, $\Delta^{(1)}(z) > 0$ holds for any z in $\gamma^{(c)}(\bar{x})$. Thus, for almost every $\bar{x}$ and for every z in $\gamma^{(c)}(\bar{x})$, $\Delta^{(1)}(z) > 0$.

In order to show that $\gamma^{(c)}(\bar{x})$ belongs to C^1-class and to calculate the gradient, it is useful to prepare the following lemma. Define functions by

$$(4.5)\quad \begin{cases} b_{-1}(x;t) \equiv -k_{-1} - \dfrac{(\cos\varphi + k\tau)t - \tau}{\{k\cos\varphi_{-1} + k'_{-1}\cos\varphi + kk'_{-1}\tau\}t - (\cos\varphi_{-1} + k'_{-1}\tau)}\,, \\ b_0(x;t) \equiv t\,, \\ b_1(x;t) \equiv \dfrac{(\cos\varphi + k'\tau_1)(h+t) + \tau_1}{\{k_1\cos\varphi + k'\cos\varphi_1 + k_1k'\tau_1\}(h+t) + \cos\varphi_1 + k_1\tau_1}\,, \end{cases}$$

where $x_i = (\iota_i, r_i, \varphi_i) \equiv T_*^{-i}x$ and the notations in § 3 are used. Define a sequence of functions recursively by

$$(4.6)\quad \begin{cases} b_{-n-1}(x;t) \equiv b_{-1}(T_*^n x; b_{-n}(x;t)) \\ b_{n+1}(x;t) \equiv b_1(T_*^{-n}x; b_n(x;t)) \end{cases}$$

for $n \geq 1$.

LEMMA 4.3. (i) *Let γ be a curve of C^1-class given by the equation $r = u(\varphi)$. Suppose that $\gamma_i \equiv T_*^{-i}\gamma$ is defined by the equation $r_i = u_i(\varphi_i)$, with $(\iota_i, u_i(\varphi_i), \varphi_i) = T_*^{-i}(\iota, u(\varphi), \varphi)$. Then, for any i,*

$$\frac{du_i}{d\varphi_i} = b_i\Big((\iota, u(\varphi), \varphi)\,;\, \frac{du}{d\varphi}\Big)\,.$$

(ii) *When $t \geq 1/K_{\max}(\iota)$ and $n \geq 0$,*

$$\frac{1}{K_{\max}(\iota_n)} \leq b_n(x\,;t) \leq 1/K_{\min}$$

with $x_n = (\iota_n, r_n, \varphi_n) \equiv T_*^{-n}x$. When $t \leq 0$ and $n \leq 0$,

$$\frac{1}{K_{\max}} \leq -b_n(x\,;t) \leq \frac{1}{K_{\min}}\,.$$

(iii) *When $t \leq 0$ and $n \leq 0$,*

$$0 \leq \frac{d}{dt}b_n(T_*^n x\,;t) \leq \frac{\cos\varphi_n}{\cos\varphi}(1+\eta)^{-2n}$$

and $b_n(T_*^{-n}x\,;t)$ converges uniformly in wide sense as $n \to -\infty$ in $(M - S) \times (-\infty, 0]$ to a function independent of t which will be denoted by $1/\chi^{(c)}(x)$. Further $\chi^{(c)}(x)$ is continuous on $M - \bigcup_{j=0}^{\infty} T_*^j S$.

(iv) *When $t \geq 1/K_{\max}(\iota_n)$ and $n \geq 0$,*

$$0 \leq \frac{d}{dt}b_n(T_*^n x\,;t) \leq \frac{\cos\varphi_n}{\cos\varphi}(1+\eta)^{-2n}$$

and $b_n(T_*^n x\,;t)$ converges uniformly in wide sense as $n \to -\infty$ in $(M - S) \times [1/K_{\max}(\iota), \infty)$ to a function independent of t, which will be denoted by $1/\chi^{(e)}(x)$. Further $\chi^{(e)}(x)$ is continuous on $M - \bigcup_{j=1}^{\infty} T_* S$.

Proof. By Lemma 3.2, (i) is obviously seen. By Lemma 3.3, (ii) is obvious. Since

$$\frac{d}{dt}b_{-1}(x_{i+1}\,;t)$$
$$= \frac{\cos\varphi_i\cos\varphi_{i+1}}{[\{k_{i+1}\cos\varphi_i + k_i'\cos\varphi_{i+1} + k_{i+1}k_i'\tau_{i+1}\}t - (\cos\varphi_i + k_i'\tau_{i+1})]^2}\,,$$

$$0 \leq \frac{d}{dt}b_{-1}(x_{i+1}\,;t) \leq \frac{\cos\varphi_{i+1}}{\cos\varphi_i}(1+\eta)^{-2}$$

holds. Therefore the inequality in (iii) is true. Since

$$|b_{n-2}(T_*^{n-1}x\,;t)-b_{n-1}(T_*^{n-1}x\,;s)|$$
$$\leq \frac{\cos\varphi_{-n}}{\cos\varphi}(1+\eta)^{-2n}|b_{-2}(x_{-n+2}\,;t)-b_{-1}(x_{-n+1}\,;s)|$$

holds, $b_n(T_*^n x\,;t)$ converges uniformly in wide sense as $n\to\infty$ to a function independent of t by (ii). Since $b_n(T_*^n x\,;t)$ is continuous on $M-\bigcup_{j=0}^{\infty}T_*^{-j}S$, $\chi^{(c)}(\iota,r,\varphi)$ is continuous. The assertion (iv) is shown similarly. Q.E.D.

Fix $\bar{x}$ with $\Delta^{(1)}(\bar{x})>0$. Suppose that the curves $\gamma^{(c)}(\bar{x})$ and $T_*^{-n}\gamma^{(c)}(\bar{x})$ are represented by the equations $r=u(\varphi)$ and $r=u_n(\varphi)$ respectively. Since the curves $\gamma^{(c)}(\bar{x})$ and $T_*^{-n}\gamma^{(c)}(\bar{x})$ are K-decreasing, $u(\varphi)$ and $u_n(\varphi)$ are absolutely continuous. By Lemma 4.3 (i), it is easily seen that for almost every φ

$$\frac{du}{d\varphi}=b_{-n}\left((\iota_n,u_n(\varphi_n),\varphi_n)\,;\frac{du_n}{d\varphi_n}\right)$$

holds with $(\iota_n,u_n(\varphi_n),r_n)=T_*^{-n}(\iota,u(\varphi),r)$. By Lemma 4.3 (iii), the right hand term converges to $\chi^{(c)}(\iota,u(\varphi),\varphi)^{-1}$. Hence for almost every φ

$$\frac{du}{d\varphi}=\chi^{(c)}(\iota,u(\varphi),\varphi)^{-1} \tag{4.7}$$

holds. Since $\gamma^{(c)}(\bar{x})$ is included in $M-\bigcup_{j=0}^{\infty}T_*^{j}S$, $\chi^{(c)}(\iota,u(\varphi),\varphi)$ is continuous in φ. Therefore, $\gamma^{(c)}(\bar{x})$ is in C^1-class and has the gradient $\chi^{(c)}(x)$ at x in $\gamma^{(c)}(\bar{x})$.

Similarly, almost every element $\zeta^{(e)}=\bigvee_{i=0}^{\infty}T^i\alpha^{(e)}$ is an increasing curve passing through $\bar{x}$ which is denoted by $\gamma^{(e)}(\bar{x})$. Then $\Gamma^{(e)}(\bar{x})\equiv\bigcup T_*^i\gamma^{(e)}(T_*^{-i}\bar{x})$ is a countable union of the curves which are elements of $\zeta^{(e)}$. Furthermore $\gamma^{(e)}(\bar{x})$ is the connected component of $\bar{x}$ in $\Gamma^{(e)}(\bar{x})$. The gradient at x is given by $\chi^{(e)}(x)$, where $\chi^{(e)}(x)$ is the limit of $b_n(T_*^n x\,;t)^{-1}$ as $n\to\infty$ with $t\geq 1/K_{\max}(\iota)$. Thus the following theorem was obtained.

THEOREM 1. *Let $\zeta^{(c)}$ and $\zeta^{(e)}$ be the partitions defined by*

$$\zeta^{(c)}\equiv\bigvee_{i=0}^{\infty}T_*^i\alpha^{(c)}\quad aud\quad \zeta^{(e)}\equiv\bigvee_{i=0}^{\infty}T_*^{-i}\alpha^{(e)}\,.$$

Then almost every element of $\zeta^{(c)}$ (resp. $\zeta^{(e)}$) is a connected K-decreasing (resp. K-increasing) curve of C^1-class, on which T_*^{-n} (resp. T_*^n) is continuous for any $n\geq 0$. The curve $\gamma^{(c)}(\bar{x})$ (resp. $\gamma^{(e)}(\bar{x})$) is a solution curve of the equation

$$\frac{d\varphi}{dr} = \chi^{(c)}(\iota, r, \varphi) \qquad \left(resp.\ \frac{d\varphi}{dr} = \chi^{(e)}(\iota, r, \varphi)\right),$$

where $\chi^{(c)}(x)$ *(resp.* $\chi^{(e)}(x)$*) is defined by*

$$\chi^{(c)}(x) \equiv \frac{1}{\lim\limits_{n\to-\infty} b_n(T_*^n x\,;\,-\infty)} \qquad \left(resp.\ \chi^{(e)}(x) \equiv \frac{1}{\lim\limits_{n\to\infty} b_n(T_*^n x\,;\,\infty)}\right).$$

The curve $\gamma^{(c)}(\bar{x})$ (resp. $\gamma^{(e)}(\bar{x})$) is called *the locally contracting* (resp. *expanding*) *transversal fibre of* $\bar{x}$, and the union of curves $\Gamma^{(c)}(\bar{x})$ (resp. $\Gamma^{(e)}(\bar{x})$) is called *the complete contracting* (resp. *expanding*) *transversal fibre of* $\bar{x}$.

In order to show more precise results, refer to a theorem of V. I. Rohlin = Ya. G. Sinai [8]. The proof will be omitted, however one can refer to Appendix 9 in [6].

LEMMA 4.4. *Let T be a given measure preserving transformation on a Lebesgue space.*

(i) *Let* ξ *be a measurable partition such that*

$$T\xi > \xi\,, \qquad \bigvee T^k\xi = \varepsilon\,, \qquad h(T\xi\,|\,\xi) = h(T) < \infty\,.$$

Then $\bigwedge T^k\xi = \pi(T)$.

(ii) *Let* α *be a countable partition with entropy* $H(\alpha) < \infty$. *Put* $\xi = \bigvee_{k=-\infty}^{0} T^k\alpha$. *If* $\bigvee_k T^k\xi = \varepsilon$, *then* $h(T\xi\,|\,\xi) = h(T)$ *and* $\bigwedge_k T^k\xi = \pi(T)$.

(iii) $\pi(T) = \pi(T^{-1})$.

THEOREM 2. (i) $\alpha^{(c)}$ *and* $\alpha^{(e)}$ *have the same finite entropy.*

(ii) $\zeta^{(c)} = \bigvee_{i=0}^{\infty} T_*^i\alpha^{(c)}$ *and* $\zeta^{(e)} = \bigvee_{i=0}^{-\infty} T_*^i\alpha^{(e)}$ *satisfy*

$$T_*^{-1}\zeta^{(c)} > \zeta^{(c)}\,, \qquad T_*\zeta^{(e)} > \zeta^{(e)}$$

$$\bigvee_{i=-\infty}^{0} T_*^i\zeta^{(c)} = \bigvee_{i=0}^{\infty} T_*^i\zeta^{(e)} = \varepsilon\,,$$

$$\bigwedge_{i=1}^{\infty} T_*^i\zeta^{(c)} = \bigwedge_{i=-\infty}^{-1} T_*^i\zeta^{(e)} = \pi(T)\,.$$

(iii) $h(T_*^{-1}\zeta^{(c)}\,|\,\zeta^{(c)}) = h(T_*\zeta^{(e)}\,|\,\zeta^{(e)}) = h(T)$.

(iv) *The partition* $\zeta_\infty^{(c)} \equiv \bigwedge_{i=1}^{\infty} T_*^i\zeta^{(c)}$ *(resp.* $\zeta_{-\infty}^{(e)} \equiv \bigwedge_{i=-\infty}^{-1} T_*^i\zeta^{(e)}$*) is the measurable covering of the partition into* $\{\Gamma^{(c)}(x)\}$ *(resp.* $\{\Gamma^{(e)}(x)\}$*).*

Proof. By Lemma 3.4, the estimate

$$\frac{\nu_0}{K_{\max}} c_{11}^2 c_{13}(j+1)^{-3} \le \nu(X_{i,j}^+) \le \frac{\nu_0}{2K_{\min}} c_{12}^2 c_{12} j^{-3}$$

is true. Therefore (i) is true. Since $\theta(T_*^{-n}\gamma^{(c)}(x)) \leq \pi(1+\eta)^{-n}$ for $n \geq 0$, $\{T_*^{-n}\zeta^{(c)}; n \geq 0\}$ separates any pair of different points. Hence $\bigvee_{i=-\infty}^{0} T_*^i\zeta^{(c)} = \varepsilon$. By Lemma 4.4, the other equalities in (ii) and (iii) are shown. (iv) is obvious by definition. Q.E.D.

§ 5. Lemmas

In § 6 ~ § 8, certain measure theoretical regularities of the partition $\zeta^{(c)}$ and $\zeta^{(e)}$ will be discussed. By using those regularities, it will be shown that $\pi(T)$ is the trivial partition $\{M, \phi\}$. The fact implies that T_* is a K-system by virtue of Theorem 2. In this section several lemmas for those sections will be prepared.

Let $\{b_n(x; t); n = 0, \pm 1, \pm 2, \cdots\}$ be the sequence of functions on $M \times (-\infty, \infty)$ defined (4.5) and (4.6). Let γ be a curve of C^1-class in $M^{(\iota)}$ defined by $r = u(\varphi)$. Put

(5.1)
$$\Lambda(x;\gamma) \equiv -\frac{k_1\cos\varphi + k'\cos\varphi_1 + k_1k'\tau_1}{\cos\varphi_1}\left\{\frac{du}{d\varphi} + h\right\} - \frac{k_1\tau_1}{\cos\varphi_1} - 1$$
$$\Lambda^*(x;\gamma) \equiv \frac{k_1\cos\varphi + k'\cos\varphi_1 + k_1k'\tau_1}{\cos\varphi} b_1\left(x; \frac{du}{d\varphi}\right) - \frac{k'\tau_1}{\cos\varphi} - 1\,,$$

with $x = (\iota, u(\varphi), \varphi)$ and $x_1 = (\iota_1, u_1(\varphi_1), \varphi_1) \equiv T_*^{-1}x$.

LEMMA 5.1. *Let γ, Λ and Λ^* be as in above.*

(i) $d\varphi_1/d\varphi = \Lambda(x;\gamma) = 1/\Lambda^*(x;\gamma)$.

(ii) *If γ is K-increasing, then*

$$-\Lambda(x;\gamma) \geq 1+\eta \quad \textit{and} \quad \cos\varphi_1\Lambda(x;\gamma) \geq \eta\,.$$

(iii) *If $T_*^{-1}\gamma$ is K-decreasing, then*

$$-\Lambda^*(x;\gamma) \geq 1+\eta \quad \textit{and} \quad \cos\varphi\Lambda^*(x;\gamma) \geq \eta\,.$$

Proof. The assertions come from Lemma 3.2 and Lemma 3.3, evidently. Q.E.D.

Let γ be an either K-increasing or K-decreasing curve of C^1-class in $M^{(\iota)}$ which is defined by the equation $r = u(\varphi)$, and let $a(\iota, u(\varphi), \varphi) = a(\varphi)$ be a function defined on γ.

LEMMA 5.2. *For suitable positive constants C_{19}, C_{20} and η_1, the following holds.*

(i) *If $a < 0$, then*

$$\frac{1}{K_{\max}} \leq -b_{-1}(x\,;a(\varphi)) \leq \frac{1}{K_{\min}}\,,$$

$$\left|\frac{d}{d\varphi_1}\log(-b_{-1}(x,a))\right| \leq c_{19} + c_{20}\left|\frac{d\varphi}{d\varphi_1}\right| + (1+\eta_1)^{-1}\left|\frac{d\varphi}{d\varphi_1}\log(-a(\varphi))\right|\,.$$

(ii) *If* $a \geq 1/K_{\max}(\iota)$, *then*

$$\frac{1}{K_{\max}(\iota_{-1})} \leq b_1(x\,;a(\varphi)) \leq \frac{1}{K_{\min}}\,,$$

$$\left|\frac{d}{d\varphi_{-1}}\log b_1(x,a)\right| \leq c_{19} + c_{20}\left|\frac{d\varphi}{d\varphi_{-1}}\right| + (1+\eta_1)^{-1}\left|\frac{d}{d\varphi_{-1}}\log a(\varphi)\right|\,.$$

Remark. The equalities in Lemma 5.1 hold with the constant $\eta_1 = (K_{\min}/K_{\max})(1+\eta)^2$. However it is convenient to define η_1 by $\eta_1 \equiv \min\{\eta, K_{\min}(1+\eta)/K_{\max}\}$.

Proof. The first inequality is obviously true by Lemma 4.3 (ii). Evidently, $(\partial/\partial k_1)\log(-b_1)$, $(\partial/\partial k')\log(-b_1)$, $(\partial/\partial(\cos\varphi))\log(-b_1)$ and $(\partial/\partial(\cos\varphi_1))\log(-b_1)$ are bounded. Moreover, $dk_1/d\varphi_1$, $d\cos\varphi_1/d\varphi_1$, $dk'/d\varphi$ and $(d\cos\varphi)/d\varphi$ are bounded. The expression

$$\left|\frac{d\tau_1}{d\varphi_1}\frac{\partial}{\partial\tau_1}\log(-b_{-1})\right|$$
$$= \frac{\cos\varphi(k_1b_{-1}-1)^2|\sin(\varphi+\varphi_1)+\sin\varphi(k_1-d\varphi_1/du_1)\tau_1|}{[\xi a-\cos\varphi-k'\tau_1][\xi ha+(\cos\varphi_1+k_1\tau_1)a-(\cos\varphi+k'\tau_1)h-\tau_1]}$$

is bounded, where $\xi = k_1\cos\varphi + k'\cos\varphi_1 + k_1k'\tau_1$. Further

$$\left|\frac{\partial}{\partial a}\log(-b_{-1}(x\,;a))\right|$$
$$= \frac{\cos\varphi\cos\varphi_1}{[\xi a-\cos\varphi-k'\tau_1][\xi h+\cos\varphi_1+k_1\tau_1-\{(\cos\varphi+k'\tau_1)h+\tau_1\}1/a]\,|a|}$$
$$\leq \frac{\cos\varphi\cos\varphi_1}{[1-\xi a/(\cos\varphi+k'\tau_1)][\{(\cos\varphi+k'\tau_1)h+\tau_1\}\{\xi-(\cos\varphi+k'\tau_1)/a\}+\cos\varphi\cos\varphi_1]\,|a|}$$
$$\leq \left[1+\frac{k_{\min}}{K_{\max}}(1+\eta)^2\right]|a|^{-1}\,.$$

Therefore (i) is true. The proof of (ii) is similar. Q.E.D.

LEMMA 5.3. *For a function* $a(\varphi)$ *on* γ, *defined* a_n *by*

$$a_n(\varphi) = b_n(\iota, u(\varphi), \varphi\,; a(\varphi))\,.$$

Then for $n \geq 0$, *the following holds with a constant* c_{21}.

Case 1. *If* $a \geq 0$ *and* γ *is K-increasing, then*

$$\left|\frac{d}{d\varphi_n} \log a_n\right| \leq c_{21} + (1+\eta_1)^{-n}\left|\frac{d}{d\varphi_n} \log a\right| .$$

Case 2. *If* $a \geq 0$ *and* γ *is K-decreasing, then*

$$\left|\frac{d}{d\varphi} \log a_n\right| \leq (1+\eta_1)^{-n} c_{21} + (1+\eta_1)^{-n}\left|\frac{d}{d\varphi} \log a\right| .$$

Case 3. *If* $a \leq 0$ *and* γ *is K-increasing, then*

$$\left|\frac{d}{d\varphi} \log (-a_{-n})\right| \leq (1+\eta_1)^{-n} c_{21} + (1+\eta_1)^{-n}\left|\frac{d}{d\varphi} \log a\right| .$$

Case 4. *If* $a \leq 0$ *and* γ *is K-decreasing, then*

$$\left|\frac{d}{d\varphi_{-n}} \log (-a_n)\right| \leq c_{21} + (1+\eta_1)^{-n}\left|\frac{d}{d\varphi_{-n}} \log a\right| .$$

Proof. By using Lemma 5.1 repeatedly, one can obtain the results with $c_{21} = c_{19}/\eta_1(1+\eta_1) + c_{20}/\eta_1$. Q.E.D.

Let $\hat{\gamma}$ and $\hat{\hat{\gamma}}$ be two connected K-decreasing curves in $M^{(\iota)}$ such that $\hat{\gamma}_j \equiv T_*^{-j}\hat{\gamma}$ and $\hat{\hat{\gamma}}_j \equiv T_*^{-j}\hat{\hat{\gamma}}$ are also connected K-decreasing curves which are defined by the equations $r_j = \hat{u}_j(\varphi_j)$ and $r_j = \hat{\hat{u}}_j(\varphi_j)$ respectively, $j = 0, 1, 2, \cdots, m$. Let γ and γ' be K-increasing curves which intersect with both $\hat{\gamma}$ and $\hat{\hat{\gamma}}$ and given by the equations $r = u(\varphi)$ and $r = u'(\varphi)$ respectively. Suppose that T_*^{-m} is continuous on γ and γ'. Put $\hat{x}_j = (\iota_j, \hat{r}_j, \hat{\varphi}_j) \equiv T_*^{-j}(\gamma \cap \hat{\gamma})$, $\hat{\hat{x}}_j = (\iota_j, \hat{\hat{r}}_j, \hat{\hat{\varphi}}_j) \equiv T_*^{-j}(\gamma \cap \hat{\hat{\gamma}})$, $\hat{x}'_j = (\iota_j, \hat{r}'_j, \hat{\varphi}'_j) \equiv T_*^{-j}(\gamma' \cap \hat{\gamma})$, $\gamma_j = T_*^{-j}\gamma$ and $\gamma'_j = T_*^{-j}\gamma'$, $j = 0, 1, 2, \cdots, m$.

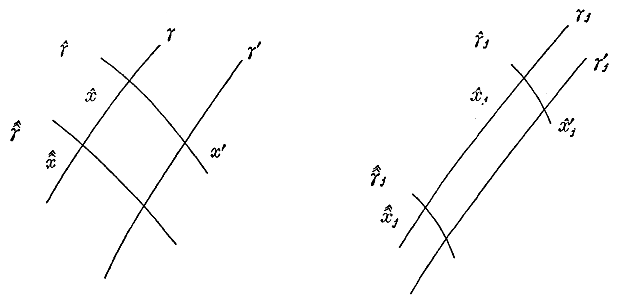

Fig. 5-1

LEMMA 5.4. *The following estimates hold with a constant* c_{22}.

(i) $$\left|\log \frac{A^*(\hat{\hat{x}}_j, \hat{\hat{\gamma}}_j)}{A^*(\hat{x}_j, \hat{\gamma}_j)}\right| \le \frac{c_{22}(1+\eta_1)^{j-m}\theta(\gamma_m)}{\min\cos(\gamma_j \cup \gamma_{j+1})} + (1+\eta_1)^{j-m}\left|\log \frac{d\hat{\hat{u}}_m}{d\hat{\hat{\varphi}}_m} \Big/ \frac{d\hat{u}_m}{d\hat{\varphi}_m}\right|$$

for $0 \le j \le m-1$.

(ii) $$\left|\log \frac{A(\hat{\hat{x}}_j, \gamma_j)}{A(\hat{x}_j, \gamma_j)}\right| \le \frac{c_{22}(1+\eta_1)^{j-m}\theta(\gamma_m)}{\min\cos(\gamma_j \cup \gamma_{j+1})} + (1+\eta_1)^{-j}\left|\log \frac{du}{d\hat{\hat{\varphi}}_0} \Big/ \frac{du}{d\hat{\varphi}_0}\right|$$

for $0 \le j \le m-1$.

(iii) $$\left|\log \frac{A(\hat{x}'_j, \gamma'_j)}{A(\hat{x}_j, \gamma_j)}\right| \le \frac{c_{22}(1+\eta_1)^{-j}\theta(\hat{\gamma})}{\min\cos(\gamma_j \cup \gamma_{j+1})} + (1+\eta_1)^{-j}\left|\log \frac{du'}{d\hat{\varphi}'_0} \Big/ \frac{du}{d\hat{\varphi}_0}\right|$$

for $0 \le j \le m-1$.

Proof. By Lemma 3.2, the following estimates are obtained:

$$|\log k(\hat{\hat{x}}_j)/k(\hat{x}_j)| \le \max_{\iota, r}\left|\frac{dk(\iota, r)}{dr}\right| \cdot \frac{(1+\eta)^{-m+j}}{k_{\min}K_{\min}}\theta(\gamma_m) \qquad 0 \le j \le m ,$$

$$|\log k'(\hat{\hat{x}}_j)/k'(\hat{x}_j)| \le \max_{\iota, r}\left|\frac{dk(\iota, r)}{dr}\right| \frac{2(1+\eta)^{-m+j}}{k_{\min}K_{\min}}\theta(\gamma_m) \qquad 0 \le j \le m ,$$

$$\left|\log \frac{\tau(\hat{\hat{x}}_j)}{\tau(\hat{x}_j)}\right| \le \frac{2+K_{\max}}{\eta}\, \frac{(1+\eta)^{-m+j}\theta(\gamma_m)}{\min\cos\theta(\gamma_{j-1})}, \qquad 1 \le j \le m .$$

$$\left|\log \frac{\cos\varphi(\hat{\hat{x}}_j)}{\cos\varphi(\hat{x}_j)}\right| \le \frac{(1+\eta)^{-m+j}\theta(\gamma_m)}{\min\cos\theta(\gamma_j)}, \qquad 0 \le j \le m .$$

For example the estimate for τ is shown by the inequality

$$\begin{aligned}
&\left|\frac{d}{d\varphi_m}\log(-\tau(\iota_j, u_j(\varphi_j), \varphi_j))\right| \\
&= \left|\frac{1}{\tau_j}\frac{du_j}{d\varphi_j}\frac{d\varphi_j}{d\varphi_m}\right|\left(\sin\varphi_j + \sin\varphi_{j-1}\left\{\frac{\cos\varphi_j}{\cos\varphi_{j-1}} \right.\right. \\
&\qquad\qquad \left.\left. + \frac{\tau_j}{\cos\varphi_{j-1}}\left(k_1 - \frac{d\varphi_j}{du_j}\right)\right\}\right) \\
&\le \frac{1}{\tau_{\min}k_{\min}}(1+\eta)^{-m+j}\left\{1 + \frac{-1}{\cos\varphi_{j-1}}(1+K_{\max})\right\} \\
&\le \frac{2+K_{\max}}{\eta\,|\cos\varphi_{j-1}|}(1+\eta)^{-m+j} .
\end{aligned}$$

Applying Lemma 5.3 Case 3 to $a(\varphi)$ defined by

$$a(\varphi) = -\exp\left[\frac{\hat{\hat{\varphi}}_m - \varphi_m}{\hat{\hat{\varphi}}_m - \hat{\varphi}_m}\log\left(-\frac{d\hat{u}_m}{d\hat{\varphi}_m}\right) + \frac{\varphi_m - \hat{\varphi}_m}{\hat{\hat{\varphi}}_m - \hat{\varphi}_m}\log\left(-\frac{d\hat{\hat{u}}_m}{d\hat{\hat{\varphi}}_m}\right)\right],$$

the following estimate is obtained

$$\left|\frac{d}{d\varphi_m}\log(-a_{j-m})\right| \leq (1+\eta_1)^{-m+j}c_{21} + (1+\eta_1)^{-m+j}\frac{1}{|\hat{\hat{\varphi}}_m - \hat{\varphi}_m|}\left|\log\frac{d\hat{\hat{u}}_m}{d\hat{\hat{\varphi}}_m}\Big/\frac{d\hat{u}_m}{d\hat{\varphi}_m}\right|.$$

Since $a_{j-m}(\iota_m, \hat{u}_m(\hat{\varphi}_m), \hat{\varphi}_m) = d\hat{u}_j/d\hat{\varphi}_j$ and $a_{j-m}(\iota_m, \hat{\hat{u}}_m(\hat{\hat{\varphi}}_m), \hat{\hat{\varphi}}_m) = d\hat{\hat{u}}_j/d\hat{\hat{\varphi}}_j$ hold by Lemma 4.3,

$$\left|\log\frac{d\hat{\hat{u}}_j}{d\hat{\hat{\varphi}}_j}\Big/\frac{d\hat{u}_j}{d\hat{\varphi}_j}\right| \leq (1+\eta_1)^{-m+j}\left\{c_{21}\theta(\gamma_m) + \left|\log\frac{d\hat{\hat{u}}_m}{d\hat{\hat{\varphi}}_m}\Big/\frac{d\hat{u}_m}{d\hat{\varphi}_m}\right|\right\}.$$

Therefore the assertion (i) is true. Similarly, (ii) is true by Lemma 5.3 Case 1 and (iii) is true by Lemma 5.3 Case 2. Q.E.D.

Call a set G in M *a quadrilateral,* if the boundary of G consists of a pair of opposite increasing curves and a pair of opposite decreasing curves (see Fig. 5–2).

Fig. 5-3

Denote the side curves of G by $\gamma_a = \gamma_a(G)$, $\gamma_b = \gamma_b(G)$, $\gamma_c = \gamma_c(G)$ and $\gamma_d = \gamma_d(G)$ respectively as in Fig. 5–2. If some of sides shrink to points, then call such a G *a trilateral* or *a dilateral* as the case may be, and use the corresponding notations for the remaining sides. If a quadrilateral is surrounded by K-increasing curves and K-decreasing curves, then call G *a K-quadrilateral.*

If T_*^{-1} is continuous on a quadrilateral G and if $T_*^{-1}G$ is also a quadrilateral, then

$$T_*^{-1}\gamma_a(G) = \gamma_c(T_*^{-1}G) \quad\text{and}\quad T_*^{-1}\gamma_c(G) = \gamma_a(T_*^{-1}G),$$
$$T_*^{-1}\gamma_b(G) = \gamma_d(T_*^{-1}G) \quad\text{and}\quad T_*^{-1}\gamma_d(G) = \gamma_b(T_*^{-1}G)$$

hold. Of course, generally $T_*^{-1}G$ is not necessarily a quadrilateral. It is convenient to denote by $\gamma_a(T_*^{-1}G)$ (resp. $\gamma_c(T_*^{-1}G)$) the part of boundary of $T_*^{-1}G$ which joins the upper (resp. lower) ends of $T_*^{-1}\gamma_b(G)$ and $T_*^{-1}\gamma_d(G)$, and to denote $\gamma_d(T_*^{-1}G) \equiv T_*^{-1}(\gamma_b(G))$ and $\gamma_b(T_*^{-1}G) \equiv T_*^{-1}(\gamma_d(G))$. Now introduce the following notations for a quadrilateral G;

$$\|G\| \equiv \theta(\gamma_b(G)) + \theta(\gamma_c(G)) = \theta(\gamma_a(G)) + \theta(\gamma_d(G)) ,$$
$$\max \theta_{\mathrm{in}}(G) \equiv \sup \{\theta(\gamma) ; \gamma \text{ runs over all increasing curves in } G\} ,$$
$$\max \theta_{\mathrm{de}}(G) \equiv \sup \{\theta(\gamma) ; \gamma \text{ runs over all decreasing curves in } G\} ,$$
$$\min \theta_{\mathrm{in}}(G) \equiv \inf \left\{\theta(\gamma) ; \begin{matrix}\gamma \text{ runs over all } K\text{-increasing curves in } G \\ \text{which join } \gamma_a(G) \text{ and } \gamma_c(G)\end{matrix}\right\} ,$$
$$\min \theta_{\mathrm{de}}(G) \equiv \inf \left\{\theta(\gamma) ; \begin{matrix}\gamma \text{ runs over all } K\text{-decreasing curves in } G \\ \text{which join } \gamma_b(G) \text{ and } \gamma_d(G)\end{matrix}\right\} ,$$

LEMMA 5.5. *The following estimates hold.*

(i) $\max \theta_{\mathrm{in}}(G) \leq \|G\|$ *and* $\max \theta_{\mathrm{de}}(G) \leq \|G\|$,

(ii) $\min \theta_{\mathrm{in}}(G) \geq \theta(\gamma_b(G)) - \theta(\gamma_a(G))$,
$\min \theta_{\mathrm{de}}(G) \geq \theta(\gamma_a(G)) - \theta(\gamma_b(G))$.

Especially if G *is a* K*-quadrilateral, then*

$$\|G\| \leq (1 + c_2)(\theta(\gamma_a(G)) + \theta(\gamma_b(G)))/2$$

with $c_2 = K_{\max}/K_{\min}$.

The proof is easily seen by definition. Now introduce a condition on a quadrilateral G.

CONDITION (L). There exist a positive constant L and a partition which satisfy the following: Every element of the partition is a K-increasing curve which joins $\gamma_a(G)$ and $\gamma_c(G)$. Denote by $\tilde{\gamma}(x)$ the element containing x. For any K-decreasing curves $\hat{\gamma}$ and $\hat{\hat{\gamma}}$ in G which join $\gamma_b(G)$ and $\gamma_d(G)$, define a mapping $\tilde{\Psi} = \tilde{\Psi}_{\hat{\gamma},\hat{\hat{\gamma}}}$ from $\hat{\gamma}$ onto $\hat{\hat{\gamma}}$ by

$$\begin{array}{rcl}\tilde{\Psi} ; \hat{\gamma} & \longrightarrow & \hat{\hat{\gamma}} \\ \cup\!\!\!\!\!\!\;\text{\tiny$\ni$} & & \cup\!\!\!\!\!\!\;\text{\tiny$\ni$} \\ x & \longrightarrow & \tilde{\gamma}(x) \cap \hat{\hat{\gamma}} .\end{array}$$

Then for every segment $\hat{\gamma}'$ of $\hat{\gamma}$, the following inequality holds

$$e^{-L} \leq \frac{\theta(\tilde{\Psi}\hat{\gamma}')}{\theta(\hat{\gamma}')} \leq e^{L} .$$

The following lemma is easily seen (see Appendix 6 in [6]).

LEMMA 5.6. *Let G be a K-quadrilateral such that* $\theta(\gamma_a(G)) \geq c_2(1+c_2)^{-1}\theta(\gamma_b(G))$. *Then G satisfies the condition* (L) *with* $L = c_3 \equiv \log 16C_2^4$.

LEMMA 5.7. *Let $\tilde{G}$ be a K-quadrilateral which satisfies the condition (L). Let $\tilde{\tilde{G}}$ be a sub-K-quadrilateral such that $\tilde{\tilde{G}} \subset \tilde{G}$, $\gamma_b(\tilde{\tilde{G}}) \subset \gamma_b(\tilde{G})$ and $\gamma_d(\tilde{\tilde{G}}) \subset \gamma_d(\tilde{G})$. Assume that T_*^{-m} is continuous on $\tilde{G}$ and that $\tilde{G}_m \equiv T_*^{-m}\tilde{G}$ and $\tilde{\tilde{G}}_m = T_*^{-m}G$ are also K-quadrilaterals. Then the following estimate of the ratio $\nu(\tilde{\tilde{G}})/\nu(\tilde{G})$ holds with some constants c_{24} and c_{25};*

$$\frac{\nu(\tilde{\tilde{G}})}{\nu(\tilde{G})} = \frac{\nu(\tilde{\tilde{G}}_m)}{\nu(\tilde{G}_m)} \leq \frac{\max \theta_{\text{in}}(\tilde{\tilde{G}}_m)}{\min \theta_{\text{in}}(\tilde{G}_m)} \exp\left[L + c_{24} + c_{25}\sum_{j=0}^{m} \frac{(1+\eta_1)^{-m+j}\|\tilde{G}_m\|}{\min \cos(\tilde{G}_j)}\right].$$

Proof. Since $d\nu = -\nu_0 \cos\varphi d\varphi dr d\iota$, the estimates

$$\nu(\tilde{\tilde{G}}_m) \leq \frac{2\nu_0}{K_{\min}} \max\cos(\tilde{\tilde{G}}_m)\max\theta_{\text{in}}(\tilde{\tilde{G}})\max\theta_{\text{de}}(\tilde{\tilde{G}}_m),$$

$$\nu(\tilde{G}_m) \geq \frac{2\nu_0}{K_{\max}} \min\cos(\tilde{G}_m)\min\theta_{\text{in}}(\tilde{G}_m)\min\theta_{\text{de}}(\tilde{G}_m)$$

hold. Easily, the estimate

$$\frac{\max\cos(\tilde{\tilde{G}}_m)}{\min\cos(\tilde{G}_m)} \leq \frac{\max\cos(\tilde{G}_m)}{\min\cos(\tilde{G}_m)} \leq \exp\frac{\|\tilde{G}_m\|}{\min\cos(\tilde{G}_m)}$$

is obtained. Now in order to estimate the ratio $\max\theta_{\text{de}}(\tilde{\tilde{G}}_m)/\min\theta_{\text{de}}(\tilde{G}_m)$, let $\hat{\hat{\gamma}}_m$ and $\hat{\gamma}_m$ be K-decreasing curves in $\tilde{G}_m$ which join $\gamma_b(\tilde{G}_m)$ and $\gamma_d(\tilde{G}_m)$. The inequality

$$\begin{aligned}\theta(\hat{\hat{\gamma}}_m) &= \int_{\hat{\hat{\gamma}}}\sum_{j=0}^{m-1}(-\Lambda^*(\iota_j, \hat{\hat{u}}_j(\hat{\hat{\varphi}}), \hat{\hat{\varphi}}; T_*^{-j}\hat{\hat{\gamma}})^{-1}d\hat{\hat{\varphi}} \\ &\leq \exp\left[L + \sum_{j=0}^{m-1}(1+\eta_1)^{j-m}\frac{\{c_{22}\max\theta_{\text{in}}(\tilde{G}_m) + \log c_2\}}{\min\cos(\tilde{G}_j \cup \tilde{G}_{j+1})}\right] \\ &\quad\times \int_{\hat{\gamma}}\prod_{j=0}^{m-1}(-\Lambda^*(\iota_j, \hat{u}_j(\hat{\varphi}), \hat{\varphi}; T_*^{-j}\hat{\gamma})^{-1}d\hat{\varphi}\end{aligned}$$

is obtained by Lemma 5.4 and the condition (L). Therefore

$$\theta(\hat{\hat{\gamma}}_m) \leq \theta(\hat{\gamma}_m)\exp\left[L + c'_{24} + c'_{25}\sum_{j=0}^{m}\frac{(1+\eta_1)^{-m+j}\|\tilde{G}_m\|}{\min\cos(\tilde{G}_j)}\right]$$

with some constants c'_{24} and c'_{25}. Hence the assertion was proved.

Q.E.D.

§ 6. The Main Lemma

The Main Lemma which will be proved in this section is the key for ergodicity, K-property and Bernoullian property. The proof of the lemma is essentially identical with that of the corresponding lemma for Sinai billiard systems. Hence one can refer to [6], in which more precise interpretations are given.

Let γ and γ' be any pair of K-increasing (resp. K-decreasing) curves. Define *the canonical mapping* $\Psi^{(c)}_{\gamma',\gamma}$ (resp. $\Psi^{(e)}_{\gamma',\gamma}$) by

$$\Psi^{(c)}_{\gamma',\gamma}x \equiv \gamma^{(c)}(x) \cap \gamma' \qquad (\text{resp. } \Psi^{(e)}_{\gamma',\gamma}x \equiv \gamma^{(e)}(x) \cap \gamma')\ ,$$

for x in the subset $\{x \in \gamma\,;\, \gamma^{(c)}(x) \cap \gamma' \neq \emptyset\}$ (resp. $\{x \in \gamma\,;\, \gamma^{(e)}(x) \cap \gamma' \neq \emptyset\}$) (see Fig. 6–1).

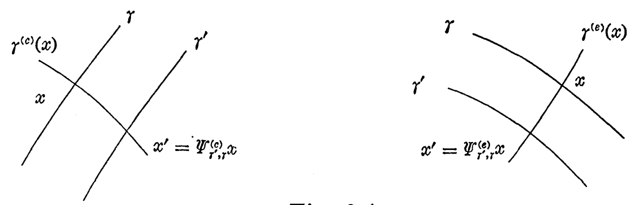

Fig. 6–1

Let $\sigma = \sigma_\gamma$ be the measure on γ induced by θ, that is,

$$\sigma_\gamma(\tilde{\gamma}) \equiv \int_{\tilde{\gamma}} d\varphi \tag{6.1}$$

for any Borel subset $\tilde{\gamma}$ of γ. The measure $\sigma_{\gamma'}$ on γ' is defined by the same way. Define a measure $\Psi^{(c)}_{\gamma,\gamma'}\sigma_{\gamma'}$ (resp. $\Psi^{(e)}_{\gamma,\gamma'}\sigma_{\gamma'}$) by

$$\Psi^{(c)}_{\gamma,\gamma'}\sigma_{\gamma'}(\tilde{\gamma}) \equiv \sigma_{\gamma'}(\Psi^{(c)}_{\gamma',\gamma}\tilde{\gamma}) \qquad (\text{resp. } \Psi^{(e)}_{\gamma,\gamma'}\sigma_{\gamma'}(\tilde{\gamma}) \equiv \sigma_{\gamma'}(\Psi^{(e)}_{\gamma',\gamma}\tilde{\gamma}))\ . \tag{6.2}$$

The canonical mapping $\Psi^{(c)}_{\gamma',\gamma}$ (resp. $\Psi^{(e)}_{\gamma',\gamma}$) is said to be *absolutely continuous* on a set A, if the restrictions of σ_γ and $\Psi^{(c)}_{\gamma,\gamma'}\sigma_{\gamma'}$ (resp. $\Psi^{(e)}_{\gamma,\gamma'}\sigma_{\gamma'}$) to A are mutually absolutely continuous. Set

$$V_m(a) \equiv \{(\iota, r, \varphi) \in M\,;\, |\cos\varphi| \leq a(1+\eta_1)^{-m/32}\}\ .$$

Now the main lemma can be stated:

LEMMA 6.1 (Main Lemma). *For given* α $(0 < \alpha < 1)$, Ω $(\Omega \geq 1)$ *and* ω $(0 < \omega < 1)$, *there exists an even natural number* $\ell_0 = \ell_0(\alpha, \Omega, \omega)$ *for which the following property holds: Let G be a K-quadrilateral satisfying the assumptions*

(A–1) $\min\cos(G) > \omega$,

(A–2) $\theta(\gamma_a(G)) \leq \Omega\theta(\gamma_b(G))$ (*resp.* $\theta(\gamma_c(G)) \leq \Omega\theta(\gamma_d(G))$) ,

(A–3) $T_*^{-j}G \cap V_j(\delta_0) = \emptyset$

$0 \leq j \leq \ell_0$ *with* $\delta_0 \equiv \theta(\gamma_b(G))$ (*resp.* $\delta_0 \equiv \theta(\gamma_d(G))$) ,

(A–4) $T_*^{-\ell_0}$ *is continuous on* G *and* $T_*^{-\ell_0}G$ *is also a K-quadrilateral.*

Then there exists a measurable subset $G^{(c,\alpha)}$ *of* G *such that*

(C–1) *for any* x *in* $G^{(c,\alpha)}$, $\gamma^{(c)}(x) \cap G^{(c,\alpha)}$ *is a connected segment of* $\gamma^{(c)}(x)$ *which joins* $\gamma_b(G)$ *and* $\gamma_d(G)$,

(C–2) $\nu(G^{(c,\alpha)}) \geq (1-\alpha)\nu(G)$,

(C–3) *for any pair* γ, γ' *of K-increasing curves in* G *which join* $\gamma_a(G)$ *and* $\gamma_c(G)$, *the canonical mapping* $\Psi^{(c)}_{\gamma',\gamma}$ *is absolutely continuous on* $\gamma \cap G^{(c,\alpha)}$. *Moreover there exists a constant* $\beta(\Omega)$ *independent of* α, ω *and* G *such that for* x *in* $\gamma \cap G^{(c,\alpha)}$

$$\frac{1}{\beta(\Omega)} \leq \frac{d\Psi_{\gamma,\gamma'}\sigma_{\gamma'}}{d\sigma_\gamma} \leq \beta(\Omega)\ .$$

Proof. One may assume that $\Omega \geq c_2^2$ without loss of generality. First, the proof will be given for the case

$$\frac{c_2}{1+c_2} \leq \frac{\theta(\gamma_a(G))}{\theta(\gamma_b(G))} \leq \Omega\ . \tag{6.3}$$

Let ℓ_0 be a sufficiently large even number, whose actual value will be given laler.

Consider a K-quadrilateral G which satisfies the assumptions (A–1), (A–2), (A–3), (A–4) and the inequality (6.3). A sequence of partitions $\pi_m^{(0)} = \{G_{m,s}^{(0)}, H_{m,t}^{0}\}$, $m \geq \ell_0$, of G which has the following properties will be constructed:

(π–1) $\{\pi_m^{(0)}\}$ is an increasing sequence of partitions.

(π–2) Set $P_m^{(0)} \equiv \bigcup_s G_{m,s}^{(0)}$ and $P_\infty^{(0)} \equiv \bigcap_{m\geq\ell_0} P_m^{(0)}$, then $P_m^{(0)}$ is monotone decreasing and the relations

$$\bigvee_{m=\ell_0}^{\infty} \pi_m^{(0)}|_{P_\infty^{(0)}} = \zeta^{(c)}|_{P_\infty^{(0)}}\ , \qquad \pi_{m+1}^{(0)}|_{G-P_m^{(0)}} = \pi_m^{(0)}|_{G-P_m^{(0)}}$$

hold.

(π–3) A point x is in $P_m^{(0)}$ if $\gamma^{(c)}(x) \cap G$ is a connected segment of $\gamma^{(c)}(x)$ which joins $\gamma_b(G)$ and $\gamma_d(G)$.

(π–4) $G_{m.s}^{(0)}$ and $G_{m,s} \equiv T^{-m}G_{m,s}^{(0)}$ are K-quadrilaterals.

(π–5) A point x is in $P_\infty^{(0)}$ if and only if $\gamma^{(c)}(x) \cap G$ is a connected segment of $\gamma^{(c)}(x)$ which joins $\gamma_b(G)$ and $\gamma_d(G)$.

(π–6) The sum of the measures $\nu(G_{m,s})$ over all $G_{m,s}$'s which satisfy

$$\delta_0(1+\eta)^{-m/2} \le \theta(\gamma_b(G_{m,s})) \le 5\delta_0(1+\eta)^{-m/8}$$

is greater than $(1-\alpha)\nu(G)$.

By Lemma 3.2 and Lemma 5.5, the inequality

$$\theta(\gamma_b(T_*^{-m}G)) \ge (1+\eta)^m\delta_0 - c_4\Omega(1+\eta)^{-m}\Omega\delta_0 \tag{6.4}$$

holds for m, $0 \le m \le \ell_0$. The quadrilateral $T_*^{-\ell_0}G$ can be divided into several K-quadrilaterals $\{G_{\ell_0,s}\}$ in such a way that

$$T_*^{-\ell_0}G = \bigcup_s G_{\ell_0,s}\,, \qquad (1+\eta)^{-\ell_0/8}\delta_0 \le \theta(\gamma_b(G_{\ell_0,s})) \le 5(1+\eta)^{-\ell_0/8}\delta_0$$

and that $\gamma_a(G_{\ell_0,s})$ (resp. $\gamma_c(G_{\ell_0,s})$) coincides with $\gamma_a(T_*^{-\ell_0}G)$ (resp. $\gamma_c(T_*^{-\ell_0}G)$) or a segment of $\bigcup_{m=0}^n T^m S$ with some $n \ge 0$. Put $\pi_{\ell_0} \equiv \{G_{\ell_0,s}\}$, $P_{\ell_0} \equiv \bigcup_s G_{\ell_0,s}$, $G_{\ell_0,s}^{(0)} \equiv T_*^{\ell_0}G_{\ell_0,s}$, $\pi_{\ell_0}^{(0)} \equiv G_{\ell_0,s}^{(0)} = T_*^{\ell_0}\pi_{\ell_0}$, $P_{\ell_0}^{(0)} \equiv \bigcup_s G_{\ell_0,s}^{(0)} = T_*^{\ell_0}P_{\ell_0}$. Assume that a set $P_{m-1} = \bigcup_s G_{m-1,s}$ and a partition $\pi_{m-1} = \{G_{m-1,s}, H_{m-1,t}\}$ which satisfy (π–1) $\sim$ (π–4) have been constructed. Every component of the restriction $\alpha^{(c)}|_{G_{m,s}}$ of $\alpha^{(c)}$ to $G_{m,s}$ is expressed in the form $G_{m-1,s} \cap X_j^{(c)}$. Obviously, $G_{m-1,s} \cap X_j^{(c)}$ is a K-quadrilateral (or a trilateral or a dilateral). If it is a K-quadrilateral, denote it by $O_{m-1,s,j}$. If there exist two tri-or dilaterals which have a common side of them, then joint them together. After that, if there still exist tri-or dilaterals which have a common side, then joint them again. Continue such a procedure repeatedly. Denote such a maximal jointed set by $Q_{m-1,s,\ell}$ (see Fig. 6–2). Then it is easily seen that

$$\theta(\gamma_b(Q_{m-1,s,\ell})) \le \theta(\gamma_a(G_{m-1,s,\ell}))\,. \tag{6.5}$$

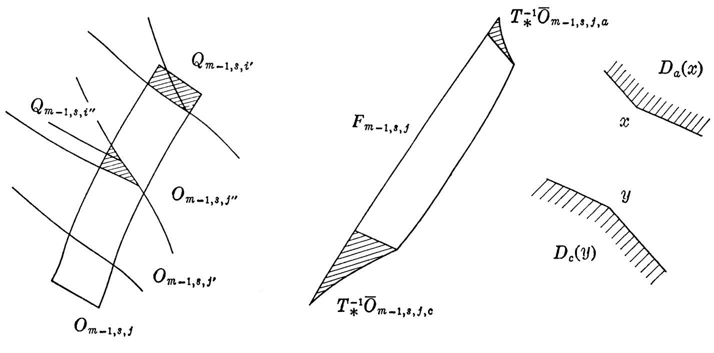

Fig. 6–2

Let $D_a(x)$ (resp. $D_c(x)$) be the set of all points which lie over (resp. below) the two lines passing through x with inclinations $-K_{\max}$ and $-K_{\min}$, respectively. Set

$$\bar{O}_{m-1,s,j,a} \equiv O_{m-1,s,j} \cap \bigcup_{x \in \gamma_c(T_*^{-1} O_{m-1,s,j})} T_* D_c(x) ,$$
$$\bar{O}_{m-1,s,j,c} \equiv O_{m-1,s,j} \cap \bigcup_{x \in \gamma_a(T_*^{-1} O_{m-1,s,j})} T_* D_a(x)$$

and $O'_{m-1,s,j} \equiv O_{m-1,s,j} - \bar{O}_{m-1,s,j,a} - \bar{O}_{m-1,s,j,c}$. Then $\bar{O}_{m-1,s,j,a}$ and $\bar{O}_{m-1,s,j,c}$ are K-trilaterals (or K-dilateral). The sets $O'_{m-1,s,j}$ and $F_{m-1,s,j} \equiv T_*^{-1} O'_{m-1,s,j}$ are K-quadrilaterals. If

$$\theta(\gamma_b(F_{m-1,s,j})) < 5\delta_0(1+\eta)^{-m/8} ,$$

then put $G_{m-1,s,j,1} \equiv F_{m-1,s,j}$. If

$$\theta(\gamma_b(F_{m-1,s,j})) \geq 5\delta_0(1+\eta)^{-m/8} ,$$

then $F_{m-1,s,j}$ can be divided into K-quadrilaterals $\{G_{m-1,s,j,q}\,;\ q = 1, 2, \cdots\}$ such that $\gamma_b(G_{m-1,s,j,q}) \subset \gamma_b(G)$, $\gamma_d(G_{m-1,s,j,q}) \subset \gamma_d(G)$,

$$\delta_0(1+\eta)^{-m/8} \leq \theta(\gamma_b(G_{m-1,s,j,q})) \leq 5\delta_0(1+\eta)^{-m/8} \tag{6.6}$$

and that $\gamma_a(G_{m-1,s,j,q})$ coincides with either $\gamma_a(G_{m-1,s})$ or a segment of $\bigcup_{i=1}^{n} T^i S$ for $n \geq 1$. Now change the numbering of $\{G_{m-1,s,j,q}\,;\ s, j, q\}$ and denote them by $\{G_{m,s'}\}$. Moreover, denote $\{T_*^{-1} Q_{m-1,s,t},\ T_*^{-1}\bar{O}_{m-1,s,j,a},\ T_*^{-1}\bar{O}_{m-1,s,j,c},\ T_*^{-1} H_{m-1,t'}\}$ by $\{H_{m,t'}\}$. Put

$$\pi_m \equiv \{G_{m,s'}, H_{m,t'}\} , \qquad \pi_m^{(0)} \equiv T_*^m \pi_m$$
$$P_m \equiv \bigcup_{s'} G_{m,s'} \quad \text{and} \quad P_m^{(0)} \equiv T_*^m P_m .$$

Then $\{\pi_m\}$ satisfies (π–1) $\sim$ (π–5) as desired; in fact the proofs for (π–1), (π–2) and (π–4) are obvious, while (π–3) and (π–5) can be shown as follows. Since $T_*^{-m}\gamma^{(c)}(x)$ is K-decreasing for any $m \geq 0$, if $\gamma^{(c)}(x) \cap G$ joins $\gamma_b(G)$ and $\gamma_d(G)$, then $T_*^{-m}(\gamma^{(c)}(x) \cap G)$ is included in an element $G_{m,s}$ for any $m \geq 0$. Therefore (π–3) is true. Conversely, if x is in $P_\infty^{(0)}$, then there exists a K-decreasing curve $\gamma_m^{(m)}$ passing through $T_*^{-m}x$ such that $\gamma_m^{(m)}$ is included in a certain element $G_{m,s}$ and that $\gamma_m^{(m)}$ joins $\gamma_b(G_{m,s})$ and $\gamma_d(G_{m,s})$. Since T_*^m is continuous on $G_{m,s}$, $\gamma_0^{(m)} \equiv T_*^m \gamma_m^{(m)}$ is a connected K-decreasing curve which joins $\gamma_b(G)$ and $\gamma_d(G)$. Further, it is easily seen by the same way as the proof of Theorem 1 that $\gamma_0^{(m)}$ converges to a curve which joins $\gamma_a(G)$ and $\gamma_b(G)$ and that the limitting curve is identical with $\gamma^{(c)}(x) \cap G$.

Now the measure of the rejected sets

$$R_{m-1}(1) \equiv \bigcup_{s,j} (\bar{O}_{m-1,s,j,a} \cup \bar{O}_{m-1,s,j,c}) ,$$
$$R_{m-1}(2) \equiv \bigcup_{s,j} Q_{m-1,s,j}$$

will be evaluated. In order to evaluate them, it is convenient to classify $\{G_{m,s}\}$ as follows.

DEFINITION. A piece $O_{m-1,s,j}$ is said to be *docile*, if either $\gamma_a(T_*^{-1}O_{m-1,s,j})$ or $\gamma_c(T_*^{-1}O_{m-1,s,j})$ intersects with S.

DEFINITION. A piece $G_{m,s}$ is said to be *narrow* if

$$\theta(\gamma_b(G_{m,s})) \leq \delta_0(1+\eta)^{-m/4} .$$

A piece $G_{m,s}$ is said to be *wide* if

$$\theta(\gamma_b(G_{m,s})) \geq \delta_0(1+\eta)^{-m/8} .$$

Put

$$R_m(3) \equiv \{G_{m,s}\,;\, G_{m,s} \cap V_m(\delta_0) \neq \emptyset\} ,$$
$$R_m(4) \equiv \{G_{m,s}\,;\, G_{m,s} \text{ is narrow}\} .$$

It is convenient to denote by the same notation $R_m(j)$ the union of the sets contained in the family $R_m(j)$ $(j = 1, 2, 3, 4)$.

(1°) Estimation for $R_m^*(3) \equiv R_m(3) \cup \{T_*^{-1}\bar{O}_{m,s,j,\cdot}\,;\, T_*^{-1}\bar{O}_{m,s,j,\cdot} \subset V_m(\delta_0)\}$.

It is easily seen by (6.6), Lemma 5.5 and Lemma 3.3 that

$$\|G_{m,s}\| \leqq 5\delta_0(1+\eta)^{-m/8} + c_4\Omega\delta_0(1+\eta)^{-m}$$

with $c_4 = 1 + K_{\max}/K_{\min}$. Hence if

$$(5 + c_4\Omega)(1+\eta)^{-\ell_0/16} < 1 , \tag{ℓ_0–1}$$

then every $G_{m,s}$ in $R_m^*(3)$ is included in $V_m(2\delta_0)$. Therefore, $R_m^*(3)$ is included in $V_m(2\delta_0)$. Hence

$$\nu(R_m^*(3)) \leq \nu(V_m(2\delta_0)) \leq 2(1+\eta_1)^{-m/16}\delta_0^2 . \tag{6.7}$$

(2°) Estimation for $R_m^*(4) \equiv R_m(4) - \bigcup_{\ell=\ell_0}^m T^{-m+\ell}R_\ell(3)$.

By Lemma 3.4 (iv), if

$$10\pi(1+\eta)^{-\ell_0/16} < c_{10} , \tag{ℓ_0–2}$$

then for any component of $\{O_{m-1,s_1,j_1}\,;\, j_1 = 1, 2, \cdots\}$, the case where sign $(\gamma_a(O_{m-1,s_1,j_1})) = (-)$ and sign $(\gamma_c(O_{m-1,s_1,j_1})) = (+)$ at the same time does not happen. Therefore one can see the following properties (G–1) $\sim$ (G–4) for a given triple

$$G_{m-1,s_1} \supset O_{m-1,s_1,j_1} \supset T_* G_{m,s}:$$

(G–1) G_{m-1,s_1} contains at most one component which is not docile.

(G–2) If $G_{m,s}$ is not contained in $R_m(3)$ and if O_{m-1,s_1,j_1} is docile, then the inequality

$$\theta(\gamma_b(G_{m,s})) \geq \delta_0(1+\eta)^{-m/8}$$

holds, namely, $G_{m,s}$ is wide.

(G–3) $T_*^{-1}G_{m-1,s_1}$ contains at most one component $G_{m,s}$ which is not wide and not contained in $R_m(3)$.

(G–4) For each wide G_{n,s_n}, there exists at most one series $\{G_{n+1,s_{n+i}};\ 0 \leq i \leq p\}$ such that

$$G_{n,s_n} \supset T_* G_{n+1,s_{n+1}} \supset \cdots \supset T_*^{p-1} G_{n+p-1,s_{n+p-1}} \supset T_*^p G_{n+p,s_{n+p}}\ ,$$

where $G_{n+i,s_{n+i}}$ is not wide, not contained in $R_{n+i}(3)$, $1 \leq i \leq p$, and $G_{n+p,s_{n+p}}$ is narrow.

The properties (G–1) $\sim$ (G–4) can be proved easily. For each fixed wide $G_{n,s}$, there exists at most one series as in (G–4). Let $G_{n+p,s_{n+p}}$ be the first narrow K-quadrilaterals in the series. Then

$$\theta(\gamma_b(G_{n+p,s_{n+p}})) \leq \delta_0(1+\eta)^{-(n+p)/4}$$
$$\theta(\gamma_c(G_{n+p,s_{n+p}})) \leq c_4\Omega\delta_0(1+\eta)^{-(n+p)}$$

hold. Hence by Lemma 5.5 and (ℓ_0–1)

$$\max \theta_{\text{in}}(T_*^p(G_{n+p,s_{n+p}})) \leq 2\delta_0(1+\eta)^{-n/4-5p/4}\ .$$

Put $\tilde{G} \equiv T_*^n G_{n,s_n}$ and $\tilde{\tilde{G}} \equiv T_*^{n+p} G_{n+p,s_{n+p}}$. Then one can apply Lemma 5.7 to the pair $\tilde{G}$ and $\tilde{\tilde{G}}$. Since the inequalities

$$\begin{aligned}
&\min \cos (T_*^{-\ell}\tilde{G}) \geq \delta_0(1+\eta_1)^{-\ell/32} \qquad \text{for } \ell_0 \leq \ell \leq n\ ,\\
&\|T_*^{-n}\tilde{G}\| \leq 5\delta_0(1+\eta)^{-n/8} + c_4\delta_0(1+\eta)^{-n} \leq \delta_0(1+\eta)^{-n/16}\ ,\\
&\min \theta_{\text{in}}(T_*^{-n}\tilde{G}) \geq \theta(\gamma_b(T_*^{-n}\tilde{G})) - \theta(\gamma_a(T_*^{-n}\tilde{G}))\\
&\qquad\qquad \geq \delta_0(1+\eta)^{-n/8} - c_4\delta_0(1+\eta)^{-n} \geq \tfrac{1}{2}\delta_0(1+\eta)^{-n/8}
\end{aligned}$$

hold by Lemma 5.5, the estimate

$$\frac{\nu(\tilde{\tilde{G}})}{\nu(\tilde{G})} \leq 4(1+\eta)^{-m/8-5p/4} \exp\,[c_3 + c_{24} + c_{25}]$$

is obtained by Lemma 5.7. Hence

$$\nu\Big(\bigcup_{m=\ell_0}^{\infty} T_*^m R_m^*(4)\Big) \leq 4 \exp\,[c_3 + c_{24} + c_{25}] \sum_{m=\ell_0}^{\infty} (1+\eta)^{-m/8}\nu(G)\ . \tag{6.8}$$

(3°) Estimation for $R_m^*(2) \equiv \{Q_{m,s,\ell}\,;\, G_{m,s}$ is not in $R_m^*(4) \cup \bigcup_{k=\ell_0}^{m} T^{-m+k} R_k(3)\}$.

Let G' be a K-quadrilateral. Then one can define a family of sets $\{Q'_\ell\,;\, \ell = 1, 2, \cdots\}$ by the same way as $Q_{m-1,s}$, in the construction of π_m. Let $U(i)$ be a sufficiently small neighbourhood of $z(i)$ where $\{z(i)\,;\, i = 1, 2, \cdots, I_1\} = \bigcap_{j=-\infty}^{\infty} T_*^j S$. Then the branching points of T_*S outside $\bigcup_{i=1}^{I_1} U(i)$ are discrete. Hence there exists a constant c'_9 such that for G' with $\|G'\| \leq c'_9$ G' contains at most one branching point outside $\bigcup_{i=1}^{I_1} U(i)$. If G' is included in $U(i)$, then G' includes at most two components $\{Q'_1, Q'_2\}$ as is seen in Fig. 6–3. Therefore there exists a constant c_9 such that for every G' with $\|G'\| \leq c_9$, G' includes at most two components $\{Q'_1, Q'_2\}$.

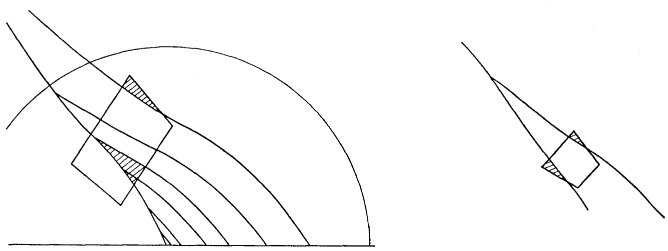

Fig. 6–3

Since $G_{m,s}$ is not narrow, by definition it holds that

$$\theta(\gamma_b(G_{m,s})) \geq \delta_0(1+\eta)^{-m/4}\ .$$

From the inequality (6.5), the inequality

$$\max \theta_{\text{in}}(Q_{m,s,\ell}) \leq \delta_0 c_4 \Omega (1+\eta)^{-m}$$

follows. Therefore, applying Lemma 5.7, the estimate

$$\frac{\nu(Q_{m,s,\ell})}{\nu(G_{m,s})} \leq 4 \exp\,[c_3 + c_{24} + c_{25}](1+\eta)^{-3m/4}$$

is obtained. If the inequality

$$(\ell_0\text{–}3) \qquad \pi(1+\eta)^{-\ell_0/16} < c_9$$

is fulfilled, the estimate

$$(6.9) \qquad \nu(R_m^*(2)) \leq 4 \exp [c_3 + c_{24} + c_{25}](1+\eta)^{-3m/4}\nu(G)$$

is obtained.

(4°) Estimation for $R_m(1)$.

Divide $R_m(1)$ into three classes;

$$R_m^*(5) \equiv \left\{ \bar{O}_{m,s,j};\ O_{m,s,j} \text{ is not docile and } G_{m,s} \text{ is not in } \bigcup_{\ell=\ell_0}^{m} T_*^{-m+\ell} R_\ell(3) \cup R_m^*(4) \right\},$$

$$R_m^*(6) \equiv \left\{ \bar{O}_{m,s,j} \notin R_m^*(5);\ \theta(\gamma_b(O_{m,s,j})) \geq \delta_0(1+\eta)^{-m/2} \text{ and } G_{m,s} \text{ is not in } \bigcup_{\ell=\ell_0}^{m} T_*^{-m+\ell} R_\ell(3) \cup R_m^*(4) \right\},$$

$$R_m^*(7) \equiv \left\{ \bar{O}_{m,s,j} \notin T_* R_m^*(3) \cup R_m^*(5);\ \theta(\gamma_b(O_{m,s,\ell})) \leq \delta_0(1+\eta)^{-m/2} \text{ and } G_{m,s} \text{ is not in } \bigcup_{\ell=\ell_0}^{m} T^{-m+\ell} R_\ell(3) \cup R_m^*(4) \right\}.$$

Since by (ℓ_0–2) $G_{m,s}$ contains at most one component which is not docile and since $G_{m,s}$ is not narrow, the estimate

$$(6.10) \qquad \nu(R_m^*(5)) \leq 8 \exp [c_3 + c_{24} + c_{25}](1+\eta)^{-3m/4}\nu(G)$$

is obtained by Lemma 5.7. By applying Lemma 5.7 again, the estimate

$$(6.11) \qquad \nu(R_m^*(6)) \leq 8 \exp [c_3 + c_{24} + c_{25}](1+\eta)^{-m/2}\nu(G)$$

is obtained. Lastly, one must estimate the measure of $R_m^*(7)$. Except for a finite number of $X_j^{(c)}$'s, say $X_j^{(c)}$, $j = 1, 2, \cdots, \hat{I}$, $X_j^{(c)}$ coincides with $X_{i,j'}^+$ with some i and j' (see § 3). There are two cases depending on the sign of $\Sigma_{i,j'}^+$. Only the case of (+) will be explained here, the case of (−) goes the same way. Since $O_{m,s,j}$ is docile, $T_*^{-1}\bar{O}_{m,s,j,c}$ is included in $V_m(\delta_0)$ and hence in $R_m^*(3)$. In order to estimate the measure $\nu(\bar{O}_{m,s,j,a}) = \nu(T_*^{-1}\bar{O}_{m,s,j,a})$, note that the inequality

$$c_{13}j^{-2} \leq \theta(\gamma_b(O_{m,s,j})) \leq \delta_0(1+\eta)^{-m/2}$$

which is obtained by Lemma 3.5, implies that $j \geq j_m$ where j_m is the minimum natural number greater than $c_{13}^{-1/2}\delta_0^{-1/2}(1+\eta)^{m/4}$. Put $\gamma \equiv \gamma_c(\bar{O}_{m,s,j,a})$ and $\gamma_1 \equiv T_*^{-1}\gamma$. Then $\theta(\gamma) \leq c_4\Omega\delta_0(1+\eta)^{-m}$. By Lemma 3.5 for x in γ

$$-\tau(T_*^{-1}x) \geq c_{15}j \quad \text{and} \quad -\cos\varphi(x) \leq c_{12}j^{-1/2}$$

hold. Therefore, by Lemma 3.2 the estimate

$$\theta(\gamma_1) = \int_{\gamma} \left| \frac{d\varphi_1}{d\varphi} \right| d\varphi \leq \frac{c_4 \Omega \delta_0 (1+\eta)^{-m}}{1 + k_{\min} c_{15} c_{12} j^{3/2}}$$

is obtained. Hence the rejected sets are included in the domain indicated by the hatching in Fig. 6–4.

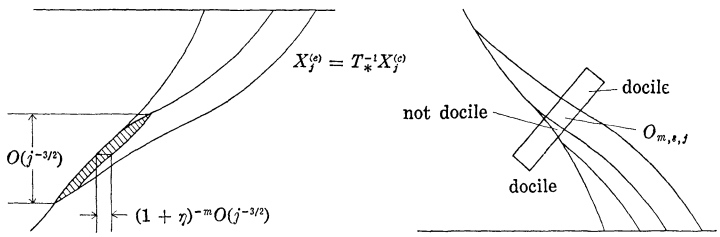

Fig. 6-4

The measure of the domain is less than

$$\frac{2\nu_0 c_4 c_{12} \delta_0 (1+\eta)^{-m}}{K_{\min} k_{\min} c_{11} c_{15}} j^{-7/2} .$$

On the other hand, by the same reason as in the estimation (3°) for j, $1 \leq j \leq \hat{l}$, at most two components $O_{m,s,j}$'s belong to $R_m^*(7)$. Since $G_{m,s}$ is not narrow and $\max \theta_{\text{in}}(\bar{O}_{m,s,j,a}) \leq c_4 \Omega \delta_0 (1+\eta)^{-m}$, by Lemma 5.7 the estimate

$$\nu(\bar{O}_{m,s,j,a}) \leq 4 \exp[c_3 + c_{24} + c_{25}](1+\eta)^{-3m/4} \nu(G_{m,s})$$

holds. Therefore the estimate

$$\begin{aligned} \nu(R_m^*(7)) &\leq \sum_{j=j_m}^{\infty} \frac{2\nu_0 c_4 c_{12} \delta_0 (1+\eta)^{-m}}{K_{\min} k_{\min} c_{11} c_{15}} j^{-7/2} \\ &\quad + 8 \exp[c_3 + c_{24} + c_{25}](1+\eta)^{-3m/4} \nu(G) \\ &\leq \frac{2}{3} \frac{\nu_0 c_4 c_{12} \Omega c_{13}^{5/4}}{K_{\min} k_{\min} c_{11} c_{15}} (1+\eta)^{-13m/8} \delta_0^{9/4} \\ &\quad + 8 \exp[c_3 + c_{24} + c_{25}](1+\eta)^{-3m/4} \nu(G) . \end{aligned} \tag{6.12}$$

is obtained.

This completes the estimations of all rejected sets. Since the estimate

$$\frac{\nu_0 c_2 \min \cos(G)}{2K_{\max}(1+c_2)^2} \delta_0^2 \leq \nu(G) \leq \frac{\nu_0 (1+c_2) \Omega}{K_{\min}} \max \cos(G) \delta_0^2$$

is true for any K-quadrilateral G, by (6.7) ~ (6.12)

$$\nu\Big(\bigcup_{m=\ell_0}^{\infty}\bigcup_{k=2}^{7} T_*^{-m}R_m^*(k)\Big) \le c_{26}\Big(\frac{1}{\omega}+1\Big)(1+\eta)^{-\ell_0/16}\nu(G)$$

holds with some constant c_{26} for a sufficiently large ℓ_0 which satisfies $(\ell_0\text{–}1), (\ell_0\text{–}2)$ and $(\ell_0\text{–}3)$. Hence if an additional condition

$$(\ell_0\text{–}4) \qquad c_{26}\Big(\frac{1}{\omega}+1\Big)(1+\eta_1)^{-\ell_0/16} < \alpha$$

is fulfilled, then the set

$$G^{(c,\alpha)} \equiv G - \bigcup_{m=\ell_0}^{\infty}\bigcup_{k=2}^{7} T_*^{m}R_m^*(k)$$

is greater than $(1-\alpha)\nu(G)$. Furthermore, $G^{(c,\alpha)}$ satisfies the conditions (C–1), (C–2) and (C–3). The conditions (C–1) and (C–2) were already seen. Now to show (C–3), define partitions $\xi(m)$ of γ(resp. $\xi'(m)$ of γ'), $m \ge \ell_0$, by

$$\xi(m) \equiv \pi_m^{(0)}|_\gamma \qquad (\text{resp. } \xi'(m) \equiv \pi_m^{(0)}|_{\gamma'}) .$$

Put $\pi_\infty^{(0)} \equiv \bigvee_m \pi_m^{(0)}$, $\xi(\infty) \equiv \pi_\infty^{(0)}|_\gamma$ and $\xi'(\infty) \equiv \pi_\infty^{(0)}|_{\gamma'}$. Then $\xi(m)$ increases to $\xi(\infty)$ and $\xi'(m)$ increases to $\xi'(\infty)$ as $m \to \infty$. Further $\xi(\infty)|_{P_\infty^{(0)}}$ (resp. $\xi'(\infty)|_{P_\infty^{(0)}}$) is the partition of $\gamma \cap P_\infty^{(0)}$ (resp. $\gamma' \cap P_\infty^{(0)}$) into the individual points. Conventionally, put $\Psi \equiv \Psi_{\gamma'\gamma}^{(c)}$. For x in $P_\infty^{(0)}$, there exists a K-quadrilateral $G_{m,s}^{(0)}$ in $\pi_m^{(0)}$ which contains x. Denote by $G_m^{(0)}(x)$ the $G_{m,s}^{(0)}$ and put $G_m(x) \equiv T_*^{-m}G_m^{(0)}(x)$. For x in γ, denote by $C_m(x)$ (resp. $C'_m(x')$) the element of $\xi(m)$ (resp. $\xi'(m)$) which contains x (resp. x'). Then for x in $P_\infty^{(0)} \cap \gamma$

$$C_m(x) = G_m^{(0)}(x) \cap \gamma \quad \text{and} \quad C'_m(\Psi x) = G_m^{(0)} \cap \gamma' .$$

In particular, if x is in $G^{(c,\alpha)}$,

$$(6.14) \qquad \begin{cases} \delta_0(1+\eta)^{-m/2} \le \theta(\gamma_b(G_m(x))) \le (5+c_4\Omega)\delta_0(1+\eta)^{-m/8} , \\ \max \theta_{\mathrm{de}}(G_m(x)) \le c_4\Omega\delta_0(1+\eta)^{-m} , \\ \min \cos (G_j(x)) \ge \delta_0(1+\eta)^{-j/32} , \qquad 0 \le j \le m . \end{cases}$$

For x in $\gamma \cap P_\infty^{(0)}$ with $x' = \Psi x$, it holds that

$$(6.15) \qquad \begin{aligned} \theta(C_m(x)) &= \int_{T_*^{-m}C_m(x)} \prod_{i=0}^{m-1} |\Lambda(\iota_i, u_i(\varphi_i), \varphi_i\,;\, T_*^{-i}\gamma)|^{-1} d\varphi_m \\ \theta(C'_m(x')) &= \int_{T_*^{-m}C'_m(x')} \prod_{i=0}^{m-1} |\Lambda(\iota_i, u'_i(\varphi_i), \varphi_i\,;\, T_*^{-i}\gamma')|^{-1} d\varphi_m , \end{aligned}$$

where $r_i = u_i(\varphi_i)$ and $r_i = u'_i(\varphi_i)$ are equations of $T_*^{-i}\gamma$ and $T_*^{-i}\gamma'$ respectively. By Lemma 5.4 for any pair $\hat{y}, y$ in $C_m(x)$,

$$
\begin{aligned}
&\prod_{i=0}^{m-1} \frac{\Lambda(y_i, T_*^{-i}\gamma)}{\Lambda(\hat{y}_i, T_*^{-i}\gamma)} \\
&\quad \leq \exp\left[c_{22} \sum_{j=1}^{m} \frac{(1+\eta_1)^{-m+j}\theta(T_*^{-m}C_m(x))}{\min\cos(T_*^{-j}C_m(x) \cup T_*^{-j-1}C_m(x))} \right. \\
&\qquad\qquad \left. + \left(1 + \frac{1}{\eta_1}\right)\left|\log \frac{du_0}{d\varphi}(\hat{y}) \Big/ \frac{du_0}{d\varphi}(\hat{y})\right|\right] \qquad (6.16) \\
&\quad \leq \exp\left[\left(\frac{1}{\eta_1} + 1\right)^2 \left\{c_{22}(1+\eta_1)^{-m/32} + \left|\log \frac{du_0}{d\varphi}(\hat{y}) \Big/ \frac{du_0}{d\varphi}(\hat{y})\right|\right\}\right] \\
&\quad \leq \exp\left[\left(\frac{1}{\eta_1} + 1\right)^2 (c_{22} + \log c_2)\right] = \exp c_{27}
\end{aligned}
$$

is obtained by (ℓ_0–1). Therefore

$$
\exp(-c_{27}) \leq \frac{\theta(C_m(x))}{\theta(T_*^{-m}C_m(x))} \prod_{i=0}^{m-1} |\Lambda(x_i, T_*^{-i}\gamma)| \leq \exp c_{27} \qquad (6.17)
$$

is obtained. Alternatively, the estimate

$$
\exp(-c_{27}) \leq \frac{\theta(C'_m(x'))}{\theta(T_*^{-m}C'_m(x'))} \prod_{i=0}^{m-1} |\Lambda(x'_i, T_*^{-i}\gamma')| \leq \exp c_{27} \qquad (6.17)'
$$

is obtained for $x' = \Psi x$. On the other hand,

$$
1 - 2(1+\eta)^{-m/16} \leq \frac{\theta(T_*^{-m}C'_m(x'))}{\theta(T_*^{-m}C_m(x))} \leq 1 + 2(1+\eta)^{-m/16} \qquad (6.18)
$$

holds by (6.14). By Lemma 5.4, the estimate

$$
\begin{aligned}
&\left|\log \frac{\Lambda(x'_i, T_*^{-i}\gamma')}{\Lambda(x_i, T_*^{-i}\gamma)}\right| \qquad (6.19) \\
&\quad \leq \frac{c_{22}(1+\eta_1)^{-i}|\varphi(x) - \varphi(x')|}{\delta_0(1+\eta_1)^{-i/32}} + (1+\eta_1)^{-i}\left|\log \frac{du'}{d\varphi'} \Big/ \frac{du}{d\varphi}\right|
\end{aligned}
$$

for $i \geq 0$ is obtained, since for $m \geq \ell_0$ $T^{-m}x$ and $T^{-m}x'$ are in the same $G_{m,s}$ which does not intersect with $V_m(\delta_0)$, further for $\ell_0 \geq i \geq 0$ $T^{-i}G$ does not intersect with $V_i(\delta_0)$. By using (6.19) and

$$
\left|\log \frac{du'}{d\varphi'} \Big/ \frac{du}{d\varphi}\right| \leq \log c_2
$$

it is proved that the infinite product

$$g(x) \equiv \prod_{i=0}^{\infty} \frac{\Lambda(x_i, T_*^{-i}\gamma)}{\Lambda(x_i', T_*^{-i}\gamma')}$$

converges absolutely and uniformly in $\gamma(\infty)$. Moreover by the assumption (A–2), $g(x)$ is bounded as

$$\frac{1}{\beta_1(\Omega)} \leq g(x) \leq \beta_1(\Omega) \tag{6.20}$$

with $\beta_1(\omega) = \exp[(1 + \eta_1^{-1})(2c_4c_{22}\Omega + \log c_2)$. By (6.16) $\sim$ (6.18),

$$\frac{1}{\beta(\Omega)} \leq \frac{\theta(C_m'(x'))}{\theta(C_m(x))} \leq \beta(\Omega) \tag{6.21}$$

holds with $\beta(\Omega) = 2e^{2c_{27}}\beta_1(\Omega)$.

Let A be a Borel subset of $\gamma \cap G^{(c,\alpha)}$ with $\sigma_\gamma(A) = 0$. Then, for any $\varepsilon > 0$ there exists a covering $\{C_i\}$ of A, such that $C_i = C_{m_i}(y(i))$ with some $y(i)$ in $G^{(c,\alpha)} \cap \gamma$, $A \subset \bigcup_i C_i$ and $\sum_{i=1}^{\infty} \theta(C_i) < \varepsilon$. Since $\Psi A \subset \bigcup_i C'_{m_i}(\Psi(y(i)))$, it is shown that

$$\sigma_{\gamma'}(\Psi A) \leq \sum_i \theta(C'_{m_i}(\Psi(y(i))) \leq \beta(\omega) \sum_i \theta(C_{m_i}(y(i))) < \beta(\omega)\varepsilon .$$

Hence $\sigma_{\gamma'}(\Psi A) = 0$. In the same way, one can show the converse assertion. Hence the canonical mapping $\Psi = \Psi_{\gamma',\gamma}$ is absolutely continuous. Also

$$\frac{1}{\beta(\Omega)} \leq \frac{d\Psi_{\gamma,\gamma'}^{(c)}\sigma_{\gamma'}}{d\sigma_\gamma} \leq \beta(\Omega)$$

can be shown by the above discussions. Thus the proof is completed for the case $\theta(\gamma_a(G)) \geq (c_2/(1 + c_2))\theta(\gamma_b(G))$. In case $\theta(\gamma_b(G)) \leq (c_2/(1 + c_2)) \cdot\theta(\gamma_b(G))$, one can divide G into small K-quadrilaterals F_j's each of which satisfies the assumptions (A–1), (A–2), (A–3), (A–4) and the inequality $\theta(\gamma_a(F_j)) \geq (c_2/(1 + c_2))\theta(\gamma_b(F_j))$. Then there exists a subset $F_j^{(c,\alpha)}$ which satisfies (C–1), (C–2) and (C–3). Put $G^{(c,\alpha)} \equiv \bigcup_j F_j^{(c,\alpha)}$. Then $G^{(c,\alpha)}$ satisfies the conditions (C–1), (C–2) and (C–3), obviously. Q.E.D.

In a similar manner the following lemma can be shown.

LEMMA 6.1′. *For given* α $(0 < \alpha < 1)$, Ω $(\Omega \geq 1)$ *and* ω $(0 < \omega < 1)$, *there exists an even natural number* $\ell_0 = \ell_0(\alpha, \Omega, \omega)$ *for which the following holds: Let G be a K-quadrilateral satisfying*

(A–1) $\min \cos(G) > \omega$,

(A–2)′ $\theta(\gamma_b(G)) \leq \Omega\theta(\gamma_a(G))$ (*resp.* $\theta(\gamma_d(G)) \leq \Omega\theta(\gamma_c(G))$),

(A–3)′ $T_*^j G \cap V_j(\delta_0) = \emptyset \quad 0 \leq j \leq \ell_0$

with $\delta_0 \equiv \theta(\gamma_a(G))$ (*resp.* $\theta(\gamma_c(G))$),

(A–4)′ $T_*^{\ell_0}$ *is continuous on* G *and* $T_*^{\ell_0}G$ *is also a K-quadrilateral.*

Then there exists a measurable subset $G^{(e,\alpha)}$ *of* G *such that*

(C–1)′ *for any* x *in* $G^{(e,\alpha)}$, $\gamma^{(e)}(x) \cap G^{(e,\alpha)}$ *is a connected segment of* $\gamma^{(e)}(x)$ *which joins* $\gamma_a(G)$ *and* $\gamma_c(G)$,

(C–2)′ $\nu(G^{(e,\alpha)}) \geq (1-\alpha)\nu(G)$,

(C–3)′ *let* γ *and* γ' *be any pair of K-decreasing curves in* G *which join* $\gamma_b(G)$ *and* $\gamma_d(G)$. *Then the canonical mapping* $\Psi_{\gamma',\gamma}^{(e)}$ *is absolutely continuous on* $\gamma \cap G^{(e,\alpha)}$. *Moreover for* x *in* $\gamma \cap G^{(e,\alpha)}$, *it holds that*

$$\frac{1}{\beta(\Omega)} \leq \frac{d\Psi_{\gamma,\gamma'}^{(e)}\sigma_{\gamma'}}{d\sigma_\gamma} \leq \beta(\Omega)\ .$$

§ 7. Canonical mapping

In order to apply Lemma 6.1, it is useful to note the following lemma.

LEMMA 7.1. *Fix* $\alpha(0<\alpha<1)$, $\Omega(\Omega>1)$ $\omega(0<\omega<1)$. *Let* $\ell_0 = \ell_0(\alpha, \Omega, \omega/4)$ *be the number which was given in Lemma* 6.1 *and Lemma* 6.1′. *Then there exist positive functions* $\varepsilon_0 = \varepsilon_0(x_0, \alpha, \Omega, \omega)$ *and* $\varepsilon_1 = \varepsilon_1(x_0, \alpha, \Omega, \omega)$ *such that; for* x_0 *not in* $\bigcup_{i=0}^{\ell_0} T_*^i S$ (*resp.* $\bigcup_{i=0}^{\ell_0} T_*^{-i} S$) *with* $-\cos\varphi(x_0) \geq \omega$

(i) $T_*^{-\ell_0}$ (*resp.* $T_*^{\ell_0}$) *is continuous on the* ε_0*-neighbourhood* $U_{\varepsilon_0}(x_0)$ *of* x_0 *and for* $0 \leq j \leq \ell_0$

$$T_*^{-j} U_\varepsilon(x_0) \cap V_j(2\varepsilon_0) = \emptyset \qquad (\textit{resp. } T_*^j U_\varepsilon(x_0) \cap V_j(2\varepsilon_0) = \emptyset)\ ,$$

$$\min \cos (U_{\varepsilon_0}(x_0)) \geq \frac{\omega}{4}\ ,$$

(ii) *for any positive* $\Omega_1 (\leq \Omega)$ *and for any K-increasing* (*resp. K-decreasing*) *curve in* $U_{\varepsilon_1}(x_0)$, *there exists a K-quadrilateral* G *in* $U_{\varepsilon_0}(x)$ *such that* $T_*^{-\ell_0}G$ (*resp.* $T_*^{\ell_0}G$) *is also a K-quadrilateral with* $\gamma_b(G) = \gamma$ *and* $\theta(\gamma_a(G)) = \Omega_1\theta(\gamma)$ (*resp. with* $\gamma_a(G) = \gamma$ *and* $\theta(\gamma_b(G)) = \Omega_1\theta(\gamma)$).

Proof. Put

$$\delta(x_0, \ell_0) \equiv \min_{0 \leq j \leq \ell_0} \tfrac{1}{2} |\cos\varphi(T_*^{-j} x_0)|\ .$$

Denote by Y the element of $\bigvee_{i=0}^{\ell_0-1} T_*^i \alpha^{(c)}$ which contains x_0. By (5°) in § 3 and by Lemma 4.1

$$Y' \equiv Y - \bigcup_{i=0}^{\ell_0-1} T_*^i V_i(\delta(x_0, \ell_0))$$

is a connected open set which contains x_0. Hence one can choose $\varepsilon_0 (< \delta(x_0, \ell_0)/4)$ in such a way that $U_{\varepsilon_0}(x_0)$ is included in Y'. Then $T_*^{-\ell_0}$ is continuous on $U_{\varepsilon_0}(x_0)$ and it is proved that $\min \cos (U_{\varepsilon_0}(x_0)) \geq (\omega/4)$ and for $0 \leq j \leq \ell_0$

$$T_*^{-j} U_{\varepsilon_0}(x_0) \cap V_j(\delta(x_0, \ell_0)) = \emptyset .$$

If ε_1 is taken to be so small that

$$U_{\varepsilon_1}(x_0) \subset T_*^{\ell_0} U_{\varepsilon_2}(T_*^{-\ell_0} x_0) \quad \text{and} \quad U_{a\varepsilon_2}(T_*^{-\ell_0} x_0) \subset T_*^{-\ell_0} U_{\varepsilon_0}(x_0)$$

with a suitable ε_2 and $a \equiv 4(c_4 + 1/K_{\min})$, then (ii) is true. Q.E.D.

Let γ and γ' be two K-increasing curves of C^1-class and let $\Psi = \Psi^{(c)}_{\gamma',\gamma}$ be the canonical mapping with domain Φ and range Φ'. Then there exists a K-quadrilateral G such that

$$\Phi \subset \gamma_b(G) \subset \gamma , \qquad \Phi' \subset \gamma_d(G) \subset \gamma' ,$$

and that both $\gamma_a(G)$ and $\gamma_c(G)$ intersect with no-elements of $\zeta^{(c)}$. Put

(7.1) $$G^0 \equiv \{x \in G ; \gamma^{(c)}(x) \cap \gamma \neq \phi \quad \text{and} \quad \gamma^{(c)}(x) \cap \gamma' \neq \phi\} .$$

Then $\Phi = G^0 \cap \gamma$ and $\Phi' = G^0 \cap \gamma'$.

LEMMA 7.2. *Let γ and γ' be K-increasing curves. Let G and G^0 be as in above. Then G^0 is measurable and there exists a measurable subset $G^{(c)}$ of G^0 with $\nu(G^{(c)}) = \nu(G^0)$ such that*

$$G \cap \gamma^{(c)}(x) \subset G^{(c)} \qquad \textit{for } x \in G^{(c)}$$

holds and that for any K-increasing curves $\tilde{\gamma}$ and $\tilde{\gamma}'$ of C^1 class in G which join $\gamma_a(G)$ and $\gamma_c(G)$, the canonical mapping $\Psi^{(c)}_{\tilde{\gamma}',\tilde{\gamma}}$ is absolutely continuous on $\tilde{\gamma} \cap G^{(c)}$.

Proof. Fix α_0 $(0 < \alpha_0 < 1)$ and put $\alpha \equiv \alpha_0 \nu^*(G^0)/4$, where $\nu^*(G^0)$ is the outer measure of the set G^0. Then Lemma 6.1 gives a natural number $\ell_0 = \ell_0(\alpha, 1 + c_2, \omega/4)$. Now construct a sequence of families of K-quadrilaterals $\{F_{m,s}\}$ like $\{G_{m,s}\}$ in the proof of Lemma 6.1 as follows. Put $F_0 \equiv G$ and suppose that $\{F_{m-1,s}\}$ is suitablly constructed. Then put

$$O_{m-1,s,j} \equiv F_{m-1,s} \cap X_j^{(c)} ,$$
$$F_{m-1,s,j} \equiv T_*^{-1} O_{m-1,s,j} - \bigcup_{x \in \gamma_a(T_*^{-1} O_{m-1,s,j})} D_a(x) - \bigcup_{x \in \gamma_c(T_*^{-1} O_{m-1,s,j})} D_c(x) .$$

After a suitable renumbering $\{F_{m-1,s,j}\}$, denote them by $\{F_{m,s'}\}$. It is obvious that

$$G^0 = \bigcap_{n=m_0}^{\infty} \bigcup_{s} T_*^m F_{m,s} \subset \bigcup_{s} T_*^m F_{m,s} \subset G .$$

Hence G^0 is measurable and $\nu^*(G^0) = \nu(G^0)$. A piece $F_{m,s}$ is said to be docile if $F_{m,s}$ touches to S. A piece $F_{m,s}$ is said to be *wide* or *narrow* according as

$$\theta(\gamma_b(F_{m,s})) \geq \pi(1 + \eta)^{-m/8}$$

or

$$\theta(\gamma_b(F_{m,s})) \leq \pi(1 + \eta)^{-m/4} .$$

Define $\Delta(x) \equiv \inf\{(1 + \eta)^{-i/4} d^{(-1)}(T_*^{-i}x)/c_1 ; i \geq 0\}$, then one can choose ω and m_* so that $\nu(G^0 - E) < \alpha$ for $E \equiv \{x \in G^0 ; -\cos\varphi(x) \geq 4\omega,\ \Delta(T_*^{-k}x) \geq 4\pi(1 + \eta^{-1})(1 + \eta)^{-k/4}$ for $k \geq m_*\}$. Put

$$W_m(a) \equiv \{x, d^{(-\ell_0)}(x) \leq a(1 + \eta)^{-m/16}\} .$$

Now let m_0 be a sufficiently large natural number whose actual value will be determined later. Fix $m\,(\geq m_0)$ and suppose that $F_{m,s}$ is not narrow, then one can find a family of K-quadrilaterals $\{G_{m,s,j}\}$ such that $T_*^{-\ell_0}$ is continuous on $G_{m,s,j}$, $T_*^{-\ell_0}G_{m,s,j}$ are also K-quadrilaterals, and the following relations hold;

$$\begin{aligned}
&F_{m,s} - \textstyle\bigcup_j G_{m,s,j} \subset W_m(2(1 + c_2)^2\pi) ,\\
&G_{m,s,j} \cap W_m((1 + c_2)^2\pi) = \emptyset ,\\
&\theta(\gamma_b(G_{m,s,j})) \leq \theta(\gamma_a(G_{m,s,j})) \leq (1 + c_2)\theta(\gamma_b(G_{m,s,j}))
\end{aligned}$$

(see § 7 in [6]). If

$$\min\cos(G_{m,s.j}) \geq \omega/4$$

holds, then one can apply Lemma 6.1 to each $G_{m,s,j}$, to prove that there exist measurable subsets $G_{m,s,j}^{(c,\alpha)}$ which satisfy the conditions (C–1), (C–2) and (C–3) in Lemma 6.1. Since T_*^m is a C^2-diffeomorphism from $G_{m,s,j}$ into G, the canonical mapping $\Psi_{\gamma,\gamma'}^{(c)}$ is absolutely continuous on $T_*^m G_{m,s,j}^{(c,\alpha)} \cap \gamma$. Put

$$G_m \equiv \bigcup_{s,j} \{G_{m,s,j} ;\ F_{m,s} \text{ is not narrow and } \min\cos(G_{m,s,j}) \geq \omega/4\} .$$

Note that the measure of the set $N \equiv \{x \in G_{m_0,s_0} ; T^k x$ is contained in not-wide and not-docile F_{k,s_k} for any $k \geq m_0\}$ is equal to zero by Lemma 5.5 (cf. §6). In other words, for almost every $x \in E^0 \equiv E - \bigcup_{m=m_0}^{\infty} T^m W_m(2(1+c_2)^2)$ with $T^m x \in F_{m,s_m}$, $m = 0, 1, 2, \cdots$, there exist infinitely many wide F_{m,s_m}'s. Note the estimate $\theta(\gamma_b(F_{m+1,s_{m+1}})) \geq (1+\eta)(\min\{\theta(\gamma_b(F_{m,s_m})), d^{(-1)}(T^{-m}x)/c_1\} - 2\max\theta_{de}(F_{m,s_m}))$. If F_{m,s_m} is wide, then for $n \geq m \geq m_*$ the estimate $\theta(\gamma_b(F_{n,s_n})) \geq 2\pi(1+\eta)^{-m/4}$ holds; that is, F_{n,s_n} is not narrow. By Poincaré's recurrent theorem, for almost every $x \in E^0$ there exist infinitely many $\{m_k\}$ with $T^{m_k}x \in G_{m_k}$. Thus one has the estimate

$$\nu\left(G^0 - \bigcup_{n=m_0}^{\infty} T_*^n G_n\right) \leq \text{const.}\ (1+\eta)^{-m_0 b(\ell_0)/16} + \nu(G^0 - E)\ ,$$

where const. is an absolute constant. Let m_0 be a natural number for which the right hand side of the above inequality is less than 2α. Put $G_m^{(c,\alpha)} \equiv \bigcup_{s,j}\{G_{m,s,j}^{(c,\alpha)}\,;\, G_{m,s,j} \subset G_m\}$ and $G(\alpha_0) \equiv \bigcup_{m=m_0}^{\infty} T_*^m G_m^{(c,\alpha)}$. Then

$$\begin{aligned}\nu(G^0 - G(\alpha_0)) &\leq \nu\left(G^0 - \bigcup_{m=m_0}^{\infty} T_*^m G_m\right) + \sum_{m=m_0}^{\infty}\ \sum_{G_{m,s,j}\subset G_m} \nu(G_{m,s,j} - G_{m,s,j}^{(c,\alpha)})\\ &\leq 3\alpha \leq \alpha_0 \nu(G_0)\ .\end{aligned}$$

Put $G^{(c)} \equiv \bigcup_{n=3}^{\infty} G(1/n)$. Then $G^{(c)}$ satisfies the desired conditions.

Q.E.D.

In general, denote by $\partial\gamma = \partial\gamma(x)$ the gradient of a curve γ at x and put $\bar\partial\gamma \equiv 1/\partial\gamma$. Further put

$$(7.2)\qquad \bar\partial_k\gamma(x) \equiv \bar\partial(T_*^{-k}\gamma)(T_*^{-k}x) \quad\text{and}\quad \partial_k\gamma(x) \equiv \partial(T_*^{-k}\gamma)(T_*^{-k}x)\ .$$

Then by Lemma 4.3 (i),

$$(7.3)\qquad \bar\partial_k\gamma(x) = b_k(x\,;\,\bar\partial\gamma(x))$$

holds.

Let γ and γ' be increasing curves of C^1-class as in Lemma 7.2. Suppose that they are given by the equations

$$r = u(\varphi) \quad\text{and}\quad r = u'(\varphi)\ ,$$

respectively. Hereafter assume that the domain and the range of the canonical mapping $\Psi_{\gamma',\gamma}^{(c)}$ to be $\Phi_{\gamma,\gamma'}^{(c)} \equiv \gamma \cap G^{(c)}$ and $\Phi_{\gamma',\gamma}^{(c)} \equiv \gamma' \cap G^{(c)}$ respectively, where $G^{(c)}$ is the set given in Lemma 7.2.

LEMMA 7.3. *Let γ and γ' be K-increasing curves of C^1-class as in Lemma 7.2, and let $g^{(c)}_{\gamma,\gamma'}(\iota, r, \varphi)$ be the Radon-Nikodym density:*

$$g^{(c)}_{\gamma,\gamma'}(\iota, r, \varphi) = \frac{d\Psi^{(c)}_{\gamma,\gamma'}\sigma_{\gamma'}}{d\sigma_\gamma} \qquad \text{on } \Phi^{(c)}_{\gamma,\gamma'} . \tag{7.4}$$

Then $g_{\gamma,\gamma'}$ can be represented by the infinite products;

$$\begin{aligned} g^{(c)}_{\gamma,\gamma'}(\iota, r, \varphi) &= \prod_{i=0}^{\infty} \frac{\dfrac{k_{i+1}\cos\varphi_i + k'_i\cos\varphi_{i+1} + k_{i+1}k'_i\tau_{i+1}}{\cos\varphi_{i+1}}\{\bar\partial_i\gamma + h(\iota_i, \varphi_i)\} + \dfrac{k_{i+1}\tau_{i+1}}{\cos\varphi_{i+1}} + 1}{\dfrac{\hat k_{i+1}\cos\hat\varphi_i + \hat k'_i\cos\hat\varphi_{i+1} + \hat k_{i+1}\hat k'_i\hat\tau_{i+1}}{\cos\hat\varphi_{i+1}}\{\bar\partial_i\gamma' + h(\iota_i, \hat\varphi_i)\} + \dfrac{\hat k_{i+1}\hat\tau_{i+1}}{\cos\hat\varphi_{i+1}} + 1} \\ &= \frac{\partial\gamma}{\partial\hat\gamma}\,\frac{\cos\varphi}{\cos\hat\varphi}\prod_{i=0}^{\infty}\frac{1 + \dfrac{k'_i\tau_{i+1}}{\cos\varphi_i} + \left\{\left(1 + \dfrac{k'\tau_{i+1}}{\cos\varphi_i}\right)h(\iota_i, \varphi_i) + \dfrac{\tau_{i+1}}{\cos\varphi_i}\right\}\partial_i\gamma}{1 + \dfrac{\hat k_i\hat\tau_{i+1}}{\cos\hat\varphi_i} + \left\{\left(1 + \dfrac{\hat k'\hat\tau_{i+1}}{\cos\varphi_i}\right)h(\iota_i, \hat\varphi_i) + \dfrac{\hat\tau_{i+1}}{\cos\hat\varphi_i}\right\}\partial_i\gamma'} \end{aligned} \tag{7.5}$$

where $(\iota_i, r_i, \varphi_i) \equiv T_*^{-i}(\iota, r, \varphi)$ and $(\iota_i, \hat r_i, \hat\varphi_i) \equiv T_*^i\Psi^{(c)}_{\gamma',\gamma}(\iota, r, \varphi)$. Moreover, the estimate

$$g^{(c)}_{\gamma,\gamma'}(\iota, r, \varphi) \le \exp\left[c_{27}\sum_{i=0}^{\infty}\frac{(1+\eta)^{-i}|\varphi - \hat\varphi|}{\min\{-\cos\varphi_i, -\cos\hat\varphi_i\}} + c_{27}\left|\log\frac{\bar\partial\gamma'}{\bar\partial\gamma}\right|\right]$$

holds with a suitable constant c_{27}.

Proof. First recall the estimate (6.16). Since $\theta(C_m(x))$ and $\theta(C'_m(x'))$ converge to 0 as $m \to \infty$,

$$\max_{\hat y, \hat y \in C_m(x)}\left|\log\frac{du_0}{d\varphi}(\hat y)\Big/\frac{du_0}{d\varphi}(\hat y)\right| \quad \text{and} \quad \max_{\hat y, \hat y \in C'_m(x)}\left|\log\frac{du'_0}{d\varphi}(\hat y)\Big/\frac{du'_0}{d\varphi}(\hat y)\right|$$

converge to 0 as well. Hence

$$\lim_{m\to\infty}\frac{\theta(C_m(x))}{\theta(T_*^{-m}C_m(x))}\prod_{i=0}^{m-1}|\Lambda(x_i, T_*^{-i}\gamma)| = 1 ,$$

$$\lim_{m\to\infty}\frac{\theta(C'_m(x'))}{\theta(T_*^{-m}C'_m(x'))}\prod_{i=0}^{m-1}|\Lambda(x'_i, T_*^{-i}\gamma')| = 1 .$$

From (6.18) and (6.19),

$$\lim_{m\to\infty}\frac{\theta(C'_m(x'))}{\theta(C_m(x)} = \prod_{i=0}^{\infty}\frac{\Lambda(x_i, T_*^{-i}\gamma)}{\Lambda(x'_i, T_*^{-i}\gamma')} = g$$

holds. Since

$$\frac{d\varphi_{i+1}}{d\varphi_i} = \frac{d\varphi_{i+1}}{dr_{i+1}} \frac{dr_i}{d\varphi_i} \frac{dr_{i+1}}{dr_i}$$

by Lemma 3.3, the two expressions in (7.5) are obtained. By (6.19) and (6.19)′, the inequality in the lemma is obtained. Q.E.D.

§ 8. Measure theoretical properties of $\gamma^{(c)}$ and $\gamma^{(e)}$

The purpose of this section is to show that $\gamma^{(c)}$ and $\gamma^{(e)}$ play a role of a coordinate system in the sense of measure theory. Let γ be a curve. Put

$$A[\gamma] = A^{(c)}[\gamma] \equiv \bigcup_{x\in\gamma} \gamma^{(c)}(x) \qquad (8.1)$$
$$\left(\text{resp. } A^{(e)}[\gamma] \equiv \bigcup_{x\in\gamma} \gamma^{(e)}(x)\right).$$

If γ is continuous, then the expression

$$A[\gamma] = \bigcap_{k=0}^{\infty} \bigcup_{C\cap\gamma\neq\emptyset, C\in\vee_{j=0}^{k} T_*^j\alpha^{(c)}} C$$

is true. Therefore $A[\gamma]$ is a Borel set.

LEMMA 8.1. *Let γ be a K-increasing curve, then*

$$\nu(A[\gamma]) > 0 .$$

Proof. Since $\bigcup_{i=0}^{\infty} T_*^i S$ consists of a countable number of K-decreasing curves, $\gamma \cap (\bigcup_{i=0}^{\infty} T_*^i S)$ is a denumerable set. Hence there exists a point x_0 in $\gamma - \bigcup_{i=0}^{\infty} T_*^i S$. Let $\varepsilon_1 = \varepsilon_1(x_0, 1/4, 1, \omega)$ be the constant given in Lemma 7.2 with $\omega = -\cos\varphi(x_0)$. Put $\tilde\gamma \equiv \gamma \cap U_{\varepsilon_1}(x_0)$. Then there exists a K-quadrilateral G such that $\tilde\gamma$ joins $\gamma_a(G)$ and $\gamma_c(G)$, $\theta(\gamma_b(G)) = \theta(\gamma_a(G))$ holds and $T_*^{-\ell_0}G$ is also a K-quadrilateral with $\ell_0 = \ell_0(1/4, 1, \omega/4)$. Obviously, $\nu(A[\gamma]) \geq \nu(G^{(c,1/4)}) \geq (3/4)\nu(G) > 0$. Q.E.D.

Let γ be a K-decreasing curve with $\theta(\gamma) = \pi$, and let $r = u_0(\varphi)$ be the equation of γ. Put $\gamma_t \equiv \{(\iota, r + t, \varphi) ; (\iota, r, \varphi) \in \gamma\}$, that is, γ_t be the curve given by the equation $r = u_t(\varphi)$ with $u_t(\varphi) \equiv u_0(\varphi) + t$. Denote by $G_{t,s}$ the quadrilateral surrounded by S, γ_t and γ_s. Put

$$G_{t,s}^0 \equiv \{x \in G_{t,s} ; \gamma^{(c)}(x) \text{ intersects with both } \gamma_t \text{ and } \gamma_s\} .$$

Then Lemma 7.2 gives a set $G_{t,s}^{(c)}$ on which the canonical mapping $\Psi_{t,s}^{(c)} \equiv$

$\Psi^{(c)}_{\gamma_t,\gamma_s}$ is absolutely continuous. Introduce, for convenience, simplified notations:

$$\Psi^{(c)}_{t,s} \equiv \Psi^{(c)}_{\gamma_t,\gamma_s},\ \Phi^{(c)}_{t,s} \equiv \Phi^{(c)}_{\gamma_t,\gamma_s} \quad \text{and} \quad g^{(c)}_{t,s} \equiv g^{(c)}_{\gamma_t,\gamma_s}\ .$$

Suppose that the curve $\gamma^{(c)}(\iota, u_t(\varphi), \varphi)$ is represented by $r = \tilde{u}_{t,\varphi}(\psi)$. Then for a given Borel set B

$$\begin{aligned}\nu(B \cap G^{(c)}_{t,s}) &= -\nu_0 \int_t^s dr \int_{B\cap G^{(c)}_{t,s}\cap\gamma_r} \cos\varphi d\sigma_{\gamma_r}(\varphi) \\ &= -\nu_0 \int_t^s dr \int_{\Phi_{t,s}\cap\Psi^{(c)}_{t,r}(B\cap\gamma_r)} \cos\varphi_r g_{t,r}(\iota, u_t(\varphi), \varphi) d\sigma_{\gamma_t}(\varphi) \\ &= \int_{\Phi_{t,s}} d\varphi \int_{B\cap\gamma^{(c)}(\iota,u_t(\varphi),\varphi)\cap G^{(c)}_{t,s}} g_t(\varphi,\psi)d\psi\end{aligned}$$

holds, where $(\iota, u_r(\varphi_r), \varphi_r) = \Psi^{(c)}_{r,t}(\iota, u_t(\varphi), \varphi)$ and

$$g_t(\varphi, \psi) \equiv -\nu_0 \cos\psi g_{t,\tilde{u}_{t,\varphi}(\psi)-u_0(\psi)}(\iota, \tilde{u}_{t,\varphi}(\psi), \psi)[\chi^{(c)}(\iota, \tilde{u}_{t,\varphi}(\psi), \psi)]^{-1}\ . \tag{8.3}$$

Put $N^*_q \equiv \bigcup_n G^{(c)}_{n2-q,(n+1)2-q}$, $N^* \equiv \bigcup_q N^*_q$ and $A^*[\gamma] \equiv \bigcup_{q,n} (G^{(c)}_{0,n2-q} - G^{(c)}_{0,(n+1)2-q}) = A[\gamma] \cap N^*$. If $\Delta^{(1)}(x) > 0$ and $\iota(x) = \iota$, there exist q and n such that x is in $G^0_{n2-q,(n+1)2-q}$, because $\theta(\gamma^{(c)}(x)) > 0$. Hence $\nu(M^{(\iota)} - N^*) = 0$. Therefore

$$\begin{aligned}\nu(A[\gamma] \cap B) &= \nu(A^*[\gamma] \cap B) \\ &= \int_{\gamma\cap A^*[\gamma]} d\sigma_\gamma(\varphi) \int_{\gamma^{(c)}(\iota,u_0(\varphi),\varphi)\cap B} g_0(\varphi,\psi) d\sigma_{\gamma^{(c)}}(\psi)\ .\end{aligned} \tag{8.4}$$

LEMMA 8.2. *Let γ be a K-decreasing curve in $M^{(\iota)}$. Then $\sigma_\gamma(\bar{\gamma}) = 0$ if and only if $\nu(A[\bar{\gamma}]) = 0$ for any Borel subset $\bar{\gamma}$ in γ.*

Proof. Assume that $\sigma_\gamma(\bar{\gamma}) = 0$. Then by (8.4)

$$\begin{aligned}\nu(A[\bar{\gamma}]) &= \nu(A[\bar{\gamma}] \cap A^*[\gamma]) \\ &= \int_{\bar{\gamma}\cap A^*[\gamma]} d\sigma_\gamma(\varphi) \int_{\gamma^{(c)}(\iota,u_0(\varphi),\varphi)} g_0(\varphi,\psi) d\sigma_{\gamma^{(c)}}(\psi) \\ &= 0\ .\end{aligned}$$

Conversely, assume that $\sigma_\gamma(\bar{\gamma}) > 0$. Since $\gamma \cap \bigcup_{i=0}^\infty T^iS$ is a denumerable set, there exists a point x_0 in $\bar{\gamma} - \bigcup_{i=0}^\infty T^iS$ which is a density point of $\bar{\gamma}$. Then there exists a segment γ_0 of γ such that x_0 is in γ_0, where γ_0 is in $U_{\varepsilon_1}(x_0)$ with $\varepsilon_1 = \varepsilon_1(x_0, 1/4, 1, \omega)$ with $\omega = -\cos\varphi(x_0)$ and that $\sigma_\gamma(\gamma_0 \cap \bar{\gamma}) \geq (1 - 1/64\beta_1 c_2^2)\sigma_\gamma(\gamma_0)$. Let G be a K-quadrilateral with γ_0 in G such that γ_0 joins $\gamma_a(G)$ and $\gamma_d(G)$, and that $T^{-\ell_0}G$ is also a K-quadrilateral with $\ell_0 = \ell_0(1/4, 1, \omega/4)$. Then there exists a subset $\overline{G} = G^{(c,1/4)}$ which satisfies

(C–1), (C–2) and (C–3) in Lemma 6.1. Since by Lemma 6.1 and Lemma 7.1

$$-\frac{\nu_0 \cos\varphi(x_0)}{4K_{\max}\beta(1)} \leq g_0(\varphi, \psi) \leq -\frac{4\nu_0 \cos\varphi(x_0)}{K_{\min}}\beta(1)$$

and since $\max\theta_{de}(G) \leq (1 + c_2)\theta(\gamma_0)$, the following estimate is given

$$\begin{aligned}\nu(\overline{G}) &\leq -\frac{4\nu_0 \cos\varphi(x_0)\beta(1)}{K_{\min}}\int_{G\cap\gamma_0} d\sigma_\gamma(\varphi)\sigma_{\gamma^{(c)}}(\iota, u_0(\varphi), \varphi)) \\ &\leq -\frac{(1 + c_2)\nu_0 \cos\varphi(x_0)\beta(1)}{4K_{\min}}\sigma_\gamma(\overline{G} \cap \gamma_0)\theta(\gamma_0) .\end{aligned}$$

On the other hand,

$$\nu(\overline{G}) \geq \frac{3}{4}\nu(G) \geq \frac{-3\nu_0 \cos\varphi(x_0)(1 + c_2)}{16K_{\max}c^2}\theta(\gamma_0)^2 .$$

Therefore

$$\sigma_\gamma(\overline{G} \cap \gamma_0) \geq \frac{3}{64c_2^2\beta(1)}\theta(\gamma_0)$$

and hence

$$\sigma_\gamma(\overline{G} \cap \gamma_0 \cap \bar{\gamma}) \geq \frac{1}{32c_2^2\beta(1)}\theta(\gamma_0) .$$

This proves

$$\begin{aligned}\nu(A[\bar{\gamma}]) &\geq \nu(A[\gamma_0 \cap \bar{\gamma} \cap \overline{G}]) \\ &\geq \frac{-\nu_0 \cos(x_0)}{4K_{\max}c_2\beta(1)}\,\theta(\gamma_0)\sigma(\gamma_0 \cap \bar{\gamma} \cap \overline{G}) \\ &> 0 .\end{aligned}$$

Q.E.D.

Let γ be a K-decreasing (resp. K-increasing) curve of C^1-class in $M^{(\iota)}$. Let γ^* be an extension of γ which is a K-decreasing (resp. K-increasing) curve of C^1-class with $\theta(\gamma^*) = \pi$. Suppose that γ^* is defined by the equation $r = u_0(\varphi)$. Denote $\gamma^{(c)}(\iota, u_0(\varphi), \varphi)$ simply by $\gamma^{(c)}$ (resp. $\gamma^{(e)}(\iota, u_0(\varphi), \varphi)$ by $\gamma^{(e)}$) and suppose that $\gamma^{(c)}$ (resp. $\gamma^{(e)}$) is defined by the equation $r = u^{(c)}(\psi)$ (resp. $r = u^{(e)}(\psi)$). Define the functions $g_0^{(c)}(\varphi, \psi)$ and $g_0^{(e)}(\varphi, \psi)$ by

$$\begin{aligned}g_0^{(c)}(\varphi, \psi) &= \frac{\nu_0 \cos\psi}{\chi^{(c)}(\hat{x}_0)}\prod_{i=0}^{\infty}\frac{\cos\psi_{i+1}}{\cos\varphi_{i+1}} \\ &\times \frac{\{k_{i+1}\cos\varphi_i + k'_i\cos\varphi_{i+1} + k_{i+1}k'_i\tau_{i+1}\}b_i + k_{i+1}\tau_{i+1} + \cos\varphi_{i+1}}{\{\hat{k}_{i+1}\cos\psi_i + \hat{k}'_i\cos\psi_{i+1} + \hat{k}_{i+1}\hat{k}'_i\hat{\tau}_{i+1}\}\hat{b}_i + \hat{k}_{i+1}\hat{\tau}_{i+1} + \cos\psi_{i+1}}\end{aligned} \tag{8.5}$$

with $\hat{x}_i = (\iota_i, \hat{r}_i, \psi_i) \equiv T_*^{-i}(\iota, u^{(c)}(\psi), \psi)$, $\hat{k}_i \equiv k(\hat{x}_i)$, $\hat{k}'_i \equiv k'(\hat{x}_i)$, $\hat{\tau}_i \equiv \tau(\hat{x}_i)$, $\hat{b}_i \equiv \hat{b}_i(\iota, u^{(c)}(\psi), \psi; du_0/d\psi)$ and $b_i \equiv b_i((\iota, u_0(\varphi), \varphi); du_0/d\varphi)$,

$$(8.6)\quad \begin{aligned} g_0^{(e)}(\varphi, \psi) = & \frac{-\nu_0 \cos\psi}{\chi^{(e)}(\tilde{x}_0)} \prod_{i=-1}^{-\infty} \frac{\cos\psi_i}{\cos\varphi_i} \\ & \times \frac{\{k_{i+1}\cos\varphi_i + k'_i \cos\varphi_{i+1} + k_{i+1}k'_i\tau_{i+1}\}b_{i+1} - k'_i\tau_{i+1} - \cos\varphi_i}{\{\tilde{k}_{i+1}\cos\psi_i + \tilde{k}'_i\cos\psi_{i+1} + \tilde{k}_{i+1}\tilde{k}'_i\tilde{\tau}_{i+1}\}\tilde{b}_{i+1} - \tilde{k}'_i\tilde{\tau}_{i+1} - \cos\psi_i} \end{aligned}$$

with $\tilde{x}_i \equiv (\iota_i, \tilde{r}_i, \psi_i) \equiv T_*^{-i}(\iota, u^{(c)}(\psi), \psi)$, $\tilde{k}_i \equiv k(\tilde{x}_i)$, $\tilde{k}'_i \equiv k'(\tilde{x}_i)$, $\tilde{\tau}_i \equiv \tau(\tilde{x}_i)$, $\tilde{b}_i \equiv b_i(\iota, u^{(e)}(\psi), \psi; du_0/d\psi)$ and $b_i \equiv b_i(\iota, u_0(\varphi), \varphi; du_0/d\varphi)$, of course $(\iota_i, r_i, \varphi_i) \equiv T_*^{-i}(\iota, u_0(\varphi), \varphi)$, $k_i \equiv k(\iota_i, r_i)$, $k'_i \equiv k'(\iota_i, r_i)$, $\tau_i \equiv \tau(\iota_i, r_i, \varphi_i)$. Then the following lemma holds.

LEMMA 8.3. *Let γ be a K-decreasing (resp. K-increasing) curve of C^1-class in $M^{(\iota)}$. Then*

$$(8.7)\quad \begin{aligned} &\nu(B \cap A^{(c)}[\gamma]) = \int_\gamma d\sigma_\gamma(\varphi) \int_{\gamma^{(e)} \cap B} g_0^{(c)}(\varphi, \psi) d\sigma_{\gamma^{(e)}}(\psi) \\ &\left(resp.\ \nu(B \cap A^{(e)}[\gamma]) = \int_\gamma d\sigma_\gamma(\varphi) \int_{\gamma^{(e)} \cap B} g_0^{(e)}(\varphi, \psi) d\sigma_{\gamma^{(e)}}(\psi)\right). \end{aligned}$$

Proof. Put $\bar{\gamma} \equiv \gamma - \gamma \cap A^*[\gamma]$ and assume that $\sigma_\gamma(\bar{\gamma}) > 0$. Then by Lemma 8.2, $\nu(A^*[\bar{\gamma}]) > 0$. Since $\bar{\gamma} \subset \gamma$, the inclusion $A^*[\bar{\gamma}] \subset A^*[\gamma]$ holds. On the other hand, $A^*[\bar{\gamma}] \cap A[\gamma] = \emptyset$ since $A^*[\gamma] \cap \bar{\gamma} = \emptyset$. This is a contradiction. Hence $\sigma(\bar{\gamma}) = 0$ and hence Lemma 8.2 is true for the first case by the use of (8.4). The second case can be shown similarly.

Q.E.D.

LEMMA 8.4. (i) *Each conditional measure with respect to $\zeta^{(c)}$ (resp. $\zeta^{(e)}$) are equivalent to $\sigma_{\gamma^{(c)}}$ (resp. $\sigma_{\gamma^{(e)}}$) for almost every $\gamma^{(c)}$ (resp. $\gamma^{(e)}$).*

(ii) *Let $\sigma^{(e)}$ be a measure on a curve $\gamma^{(c)}$ (resp. $\gamma^{(e)}$) defined by*

$$\sigma^{(e)}(\bar{\gamma}) \equiv \nu(A^{(c)}[\bar{\gamma}]), \bar{\gamma} \subset \gamma^{(e)},$$
$$(resp.\ \sigma^{(c)}(\bar{\gamma}) \equiv \nu(A^{(e)}[\bar{\gamma}]), \bar{\gamma} \subset \gamma^{(c)}).$$

Then for almost every $\gamma^{(e)}$ in $\zeta^{(e)}$ (resp. $\gamma^{(c)}$ in $\zeta^{(c)}$) $\sigma^{(e)}$ and $\sigma_{\gamma^{(e)}}$ (resp. $\sigma^{(c)}$ and $\sigma_{\gamma^{(c)}}$) are equivalent.

Proof. The proof is clear by Lemma 8.2 and 8.3.

§ 9. A perturbed billiard transformation is a K-system

The idea of the proof of the K-property is the same as in the case of the Sinai billiard system [6], [10]. The idea due originally to E. Hopf was generalized by Ya. G. Sinai [9].

LEMMA 9.1. *$\zeta_{-\infty}^{(c)} \wedge \zeta_{\infty}^{(e)}$ is the trivial partition.*

Proof. Let $f(x)$ be a $\zeta_{-\infty}^{(c)} \wedge \zeta_{\infty}^{(e)}$-measurable function. Then there exist functions $f_1(x)$ and $f_2(x)$ such that

$$(9.1) \qquad \begin{cases} f_1(x) = f_1(y) & \text{for any } y \text{ in } \Gamma^{(c)}(x) \\ f_2(x) = f_2(z) & \text{for any } z \text{ in } \Gamma^{(e)}(x) \\ f(x) = f_1(x) = f_2(x) & \text{for almost every } x \text{ in } M . \end{cases}$$

Further there exists a measurable set $N(f)$ such that

$$(9.2) \qquad \begin{cases} \nu(N(f)) = 1 \\ \nu(N(f) \mid \gamma^{(c)}(x)) = \nu(N(f) \mid \gamma^{(e)}(x)) = 1 & \text{for } x \text{ in } N(f) \\ f(x) = f_1(x) = f_2(x) & \text{for } x \text{ in } N(f) . \end{cases}$$

Put $\alpha = (128c_2(1 + c_2))^{-1}$ and $\ell_0 = \ell_0(\alpha, 2, \omega/4)$. Denote by $\{Y_j^{(\ell_0)}\}$ the all elements of the partition $\bigvee_{j=-\ell_0-1}^{\ell_0} T_*^k \alpha^{(c)} \cap \{x\,;\, -\cos\varphi(x) \geq \omega\}$. Let x be an inner point of $Y_j^{(\ell_0)}$ and let $\varepsilon_1 = \varepsilon_1(x, \alpha, 2, \omega)$ be as in Lemma 7.1. Let V be a rectangle in $U_{\varepsilon_1/2}(x)$ such that a pair of sides is parallel with φ-axis, the length of the horizontal side is $4/K_{\min}$ times of the length of the vertical side and x is the center of V. Let $\bar{V}$ be the rectangle with the same center x, the same horizontal size as V and twice vertical size of V. Then V separates $\bar{V}$ into three rectangles. Denote by $\bar{V}_1$ the top rectangle and by $\bar{V}_2$ the bottom rectangle (see Fig. 9–1). Since $\bar{V} \subset U_{\varepsilon_1}(x)$, there exists a K-quadrilateral G such that $\gamma_a(G)$ and $\gamma_b(G)$ join the top side and the bottom side of $\bar{V}$ and that $T_*^{-\ell_0}G$ is also a K-quadrilateral. By Lemma 6.1, there exists a subset $G^{(c,\alpha)}$ of G, which satisfies (C–1), (C–2) and (C–3). Since the estimate

$$\nu(\bar{V}_1 \cap G) \geq \frac{\nu(G)}{64(1 + c_2)c_2} \geq 2\alpha\nu(G)$$

is obtained, the inequality

$$\nu(G^{(c,\alpha)} \cap \bar{V}_1 \cap N(f)) \geq \alpha\nu(G) > 0$$

holds. Hence there exists a point $\bar{x}$ in $G^{(\alpha)} \cap N(f) \cap \bar{V}_1$. Obviously, the curve $\gamma^{(c)}(\bar{x})$ intersects with the bottom side and the top side of V.

Let x_0 be an arbitrary point in $V \cap N(f)$. Let V_0 be a rectangle in V such that the vertical sides of V_0 are included in the vertical sides of

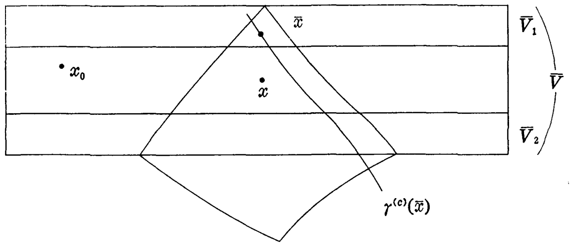

Fig. 9-1

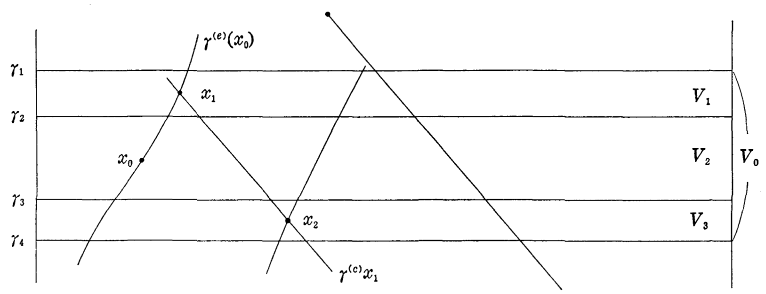

Fig. 9-2

V and the line $\varphi = \varphi(x_0)$ is the center line of V_0. Divide V_0 into three rectangles V_1, V_2 and V_3, where V_1 is the upper quarter of V_0, V_2 is the central half of V_0 and V_3 is the lower quarter of V_0. Denote by $\gamma_1, \gamma_2, \gamma_3$ and γ_4 the top side of V_1, the top side of V_2, the top side of V_3 and the bottom side of V_3, respectively (see Fig. 9–2). Suppose that x_0 lies in the left hand side of $\gamma^{(c)}(\bar{x})$. Then there exists a K-quadrilateral G_1 such that $\gamma_b(G_1) = \gamma^{(e)}(x) \cap V_0$, $\theta(\gamma_a(G_1)) = \theta(\gamma_b(G_1))$ and $T_*^{-\ell_0}G_1$ is also a K-quadrilateral. Then

$$\nu(G_1^{(c,\alpha)} \cap V_1 \cap N(f)) \geq \alpha\nu(G_1) > 0$$

holds. By Lemma 8.3 and Lemma 8.4,

$$\sigma_{\gamma^{(e)}}(G_1^{(c,\alpha)} \cap V_1 \cap N(f) \cap \gamma^{(e)}(x_0)) > 0$$

because x is in $N(f)$. Hence there exists a point x_1 in

$$G_1^{(c,\alpha)} \cap V_1 \cap N(f) \cap \gamma^{(e)}(x_0) .$$

Then $\gamma^{(c)}(x_1)$ intersects with γ_2 and γ_4. Therefore by Lemma 6.1′, there exists a K-quadrilateral G_2 such that $\gamma_c(G_2) = \gamma^{(c)}(x_1) \cap (V_2 \cup V_3)$, $\gamma_d(G)$ joins γ_1 and γ_4, and $T_*^{\ell_0}G_2$ is also a K-quadrilateral. Then similarly in the above, one can see that

$$\sigma_{\gamma^{(c)}(x_1)}(G_2^{(e,d)} \cap V_1 \cap N(f) \cap \gamma^{(c)}(x_1)) > 0 \ ,$$

and that there exists a point x_2 in $G_2^{(e,d)} \cap V_1 \cap N(f) \cap \gamma^{(c)}(x_1)$. Performing such a procedure repeatedly, one can obtain a chain $\{x_0, x_1, \cdots, x_{2n}\}$ such that x_i is in $N(f)$, x_{2i} is in $\gamma^{(c)}(x_{2i-1})$, x_{2i+1} is in $\gamma^{(e)}(x_{2i})$ and $\gamma^{(e)}(x_{2n})$ intersects with $\gamma^{(c)}(\bar{x})$. Since the canonical mapping $\Psi^{(e)}_{\gamma^{(c)}(x_{2n-1}),\gamma^{(c)}(\bar{x})}$ is absolutely continuous, there exists a point x'_{2n} in $\gamma^{(c)}(x_{2n-1}) \cap N(f)$ such that $x'_{2n+1} \equiv \gamma^{(e)}(x'_{2n}) \cap \gamma^{(c)}(\bar{x})$ is in $N(f)$. By (9.1) and (9.2), it is obtained that

$$\begin{aligned} f(x_0) &= f_2(x_0) = f_2(x_1) = f_1(x_1) = f_1(x_2) = f_2(x_2) = \cdots \\ \cdots &= f_2(x_{2n-2}) = f_2(x_{2n-1}) = f_1(x_{2n-1}) = f_1(x'_{2n}) = f_2(x'_{2n}) \\ &= f_2(x'_{2n+1}) = f_1(x'_{2n+1}) = f_1(\bar{x}) = f(\bar{x}) \ . \end{aligned}$$

Similarly, one can see that $f(x_0) = f(\bar{x})$ when x_0 lies in the right hand side of $\gamma^{(c)}(\bar{x})$. Since x_0 in $N(f) \cap V$ is arbitrary, $f(x_0)$ is equal to a constant for almost every x_0 in V_0. Since x is an arbitrary inner point in $Y_j^{(\ell_0)}$, $f(x)$ is equal to a constant for almost every x in $Y_j^{(\ell_0)}$. Assume that the intersection of the boundaries of $Y_j^{(\ell_0)}$ and $Y_{j'}^{(\ell_0)}$ includes a curve γ. Then by Lemma 4.1, one may assume that γ is either K-increasing or K-decreasing. Suppose that γ is K-increasing. Since $\gamma \cap (\bigcup_{i=0}^{\infty} T_*^i S)$ is a denumerable set, there exists a point x_0 in γ which is not in $\bigcup_{i=0}^{\infty} T_*^i S$. Then there exists a K-quadrilateral G in $U_{\varepsilon_1}(x_0)$ with $\varepsilon_1 = \varepsilon_1(x_0, 1/4, 1, \omega)$ such that $\theta(\gamma_a(G)) = \theta(\gamma_b(G))$ holds, $T_*^{-\ell_0}G$ is also a K-quadrilateral and γ intersects with $\gamma_a(G)$ and $\gamma_c(G)$. Then $\nu(Y_j^{(\ell_0)} \cap G^{(c,1/4)} \cap N(f)) > 0$ and $\nu(Y_{j'}^{(\ell_0)} \cap G^{(c,1/4)} \cap N(f)) > 0$. By (9.1), for almost every x in $Y_j^{(\ell_0)} \cup Y_{j'}^{(\ell_0)}$ is equal to a constant. When γ is decreasing, one can show the same result. Since $\omega > 0$ is arbitrary, it is proved that for almost every x in $M^{(\iota)}$ $f(x)$ is equal to a constant $a^{(\iota)}$.

Observe a triple of boundaries $\partial Q_\iota, \partial Q_{\iota'}, \partial Q_{\iota''}$ such that there exists a point z in $M^{(\iota')} \cap S$ with $T_*^{-1}z$ in $M^{(\iota)} - S$ and $T_* z$ in $M^{(\iota'')} - S$. Let γ be the branch of $T_*^{-1}S$ which contains $T_*^{-1}z$. Suppose that γ is the common part of the boundaries of $X_j^{(e)}$ and $X_{j'}^{(e)}$. Since γ is K-increasing,

$$\nu(A^{(c)}[\gamma] \cap X_j^{(e)}) > 0 \quad \text{and} \quad \nu(A^{(c)}[\gamma] \cap X_{j'}^{(e)}) > 0 \ .$$

Since one of $X_j^{(e)}$ and $X_{j'}^{(e)}$ is mapped into $M^{(\iota')}$, and the other is mapped into $M^{(\iota'')}$, and since $f_1(x)$ is constant on $T_*\gamma^{(c)}(y)$ for y in γ, one can see that $a_{\iota'} = a_{\iota''}$. Performing this argument repeatedly, it is concluded that for almost every x in M $f(x)$ is equal to a constant. Q.E.D.

THEOREM 3. *Under the assumptions* (H–1), (H–2) *and* (H–3),
(i) T_* *is a K-system,*
(ii) $\zeta^{(c)}$ *and* $\zeta^{(e)}$ *are K-partitions,*

$$\text{(iii) } h(T_*) = \int \log\Big(1 + \frac{k_1\tau_1}{\cos\varphi_1} + \frac{k_1\cos\varphi + k'\cos\varphi_1 + k_1k'\tau_1}{\cos\varphi_1}\Big\{\frac{1}{\chi^{(e)}(\iota, r, \varphi)} + h(\iota,\varphi)\Big\}\Big)d\nu$$
$$= \int \log\Big(1 + \frac{k'\tau_1}{\cos\varphi} + \frac{k_1\cos\varphi + k'\cos\varphi_1 + k_1k'\tau_1}{\cos\varphi\chi^{(c)}(\iota_1, r_1, \varphi_1)}\Big)d\nu\ .$$

Proof. By Theorem 2 and Lemma 9.1,

$$\pi(T_*) = \pi(T_*^{-1}) = \zeta_{-\infty}^{(c)} = \zeta_{\infty}^{(e)} = \zeta_{-\infty}^{(c)} \wedge \zeta_{\infty}^{(e)}$$

is the trivial partition. Therefore (i) and (ii) are proved. The third assertion (iii) follows from a theorem of Ya. G. Sinai [10] together with Lemma 3.3 (see § 11 of [6], [5]).

§ 10. The motion of a particle in a compound central field

Appealing to Theorem 3, the ergodicity of the motion of a particle in a compound central field will be shown under some assumptions. Suppose that there exist several fixed kernels $\bar{q}(1), \cdots, \bar{q}(I)$ in a torus $\boldsymbol{T}$ and that these kernels have central potentials; $U_\iota(|q - \bar{q}(\iota)|)$, $\iota = 1, 2, \cdots, I$, where $|q - \bar{q}(\iota)|$ means the Euclidean distance between q and $\bar{q}(\iota)$. The potential field governed by

$$U(q) = \sum_{\iota=1}^{I} U_\iota(|q - \bar{q}(\iota)|) \tag{10.1}$$

is called *a compound central field.* If the potential ranges of $U_\iota(|q - \bar{q}(\iota)|)$'s do not overlap, the dynamical system of a particle in the potential field satisfies assumptions (H–1) and (H–2). Therefore Theorem 3 is applicable to the dynamical system. In order to check the assumption (H–3), it is necessary to calculate the path of the motion of a particle in a central field. A central potential function V is said to be *bell-shaped,* if

(V–1) *$V(s)$ is continuous for $s > 0$ and $V(s) = 0$ for $s \geq R$ with some R,*

(V–2) *$V(s)$ belongs to C^2-class in $(0, R)$ and there exist left derivatives $V'(R\text{–}0)$ and $V''(R\text{–}0)$,*

(V–3) *$-sV'(s)$ is monotone decreasing and $V'(R\text{–}0) < 0$.*

Now discuss the motion of a particle with mass m and energy E in the potential field governed by a bell-shaped potential function V. Then the Hamiltonian is given by

$$H(s, \beta) = \tfrac{1}{2}m(\dot{s}^2 + s^2\dot{\beta}^2)V(s) \tag{10.2}$$

using the polar coordinates (s, β). It is well known that the angular momentum of the particle

$$A = ms^2\dot{\beta} \tag{10.3}$$

is a first integral and that the equation of the motion is given by

$$ms - s\dot{\beta}^2 = -V'(s)\ . \tag{10.4}$$

Hence the equation of a path is expressed in the form

$$\beta = \int \frac{\pm As^{-2}}{(2m(E - V(s)) - A^2s^{-2})^{1/2}} ds + \text{const}\ . \tag{10.5}$$

Observe a path whose minimum value of the radial coordinate is equal to u. Suppose that the path passes $(u, 0)$. Let $(R, \alpha(u))$ be the point at which the path goes out from the potential range, and let $\psi(u)$ be the angle between the velocity and the radius vector at $(R, \alpha(u))$. Then the formula

$$H(\varphi) = 2R\alpha(\psi^{-1}(|\pi - \varphi|))\ \text{sign}\ (\varphi - \pi) \tag{10.6}$$

is obtained.

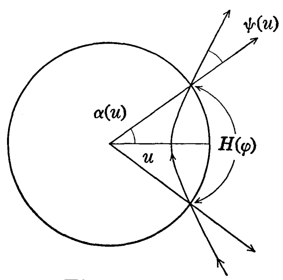

Fig. 10-1

The angular momentum A is expressed in the form

$$A = \{2m(E - V(u))\}^{1/2}u$$

by (10.2) and (10.3). By (10.5)

(10.7) $$\alpha(u)=\int_u^R\left\{\frac{u^2(E-V(u))}{s^2(E-V(s))-u^2(E-V(u))}\right\}^{1/2}ds\ .$$

Since the velocity at (s,β) is given by $(\dot{s}\cos\beta-s\dot{\beta}\sin\beta, \dot{s}\sin\beta+s\dot{\beta}\cos\beta)$, one can see

(10.8) $$\cos\psi(u)=\left.\frac{\dot{s}}{\{\dot{s}^2+s^2\dot{\beta}^2\}^{1/2}}\right|_{s=R}\ .$$

By (10.2)

(10.9) $$\left.\dot{s}^2+s^2\dot{\beta}^2\right|_{s=R}=\left.\frac{2}{m}(E-V(s))\right|_{s=R}=\frac{2E}{m}\ .$$

Since by (10.3), (10.4), (10.8) and (10.9)

$$\dot{s}^2=\frac{2E}{m}-s^2\dot{\beta}^2=\frac{2E}{m}-\frac{A^2}{ms^2}=\frac{2E}{m}-\frac{2(E-V(u))u^2}{ms^2}$$

is seen, and the expression

(10.10) $$\psi(u)=\cos^{-1}\left\{\frac{R^2E-u^2(E-V(u))}{R^2E}\right\}^{1/2}$$

is obtained.

LEMMA 10.1.

$$H(\varphi)=2R\alpha(\psi^{-1}(|\pi-\varphi|))\operatorname{sign}(\varphi-\pi)\ ,$$

where $\alpha(u)$ *and* $\psi(u)$ *are given by* (10.7) *and* (10.8) *respectively. Further* $H(\varphi)$ *belongs to* C^2*-class and*

$$\frac{dH(\varphi)}{d\varphi}=\frac{-4R(E-V(u))+2R\{R^2E-u^2(E-V(u))\}^{1/2}g(u)}{2(E-V(u))-uV'(u)}$$

with $u=\psi^{-1}(|\pi-\varphi|)$, *where*

$$g(u)\equiv\int_1^{\log R/u}\frac{[-e^{2s}(E-V(e^su))V'(u)+e^{3s}(E-V(u))V'(e^su)]}{2[E-V(u)]^{1/2}[e^{2s}(E-V(e^su))-E+V(u)]^{3/2}}ds\ .$$

Proof. The first equality was shown. Noting the expression

$$\alpha(u)=\int_1^{\log R/u}\left\{\frac{E-V(u)}{e^{2s}(E-V(e^su))-(E-V(u))}\right\}^{1/2}ds\ ,$$

$h(\varphi)=dH(\varphi)/d\varphi$ can be calculated and it can be shown that $h(\varphi)$ is continuously differentiable. Q.E.D.

Denote by R_ι the range of the potential U_ι and denote by $L_{\min}$ the minimum distance between the domains $\overline{Q}_\iota \equiv \{q; |q - \bar{q}(\iota)| < R_\iota\}$, $\iota = 1, 2, \cdots, I$.

THEOREM 4. *If every U_ι is bell-shaped and if energy E satisfies the condition*

$$(10.11) \qquad 0 < E < \frac{1}{4} \min_\iota \left\{ -\frac{R_\iota L_{\min}}{R_\iota + L_{\min}} U'_\iota(R - 0) \right\},$$

then $\{S_t\}$ is ergodic. Moreover the transformation T_* is a K-system, of course T_* is ergodic.

Proof. Since the curvature of ∂Q_ι is equal to $1/R_\iota$ and $|\tau|_{\min} = L_{\min}$, the assumption (H–3) is equivalent to

$$\min \left\{ \frac{dH(\iota, \varphi)}{d\varphi} + \left(\frac{1}{R_\iota} + \frac{1}{L_{\min}} \right)^{-1} \right\} > 0 .$$

If U_ι is bell-shaped,

$$\min \frac{dH(\iota, \varphi)}{d\varphi} \geq \frac{4E}{U_\iota(R_\iota - 0)}$$

holds by Lemma 10.1. Therefore if E satisfies the inequality (10.11), then the assumption (H–3) is fulfilled. Q.E.D.

EXAMPLE. The following central potentials are bell-shaped.

$$\text{(a)} \qquad V^\alpha(s) = \begin{cases} as^\alpha - aR^\alpha & 0 < s < R, \\ 0 & R \leq s, \end{cases}$$

for $\alpha < 0$,

$$\text{(b)} \qquad V^0(s) = \begin{cases} a \log R/s & 0 < s < R, \\ 0 & R \leq s. \end{cases}$$

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