H. Umemura
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# STABLE VECTOR BUNDLES WITH NUMERICALLY TRIVIAL CHERN CLASSES OVER A HYPERELLIPTIC SURFACE

HIROSHI UMEMURA

In [17], Weil studied the space of representations of certain Fuchsian groups as a generalization of Jacobian variety. The theory of stable vector bundles over a curve developed by Mumford, Seshadri and others are the theory of unitary representations of Fuchsian groups. The moduli space of stable vector bundles over a curve is the space of the irreducible unitary representations of a Fuchsian group. The moduli space is studied in detail. Recently Mumford (unpubished) and Takemoto [12] introduced the notion of $H$-stable vector bundle over a non-singular projective algebraic surface. In this paper, we study the space of the irreducible unitary representations of the fundamental group of a hyperelliptic surface. Our view point is based on the theory of $H$-stable vector bundles of Takemoto [12] and [13]. We deal only with hyperelliptic surfaces. Our results should be generalized to the vector bundles over some other surfaces (See § 3). Our main results are as follows:

1° An $H$-stable vector bundle with numerically trivial Chern classes over a hyperelliptic surface is defined by an irreducible unitary representation of the fundamental group.

2° The space of irreducible unitary representations gives the local moduli if degree $\geq 2$ with some additional assumption (see Theorem (2.18) for a precise statement).

3° There are many $H$-semi-stable vector bundles which can not be a limit of $H$-stable bundles. In certain case, it is impossible to deform an $H$-stable bundle of certain rank $r$ with numerically trivial Chern classes to any $H$-semi-stable vector bundle which is not $H$-stable.

Our theory can be regarded as a generalization of the Picard scheme. 2° shows the space of irreducible unitary representation of the fundamental group sometimes gives finer result than the Picard scheme.

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## § 1. *H*-stable vector bundles

We recall the definition and some basic results on $H$-stable vector bundles (for the proof see Takemoto [12]). We assume that all schemes, which we call varieties, are reduced and of finite type over the complex number field $\boldsymbol{C}$.

DEFINITION (1.1). Let $S$ be a non-singular projective surface. Let $H$ be an ample line bundle on $S$. Let $E$ be a vector bundle i.e. a locally free sheaf of finite rank on $S$. We say that $E$ is $H$-stable (resp. $H$-stable) if we have

$$\frac{(H\cdot c_1(E))}{r(E)} < \frac{(H\cdot c_1(F))}{r(F)}$$
$$(\text{resp. } \leq)$$

for any torsion free quotient $O_S$-Module $F$ of $E$ with $r(E) > r(F)$ where $c_1(\ )$ is the first Chern class, $r(\ )$ is the rank and $(\ \cdot\ )$ denotes the intersection number.

If we fix the line bundle $H$ and the numerical Chern classes $c_1, c_2$ and if $r = 2$, the coarse moduli space of $H$-stable vector bundles $E$ with $c_1(E) = c_1$, $c_2(E) = c_2$, exists (Maruyama [4], see also Theorem (2.16)).

Let $\tilde{S}$ be the universal covering space of $S$. Let $G$ be the fundamental group of $S$ hence $G$ acts on $\tilde{S}$ and the quotient space $\tilde{S}/G$ is isomorphic to $S$. Given a representation $\rho: G \to GL(r, \boldsymbol{C})$, we can associate a vector bundle $E$ to this representation.

PROPOSITION (1.2). *If $\rho$ is a unitary representation, $E_\rho$ is H-semi-stable for any ample line bundle H on S. If $\rho$ is an irreducible unitary representation, $E_\rho$ is H-stable for any ample line bundle H on S.*

*Proof* (essentially due to F. Takemoto). Let $H$ be an ample line bundle on $S$. Assume that $\rho$ is an irreducible unitary representation. We show that $E_\rho$ is $H$-stable. We remark that a vector bundle is $H$-stable if and only if it is $H^{\otimes m}$-stable for one positive integer $m$. Hence we may replace $H$ by $H^{\otimes m}$. Let $C_m$ be a generic hyperplane section of the linear system $|H^{\otimes m}|$. Then, taking $m$ sufficiently big, we have, by the Lefschetz theorem, the map $\pi_1(C_m) \xrightarrow{i_*} \pi_1(S)$ is surjective (Bott [2]). Hence the unitary representation $\rho \circ i_*$ is irreducible. The restriction

$E_{\rho|C_m}$ of $E_\rho$ to $C_m$ is associated to the irreducible unitary representation hence by Narasimhan and Seshadri [8] $E_{\rho|C_m}$ is stable. It follows that $E_\rho$ is $H^{\otimes m}$-stable. This is what we had to show. When $\rho$ is a unitary representation, the proof of $H$-semistability of $E_\rho$ is the same as above, in fact easier.

PROPOSITION (1.3). *Let $S$ be a non-singular projective surface. Let $T$ be a variety. Let $E$ be vector bundle over $T \times S$ such that for any (closed) point $t$, the restriction $E_t$ of $E$ to $t \times S$ has numerically trivial Chern classes and is $H_t$-stable for some ample line bundle $H_t$ on $S$. Then the set of points $\{t \in T \mid E_t$ is defined by an irreducible unitary representation$\}$ is closed.*

*Proof.* Let $G$ be the fundamental group of $S$. Let $g_1, g_2, \cdots, g_\ell$ be a set of generators of $G$. Let $\rho$ be a representation of $G$. Then it determines a point $(\rho(g_1), \rho(g_2), \cdots, \rho(g_\ell)) \in GL(r, \boldsymbol{C})^\ell$. It follows easily all the representations of degree $r$ of $G$ (not the isomorphism classes) form a closed analytic subset $W \subset GL(r, \boldsymbol{C})^\ell$. The intersection $U = W \cap U(r, \boldsymbol{C})^\ell$ is a compact subset of $W$. This is the set of all unitary representation of $G$. Let $F$ be the family of vector bundles over $S \times W$ i.e. the restriction $F_w$ of $F$ to $S \times w$ is the vector bundle associated to the representation corresponding to $w$. Consider the set $M = \{(w, k) \in W \times T \mid \mathrm{Hom}(F_w, E_k) \neq 0\}$. Then $M$ is a closed subset of $W \times T$. If we consider the projection $\mathrm{pr}_T(M \cap U \times T)$, then this is a closed subset of $T$ since $U$ is compact. Proposition now follows if we notice the following fact; Let $E_1$ be an $H$-semi-stable bundle and $E_2$ an $H$-stable bundle such that $E_1$ and $E_2$ have the same numerical Chern classes and $r(E_1) = r(E_2)$. If $\mathrm{Hom}(E_1, E_2) \neq 0$, then $E_1$ is isomorphic to $E_2$.

PROPOSITION (1.4). *Let $S$ be a compact complex manifold. Let $\rho_1$ and $\rho_2$ be unitary representations of the fundamental group of $S$. We denote by $E_{\rho_1}$, $E_{\rho_2}$ the vector bundles associated to the unitary representations $\rho_1$ and $\rho_2$. Then the vector bundle $E_{\rho_1}$ is isomorphic to $E_{\rho_2}$ if and only if the representations $\rho_1$ and $\rho_2$ are equivalent.*

*Proof.* See Seshadri [10].

## § 2. Moduli space of $H$-stable vector bundles with trivial Chern classes over hyperelliptic surfaces

We need some results on hyperelliptic surfaces. For the detail we refer to Suwa [11].

(2.1.1) A hyperelliptic surface $S$ is, by definition, an elliptic surface free from singular fibres over an elliptic curve with the first Betti number $b_1(S) = 2$.

(2.1.2) Let $K$ be the canonical bundle of a hyperelliptic surface $S$. Then according to Suwa [11], such surfaces are classified into four types:

I) $K^{\otimes 2} \simeq O_s \qquad (K \not\simeq O_s)$
II) $K^{\otimes 3} \simeq O_s \qquad (K \not\simeq O_s)$
III) $K^{\otimes 4} \simeq O_s \qquad (K^{\otimes 2} \not\simeq O_t)$
IV) $K^{\otimes 6} \simeq O_s \qquad (K^{\otimes 2}, K^{\otimes 3} \not\simeq O_s)$.

The hyperelliptic surface has an abelian variety as an unramified covering space of degree 2, 3, 4 or 6 according as $S$ is of type I, II, III or IV. Let $d(S)$ be the minimal positive integer such that $K^{\otimes d(S)}$ is isomorphic to $O_S$.

We quote

THEOREM (2.1.3) (Suwa [11] p. 473). *Any hyperelliptic surface can be expressed as the quotient space of an abelian variety $A$ by the group generated by an automorphism $g_5$ of $A$. The period matrix of $A$ and the automorphism $g_5$ are given as follows*

I) i) $\begin{pmatrix} 1 & 0 & \tau & 0 \\ 0 & 1 & 0 & \omega \end{pmatrix}$, ii) $\begin{pmatrix} 1 & 0 & \tau & 0 \\ 0 & 1 & \frac{1}{2} & \omega \end{pmatrix}$,

$$g_5: (u, \zeta) \longmapsto (u + \tfrac{1}{2}, -\zeta),$$

II) ii) $\begin{pmatrix} 1 & 0 & \tau & 0 \\ 0 & 1 & 0 & \rho^2 \end{pmatrix}$, ii) $\begin{pmatrix} 1 & 0 & \tau & 0 \\ 0 & 1 & \frac{1}{3}(1-\rho^2) & \rho^2 \end{pmatrix}$,

$$g_5: (u, \zeta) \longmapsto (u + \tfrac{1}{3}, \rho^2\zeta), \qquad \rho = \exp(\pi i/3),$$

III) iii) $\begin{pmatrix} 1 & 0 & \tau & 0 \\ 0 & 1 & 0 & i \end{pmatrix}$, iii) $\begin{pmatrix} 1 & 0 & \tau & 0 \\ 0 & 1 & \frac{1}{2}(1+i) & i \end{pmatrix}$,

$$g_5: (u, \zeta) \longmapsto (u + \tfrac{1}{4}, i\zeta),$$

IV) $\begin{pmatrix} 1 & 0 & \tau & 0 \\ 0 & 1 & 0 & \rho^2 \end{pmatrix}$, $\quad g_5: (u, \zeta) \longmapsto (u + \tfrac{1}{6}, -\rho^2\zeta)$, *where $\tau$ and $\omega$ denote arbitrary constants with non-zero imaginary parts.*

The parameters $\tau, \omega$ in the case I and $\tau$ in the other cases give the local moduli of hyperelliptic surfaces. They form complex analytic families of hyperelliptic surfaces and the parameters $\tau, \omega$ and $\tau$ are effective and complete.

In any case of seven types above, we denote by $g_i$, $1 \leq i \leq 4$, the $i$-th column vector of the period matrix. Then $\{g_1, g_2, g_3, g_4, g_5\}$ is a set of generators of the fundamental group.

(2.1.4) If we put $G = \pi_1(S)$ and $H =$ the normal subgroup of $G$ generated by $g_1, g_2, g_3$ and $g_4$, then $H$ is isomorphic to $Z^{\otimes 4}$ and we get an exact sequence

$$(*) \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & H & \longrightarrow & G & \longrightarrow & G' & \longrightarrow & 1 . \\ & & \wr\parallel & & & & & & \\ & & Z^{\otimes 4} & & & & & & \end{array}$$

The quotient group $G'$ is generated by $g_5 \bmod H$ and of order $2, 3, 4, 6$ according as $S$ is of type I, II, III, IV.

LEMMA (2.1.5). *Suppose that $S$ is of type* I i), *then we have following relations*

(0) $g_1, g_2, g_3$ *and* $g_4$ *commute each other.*

(a) $g_1 g_5 = g_5 g_1$ (b) $g_2 g_5 = g_5 g_2^{-1}$

(c) $g_3 g_5 = g_5 g_3$ (d) $g_4 g_5 = g_5 g_4^{-1}$

(e) $g_5^2 = g_1$ .

*Any other relation is deduced from the above relations.*

*Proof.* It is trivial that $g_1, g_2, g_3, g_4$ and $g_5$ satisfy the condition (0), (a), (b), (c), (d) and (e). Conversely consider the group $\Gamma$ generated $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ and $\gamma_5$ such that $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ and $\gamma_5$ satisfy the relations above. Let $N'$ be a subgroup of $\Gamma$ generated by $\gamma_1, \gamma_2, \gamma_3, \gamma_4$. $N'$ is a normal subgroup. We have

$$0 \longrightarrow N' \longrightarrow \Gamma \longrightarrow \Gamma / N' \longrightarrow 1 .$$

But the relations above also determine the cocycle of the extention above. For the same reason we have following lemmae.

LEMMA (2.1.6). *If $S$ is of type* I ii), *then we have following relations*

(0) $g_1, g_2, g_3$ *and* $g_4$ *commute each other.*

(a) $g_1g_5 = g_5g_1$ (b) $g_2g_5 = g_5g_2^{-1}$

(c) $g_3g_5 = g_5g_3g_2^{-1}$ (c) $g_4g_5 = g_5g_4^{-1}$

(e) $g_5^2 = g_1$

*Any other relation is deduced from the above relations.*

LEMMA (2.1.7). *If S is of type* II *i), then we have following relations*

(0) $g_1, g_2, g_3$ *and* $g_4$ *commute each other.*

(a) $g_1g_5 = g_5g_1$ (b) $g_2g_5 = g_5g_2^{-1}g_4^{-1}$

(c) $g_3g_5 = g_5g_3$ (d) $g_4g_5 = g_5g_2$

(e) $g_5^3 = g_1$

*Any other relation is deduced from the above relations.*

LEMMA (2.1.8). *If S is of type* II ii), *then we have following relations*

(0) $g_1, g_2, g_3$ *and* $g_4$ *commute each other.*

(a) $g_1g_5 = g_5g_1$ (b) $g_2g_5 = g_5g_2^{-1}g_4^{-1}$

(c) $g_3g_5 = g_5g_3g_2^{-1}$ (d) $g_4g_5 = g_5g_2$

(e) $g_5^3 = g_1$

*Any other relation is deduced from the above relations.*

LEMMA (2.1.9). *If S is of type* III i), *then we have following relations*

(0) $g_1, g_2, g_3$ *and* $g_4$ *commute each other.*

(a) $g_1g_5 = g_5g_1$ (b) $g_2g_5 = g_5g_4^{-1}$

(c) $g_3g_5 = g_5g_3$ (d) $g_4g_5 = g_5g_2$

(e) $g_5^4 = g_1$

*Any other relation is deduced from the above relations.*

LEMMA (2.1.10). *If S is of type* III ii), *then we have following relations*

(0) $g_1, g_2, g_3$ *and* $g_4$ *commute each other.*

(a) $g_1g_5 = g_5g_1$ (b) $g_2g_5 = g_5g_4^{-1}$

(c) $g_3g_5 = g_5g_3g_4^{-1}$ (d) $g_4g_5 = g_5g_2$

(e) $g_5^4 = g_1$

*Any other relation is deduced from the above relations.*

LEMMA (2.1.11). *If $S$ is of type* IV, *then we have following relations*

(0) $g_1, g_2, g_3$ *and* $g_4$ *commute each other.*

(a) $g_1 g_5 = g_5 g_1$ (b) $g_2 g_5 = g_5 g_2 g_4$

(c) $g_3 g_5 = g_5 g_3$ (d) $g_4 g_5 = g_5 g_2^{-1}$

(e) $g_5^6 = g_1$.

*Any other relation is deduced from the above relations.*

LEMMA (2.2) (Oda). *Let $A$ be an abelian variety. Let $E$ be a vector bundle of rank $r$ over $A$ and $p: H^1(A, \Omega_A^1) \to H^1(A, \Omega_A^1 \otimes \mathrm{End}\, E)$ be the linear map induced by the canonical inclusion $0 \to O_A \to \mathrm{End}\, E$. Let $a(E) \in H^1(A, \Omega_A^1 \otimes \mathrm{End}\, E)$ be the fundamental class of $E$. If $a(E)$ is in $(1/r)p(H^2(A, \boldsymbol{Z}) \cap H^1(A, \Omega_A^1))$, then there exists a line bundle $L$ on $A$ such that $L^{-1} \otimes r_A^* E$ is homogeneous, where $r_A$ is the multiplication by $r$.*

*Proof.* Let $\pi: \boldsymbol{C}^n \to A$ be the universal covering space of $A$. Let $\Gamma$ be the fundamental group of $A$. Hence $\boldsymbol{C}^n/\Gamma \simeq A$. Since $H^i(\boldsymbol{C}^n, \pi^*\Omega_A^1) = 0$ $i \geq 1$, we have

$$H^i(\Gamma, H^0(\boldsymbol{C}^n, \pi^*\Omega_A^1)) \simeq H^i(A, \Omega_A^1)$$

where the left hand side is the $i$-th cohomology group of $\Gamma$-module $H^0(\boldsymbol{C}^n, \pi^*\Omega_A^1)$ (see Mumford [7]). For the same reason,

$$H^i(\Gamma, H^0(\boldsymbol{C}^n, \pi^*(\Omega_A^1 \otimes \mathrm{End}\, E))) \simeq H^i(A, \Omega_A^1 \otimes \mathrm{End}\, E) .$$

Let $\rho_u(z)$ be a 1-cocycle for $\Gamma$ with coefficients in $H^0(\boldsymbol{C}^n, GL(r, O_{\boldsymbol{C}^n}))$ defining $E$. Then $a(E)$ corresponds to the cocycle $-d \log \rho_u(z) = -d\rho_u(z) \cdot \rho_u(z)^{-1}$. Hence from the hypothesis and Weil [18], there exists a 1-cocycle $\varphi_\alpha(z)$ for $\Gamma$ with coefficients in $H^0(\boldsymbol{C}^n, O_{\boldsymbol{C}^n}^*)$ such that $-d \log \rho_u(z) = -(1/r) d \log \varphi_\alpha(z) \cdot I_r$ in $H^1(\Gamma, H^0(\boldsymbol{C}^n, \pi^*(\Omega_A^1 \otimes \mathrm{End}\, E))$. It is easy to see that $r_A^*(-(1/r) d \log \varphi_\alpha(z))$ is integral i.e. $-(1/r) d \log \varphi_\alpha(z) \in H^1(\boldsymbol{C}^n, \pi^*\Omega_{A'}^1)$ corresponds to an element of the subset $H^2(A', \boldsymbol{Z}) \cap H^1(A', \Omega_{A'}^1)$ of $H^1(A', \Omega_{A'}^1)$ where $\Gamma' \subset r\Gamma$ and $A' = \boldsymbol{C}^n/\Gamma'$. By Weil [18], there exists a line bundle $L$ on $A'$ defined by a 1-cocycle $\varphi_u'$ for $\Gamma'$ with coefficients in $H^0(\boldsymbol{C}^n, O_{\boldsymbol{C}^n}^*)$ such that $d \log \varphi_u'(z) = (1/r) d \log \varphi_u(z)$, $u \in \Gamma'$. Hence $-d \log \rho_u(z) = -d \log \varphi_u'(z) I_r$, $u \in \Gamma'$. If we consider the vector bundle $L^{-1} \otimes \varphi^* E$ on $A'$, this vector bundle is defined by the 1-cocycle $\varphi_u'(z)^{-1} \rho_u(z)$ for $\Gamma$ with coefficients in $H^0(\boldsymbol{C}^n, GL(r, O_{\boldsymbol{C}^n}))$. Then $-d \log \varphi_u'(z) \rho_u(z) = 0$ in $H^1(A', \pi'^*(\Omega_{A'}^1 \otimes \mathrm{End}\, (L^{-1} \otimes E))$. Hence

$L^{-1}\otimes\varphi^*E$ has a connection (see Atiyah [1]). By the theorem of Matsushima [5], $L^{-1}\otimes\varphi^*E$ is homogeneous.

COROLLARY (2.3). *Let $A$ be an abelian variety of dimension 2 and $E$ an $H$-stable vector bundle. If $c_1=0$, $c_2=0$, then $E$ is a line bundle.*

*Proof.* Since an $H$-stable vector bundle i.e. $\dim H^0(A, \operatorname{End} E)=1$, by the Riemann-Roch theorem, the canonical linear map $H^1(A, \Omega_A^1)\to H^1(A, \Omega_A^1\otimes\operatorname{End} E)$ is an isomorphism. Hence there exist an isogeny $\varphi: A'\to A$ and a line bundle $L$ such that $L^{-1}\otimes\varphi^*E$ is homogeneous. Since the first Chern class of $E$ is trivial, we can take $L\simeq O_{A'}$. On the other hand, by Takemoto [13], $\varphi^*E$ is the direct sum of $\varphi^*H$-stable bundles. By Morimoto [6], $\varphi^*E$ is the direct sum of line bundles which are algebraically equivalent to 0. Hence $E$ is the direct image of a line bundle on $A'$. Therefore the rank should be one.

COROLLARY (2.4) (Oda). *Let $A$ be an abelian variety of Dimension 2. Let $E$ be an $H$-stable vector bundle of rank $r$ with $(r-1)c_1^2-2rc_2=0$. Then there there exist an isogeny $\varphi: A'\to A$ and a line bundle $L$ on $A'$ such that $\varphi_*L\simeq E$.*

*Proof.* Similar to the proof of Corollary (2.3).

LEMMA (2.5). *Let $A$ be an abelian variety of Dimension 2. Let $E$ be a vector bundle on $A$. Then,*

(1) *The second Chern class of* $\operatorname{End} E=-(r-1)c_1^2+2rc_2$ *where $c_1, c_2$ denote the first and the second Chern class of $E$.*
(2) *If $E$ is simple, then $(r-1)c_1^2-2rc_2\le 0$.*

*Proof.* (2) follows from the Riemann-Roch theorem for $\operatorname{End} E$. (1) is trivial.

LEMMA (2.6). *Using the notation of Theorem* (2.1.3), *let $E$ be an $H$-stable vector bundle of rank $r$ over the hyperelliptic surface $S$. If $E$ has the numerically trivial Chern classes, then $\pi^*E$ is the direct sum of line bundles which are algebraically equivalent to 0, where $\pi: A\to S$ is the natural projection.*

*Proof.* We prove the lemma when $S$ is of type I. The proofs for other cases are similar. By Takemoto [13], $\pi^*E$ is either $\pi^*H$-stable or isomorphic to $E_1 \oplus \bar{g}_5^* E_1$ where $E_1$ is a $\pi^*H$-stable over $A$. If $\pi^*E$ is $\pi^*H$-stable, then $\pi^*E$ is line bundle by Lemma (2.3). Hence we may assume $\pi^*E \simeq E_1 \oplus \bar{g}_5^* E_1$. We claim that $E_1$ has numerically trivial Chern classes. We put $r_1 = \operatorname{rank} E_1$. Let $g: \boldsymbol{C}^2 \to \boldsymbol{C}^2$ be a linear automorphism of the universal

$$(u, \zeta) \longmapsto (u, \zeta)$$

covering space of $A$. We denote by $\bar{g}$ the automorphism of $A$ induced by $g$. Putting $E_2 = \bar{g}_5^* E_1$ let $c^i$ be the $i$-th Chern class of $E_1$, $1 \le i \le 2$. Then, from the hypothesis $E \simeq E_1 \oplus \bar{g}_5^* E_1$, we get

$$c_1 + \bar{g}^* c_1 = 0 \, , \qquad (c_1 \cdot \bar{g}^* c_1) + 2c_2 = 0 \, ,$$

since a translation of a line bundle is algebraically equivalent to itself. It follows $-(c_1^2) + 2c_2 = 0$. Let $E_1 = L(\overline{H}, \alpha)$ (see Mumford [7], p. 20). Then it also follows $\overline{H} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \overline{H} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 0$. Hence $\overline{H} = \begin{pmatrix} 0 & a \\ \bar{a} & 0 \end{pmatrix}$. In view of Lemma (2.3), it is sufficient to show $a = 0$. If $a \neq 0$, the index of $L$ would be 1 since $\overline{H}$ has 1 positive eigen-value and 1 negative one (see Mumford [8] and Umemura [15]). Hence by the Riemann-Roch theorem, $c_1^2 < 0$. Hence $c_2 > 0$ since $-(c_1^2) + 2c_2 = 0$. $(r-1)c_1^2 - 2rc_2 = (r-1)(c_1^2 - 2c_2) + 2c_2 = 2c_2 > 0$ contradicting to Lemma (2.5). Hence a should be 0.

COROLLARY (2.7). *Using the notation of Lemma* (2.6), *we assume that $E$ has numerically trivial Chern classes. Then $r \mid d(S)$ and there exist an unramified covering of degree $d(s) \mid r$ of $S$ and a line bundle on it such that $E$ is isomorphic to the direct image of the line bundle.*

*Proof.* $r \mid d(S)$ is an immediate consequence of Lemma (2.6). We prove the last assertion under the hypothesis that $S$ is of type II and $E$ is of rank 2. In this case we notice that $A = \operatorname{Spec}(\oplus_{i=2}^3 K^{\otimes i})$. By Lemma (2.6), $H^0(A, \operatorname{Hom}(\pi^*E, \pi^*E)) \simeq \oplus_{i=0}^3 H^0(S, \operatorname{Hom}(E, E \otimes K^{\otimes i}))$ has dimension $\geq 2$. Hence we have $\operatorname{Hom}(E, E \otimes K) \neq 0$, $\operatorname{Hom}(E, E \otimes K)$ or $\operatorname{Hom}(E, E \otimes K^{\otimes 3}) \neq 0$ since $\operatorname{Hom}(E, E) = \boldsymbol{C}$. It follows that $\operatorname{Hom}(E, E \otimes K^{\otimes 2}) \neq 0$. Since $E$ is stable, $E \simeq E \otimes K^{\otimes 2}$. Hence by Takemoto [13], the Corollary follows. The proofs are same for other cases.

LEMMA (2.8). *Using the notation of Theorem* (2.1.3), *let $\rho$ be a* *linear representation of degree* $r$ *of the fundamental group* $H$ *of the Abelian variety* $A$. *Let* $L$ *be the line bundle defined by the represen tation* $\rho$. *Then the pull-back* $\bar{g}_5^*L$ *is defined by the representation* $\rho': H \to GL(r, C)$ $g \to \rho(g_5 g g_5^{-1})$ *where* $\bar{g}_5$ *is the automorphism of* $A$ *induced by* $g_5$.

*Proof.* The lemma is an immediate consequence of the following commulative diagram:

$$\begin{array}{ccc} H \times C^2 & \longrightarrow & C^2 \\ \downarrow & & \downarrow \\ H \times C^2 & \longrightarrow & C^2 \end{array}$$

where horizontal arrows are the action of $H$ on the universal covering space $C^2$, the left vertical arrow sends $(g, x) \in H \times C^2$ to $(g_5 g g_5^{-1}, g_5 x)$ and the right vertical arrow is the action of $g_5$ on the universal covering space $C^2$ of the hyperelliptic surface $S$.

LEMMA (2.9). *Using the notation of Theorem* (2.1.3), *let* $L$ *be a line bundle over a hyperelliptic surface* $S$. *Then* $L$ *is homologically equivalent to* 0 *if and only if it is numerically equivalent to* 0.

*Proof.* If a line bundle is homologically equivalent to 0, then generally it is numerically equivalent to 0. Hence we assume that $L$ is numerically equivallent to 0. We prove this for a hyperelliptic surface of type I i). In other cases, the proofs are similar. Let us consider the pull-back $\pi^*L$ over $A$. Over an abelian variety, the numerical equivalence coincides with the homological equivalence. Hence there exists a unitary representation $\rho$ of degree 1 of the fundamental group $G$ of $A$ such that $\pi^*L$ is associated with $\rho$. We put $\rho(g_1) = \alpha$, $\rho(g_2) = \beta$, $\rho(g_3) = \gamma$, $\rho(g_4) = \delta$' where $\alpha, \beta, \gamma$ and $\delta$ are complex numbers with $|\alpha| = |\beta| = |\gamma| = |\delta| = 1$. By Lemma (2.1.5) and Lemma (2.8), $\bar{g}_5^*\pi^*L$ is given by the unitary representation $\rho'$: $\rho'(g_1) = \alpha$, $\rho'(g_2) = \beta^{-1}$, $\rho'(g^{\varepsilon}) = \gamma$, $\rho'(g_4) = \delta^{-1}$. Since $\pi^*L$ is isomorphic to $\bar{g}_5^*\pi^*L$, we conclude $\beta = \pm 1$, $\delta = \pm 1$. If we put $\rho''(g_1)$ $\rho''(g_1) = \alpha$, $\rho''(g_2) = \beta$, $\rho''(g_3) = \gamma$, $\rho''(g_4) = \delta$, $\rho''(g_5) = \varepsilon$ with $\varepsilon^2 = \alpha$, then by Lemma (2.1.5) $\rho''$ defines a unitary representation of degree 1 of the fundamental group $G$ of $S$. Since there are two such $\varepsilon$'s, there are two line bundles on $S$ defined by unitary representations such that there pull-backs over $A$ are isomorphic to $\varphi^*L$, Hence $L$ should be one of them and defined by a unitary representation. Hence $L$ is homologically equivalent to 0.

Let us determine all the irreducible unitary representation of the fundamental group $G$ of a given hyperelliptic surface $S$. Let $\rho$ be an irreducible unitary representation of degree $r$ of the fundamental group $G$ of $S$. Then by Proposition (1.2) and Corollary (2.7), $r$ devides $d(S)$. Hence it is sufficient to consider only such $r$. We put $r\cdot s = d(S)$. Since the subgroup $H$, the fundamental group of the abelian variety $A$, is commutative we can diagonalies the restriction of $\rho$ to $H$. Hence we may assume

$$\rho(g_i) = \begin{bmatrix} \rho_1(g_i) & & & 0 \\ & \rho_2(g_i) & & \\ & & \ddots & \\ 0 & & & \rho_r(g_i) \end{bmatrix}$$

$1 \leq i \leq 4$. If $E$ denotes the vector bundle defined by the representation $\rho$, then $\pi^*E$ is the direct sum of the line bundle $L_i$ defined by the representation $\rho_i$ of the fundamental group of $A$. Since $\bar{g}_5^*\pi^*E$ is isomorphic to $\pi^*E$, by Lemma (2.8), we conclude

$$\rho_j(g) = \rho_1(g_5^{-j+1} g g_5^{j-1}) , \qquad 1 \leq j \leq r$$
$$\rho_1(g) = \rho_1(g_5^{-r} g g_5^{r}) .$$

Hence we proved

PROPOSITION (2.10). *The irreducible unitary representation is normalized as follows*

$$\rho(g_i) = \begin{bmatrix} \rho_1(g_i) & & & \\ & \rho_1(g_5^{-1} g_i g_5) & & \\ & & \ddots & \\ & & & \rho_1(g_5^{-r+1} g_i g_5^{r+1}) \end{bmatrix}$$

$1 \leq i \leq 4$, $s\cdot r = d(S)$ *where* $\rho_1$ *is a representation of degree* 1 *of the fundamental group* $H$ *of the abelian variety.*

THEOREM (2.11). *The irreducible unitary representation of the fundamental group $G$ of $S$ are classified as follows.* (*We give the values of generators* $g_1, g_2, g_3, g_5$ *and* $g_5$ *of* $G$. *All the matrices below are unitary.*)

*Case* I i). *S is of type* I i).

$r = 1$. $\rho(g_1) = \alpha$ , $\rho(g_2) = \pm 1$ , $\rho(g_3) = \gamma$ , $\rho(g_4 = \pm 1$ ,
$\rho(g_5) = \varepsilon$ *with* $|\alpha| = |\gamma| = 1$ *and* $\varepsilon^2 = \alpha$ .

$r=2.$ $\rho(g_1)=\begin{pmatrix}\alpha & 0\\ 0 & \alpha\end{pmatrix}$, $\rho(g_2)=\begin{pmatrix}\beta & 0\\ 0 & \beta^{-1}\end{pmatrix}$, $\rho(g_3)=\begin{pmatrix}\gamma & 0\\ 0 & \gamma\end{pmatrix}$,

$\rho(g_4)=\begin{pmatrix}\delta & 0\\ 0 & \delta^{-1}\end{pmatrix}$, $\rho(g_5)=\begin{pmatrix}0 & \alpha\\ 1 & 0\end{pmatrix}$

*with* $\beta^2\neq 1$ *or* $\delta^2\neq 1$ .

*There is no irreducible unitary representation of degree* $>2$.

*Case* I ii). *S is of type* I ii).

$r=1.$ $\rho(g_1)=\alpha$ , $\rho(g_2)=1$ , $\rho(g_3)=\gamma$ , $\rho(g_4)=\pm 1$ ,
$\rho(g_5)=\varepsilon$ *with* $|\alpha|=|\gamma|=1$ *and* $\varepsilon^2=\alpha$ .

$r=2.$ $\rho(g_1)=\begin{pmatrix}\alpha & 0\\ 0 & \alpha\end{pmatrix}$, $\rho(g_2)=\begin{pmatrix}\beta & 0\\ 0 & \beta^{-1}\end{pmatrix}$, $\rho(g_3)=\begin{pmatrix}\gamma & 0\\ 0 & \beta^{-1}\gamma\end{pmatrix}$,

$\rho(g_4)=\begin{pmatrix}\delta & 0\\ 0 & \delta^{-1}\end{pmatrix}$, $\rho(g_5)=\begin{pmatrix}0 & \alpha\\ \alpha & 0\end{pmatrix}$

*with* $\beta\neq 1$ *or* $\delta\neq\pm 1$ .

*There is no irreducible unitary representation of degree* $>2$.

*Case* II i). *S is of type* II i).

$r=1.$ $\rho(g_1)=\alpha$ , $\rho(g_2)=\rho(g_4)=\beta$ , $\rho(g_3)=\gamma$ , $\rho(g_5)=\varepsilon$ ,
*with* $|\alpha|=|\beta|=|\varepsilon|=1$, $\varepsilon^3=\alpha$, $\beta^3=1$ .

$r=3.$ $\rho(g_1)=\begin{pmatrix}\alpha & 0 & 0\\ 0 & \alpha & 0\\ 0 & 0 & \alpha\end{pmatrix}$, $\rho(g_2)=\begin{pmatrix}\beta & 0 & 0\\ 0 & \beta^{-1}\delta^{-1} & 0\\ 0 & 0 & \delta\end{pmatrix}$,

$\rho(g_3)=\begin{pmatrix}\gamma & 0 & 0\\ 0 & \gamma & 0\\ 0 & 0 & \gamma\end{pmatrix}$, $\rho(g_4)=\begin{pmatrix}\delta & 0 & 0\\ 0 & \beta & 0\\ 0 & 0 & \beta^{-1}\delta^{-1}\end{pmatrix}$,

$\rho(g_5)=\begin{pmatrix}0 & 0 & \alpha\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}$ *with* $\beta^3\neq 1$ *or* $\beta\neq\delta$ .

*There is no other irreducible representation.*

*Case* II ii). *S is of* II ii).

$r=1.$ $\rho(g_1)=\alpha$ , $\rho(g_2)=1$ , $\rho(g_3)=\gamma$ , $\rho(g_4)=1$ ,
$\rho(g_3)=\varepsilon$ , *with* $|\alpha|=|\gamma|=1$ *and* $\varepsilon^3=\alpha$ .

$r=3.$ $\rho(g_1)=\begin{pmatrix}\alpha & 0 & 0\\ 0 & \alpha & 0\\ 0 & 0 & \alpha\end{pmatrix}$, $\rho(g_2)=\begin{pmatrix}\beta & 0 & 0\\ 0 & \beta^{-1}\delta^{-1} & 0\\ 0 & 0 & \delta\end{pmatrix}$,

$$\rho(g_3) = \begin{pmatrix} \gamma & 0 & 0 \\ 0 & \gamma\beta^{-1} & 0 \\ 0 & 0 & \gamma\delta \end{pmatrix}, \qquad \rho(g_4) = \begin{pmatrix} \delta & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \beta^{-1}\delta^{-1} \end{pmatrix},$$

$$\rho(g_5) = \begin{pmatrix} 0 & 0 & \alpha \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad \textit{with } \beta \neq 1 \textit{ or } \delta \neq 1 .$$

*There is no other irreducible unitary representation.*

*Case* III i). *S is of type* III i).

$r = 1.$ $\rho(g_1) = \alpha$, $\rho(g_2) = \rho(g_4) = \pm 1$, $\rho(g_3) = \gamma$, $\rho(g_5) = \varepsilon$, *with* $|\alpha| = |\gamma| = |\varepsilon| = 1$ *and* $\varepsilon^4 = \alpha$ .

$r = 2.$

$$\rho(g_1) = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}, \quad \rho(g_2) = \begin{pmatrix} \beta & 0 \\ 0 & \delta \end{pmatrix}, \quad \rho(g_3) = \begin{pmatrix} \gamma & 0 \\ 0 & \gamma \end{pmatrix},$$

$$\rho(g_4) = \begin{pmatrix} \delta & 0 \\ 0 & \beta \end{pmatrix}, \quad \rho(g_5) = \begin{pmatrix} 0 & \varepsilon \\ 1 & 0 \end{pmatrix}$$

*with* $|\alpha| = |\gamma| = 1$, $\beta^2 = \delta^2 = 1$, $\varepsilon^2 = \alpha$, *and* $\beta \neq \delta$ .

$r = 4.$

$$\rho(g_1) = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha \end{pmatrix}, \qquad \rho(g_2) = \begin{pmatrix} \beta & 0 & 0 & 0 \\ 0 & \delta^{-1} & 0 & 0 \\ 0 & 0 & \beta^{-1} & 0 \\ 0 & 0 & 0 & \delta \end{pmatrix},$$

$$\rho(g_3) = \begin{pmatrix} \gamma & 0 & 0 & 0 \\ 0 & \gamma & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & \gamma \end{pmatrix}, \qquad \rho(g_4) = \begin{pmatrix} \delta & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \delta^{-1} & 0 \\ 0 & 0 & 0 & \beta^{-1} \end{pmatrix},$$

$$\rho(g_5) = \begin{pmatrix} 0 & 0 & 0 & \alpha \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

*with either* $\beta \neq \delta$ *or* $\beta \neq \pm 1$ *or* $\delta \neq \pm 1$ .

*There is no other irreducible unitary representation.*

*Case* III ii). *S is of type* III ii).

$r = 1.$ $\rho(g_1) = \alpha$, $\rho(g_2) = \rho(g_4) = 1$, $\rho(g_3) = \gamma$, $\rho(g_5) = \varepsilon$ *with* $|\alpha| = |\gamma| = 1$ *and* $\varepsilon^4 = \alpha$ .

$r = 2.$

$$\rho(g_1) = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}, \qquad \rho(g_2) = \rho(g_4) = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\rho(g_3) = \begin{pmatrix} \gamma & 0 \\ 0 & -\gamma \end{pmatrix}, \quad \rho(g_5) = \begin{pmatrix} 0 & \varepsilon \\ 1 & 0 \end{pmatrix}$$

*with* $|\alpha| = |\gamma| = 1,\ \varepsilon^2 = \alpha$ .

$r = 4.$

$$\rho(g_1) = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha \end{pmatrix}, \qquad \rho(g_2) = \begin{pmatrix} \beta & 0 & 0 & 0 \\ 0 & \delta^{-1} & 0 & 0 \\ 0 & 0 & \beta^{-1} & 0 \\ 0 & 0 & 0 & \delta \end{pmatrix},$$

$$\rho(g_3) = \begin{pmatrix} \gamma & 0 & 0 & 0 \\ 0 & \gamma\delta^{-1} & 0 & 0 \\ 0 & 0 & \gamma\beta^{-1}\delta^{-1} & 0 \\ 0 & 0 & 0 & \gamma\beta^{-1} \end{pmatrix}, \qquad \rho(g_4) = \begin{pmatrix} \delta & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \delta^{-1} & 0 \\ 0 & 0 & 0 & \beta^{-1} \end{pmatrix},$$

$$\rho(g_5) = \begin{pmatrix} 0 & 0 & 0 & \alpha \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

*with either* $\beta^2 \neq 1$ *or* $\beta \neq \delta^{-1}$ *or* $\delta \neq 1$.

*There is no other irreducible unitary representation.*

*Case* IV. *S is of type* IV.

$r = 1.$ $\rho(g_1) = \alpha\,,\quad \rho(g_2) = \rho(g_4) = 1\,,\quad \rho(g_3) = \gamma\,,\quad \rho(g_5) = \varepsilon$

*with* $|\alpha| = |\gamma| = 1$ *and* $\varepsilon^6 = \alpha$ .

$r = 2.$

$$\rho(g_1) = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}, \quad \rho(g_2) = \rho(g_4) = \begin{pmatrix} \beta & 0 \\ 0 & \beta^2 \end{pmatrix},$$

$$\rho(g_3) = \begin{pmatrix} \gamma & 0 \\ 0 & \gamma \end{pmatrix}, \quad \rho(g_5) = \begin{pmatrix} 0 & \varepsilon \\ 1 & 0 \end{pmatrix}$$

*with* $|\alpha| = |\gamma| = 1,\ \beta \neq 1,\ \beta^3 = 1$ *and* $\varepsilon^3 = \alpha$ .

$r = 3.$

$$\rho(g_1) = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix}, \qquad \rho(g_2) = \begin{pmatrix} \beta & 0 & 0 \\ 0 & \beta\delta & 0 \\ 0 & 0 & \delta \end{pmatrix},$$

$$\rho(g_5) = \begin{pmatrix} \gamma & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & \gamma \end{pmatrix}, \qquad \rho(g_4) = \begin{pmatrix} \delta & 0 & 0 \\ 0 & \beta^{-1} & 0 \\ 0 & 0 & \beta^{-1}\delta^{-1} \end{pmatrix},$$

$$\rho(g_5) = \begin{pmatrix} 0 & 0 & \varepsilon \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

*with* $|\alpha| = |\gamma| = 1,\ \beta^2 = \delta^2 = 1,\ \varepsilon^2 = \alpha$ *and either* $\beta \neq 1$ *or* $\delta \neq 1$ .

$r = 6.$ $\qquad \rho(g_1) = \alpha I_6\,, \qquad \rho(g_2) = \begin{bmatrix} \beta & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta\delta & 0 & 0 & 0 & 0 \\ 0 & 0 & \delta & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta^{-1}\delta^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \delta^{-1} \end{bmatrix},$

$$\rho(g_3) = \gamma I_6\,, \qquad \rho(g_4) = \begin{bmatrix} \delta & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta^{-1}\delta^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta\delta \end{bmatrix},$$

$$\rho(g_5) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \alpha \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

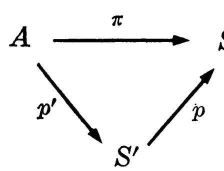
*with* $|\alpha| = |\beta| = |\gamma| = |\delta| = 1$, *except for following two cases*
(a) $\beta = \delta$, $\beta^3 = \delta^3 = 1$ (b) $\beta^2 = \delta^2 = 1$ .

*There is no other irreducible unitary representation.*

*Proof.* The theorem follows easily from Lemma (2.1.5), (2.1.6), (2.1.7), (2.1.8), (2.1.9), (2.1.10), (2.1.11), (2.8) and Proposition (2.10).

THEOREM (2.12). *Using the notation of Theorem* (2.1.3), *let $E$ be an $H$-stable vector bundle with numerically trivial Chern classes over $S$. Then $E$ is defined an irreducible unitary representation of the fundamental group $G$ of the hyperelliptic surface $S$.*

*Proof.* If $E$ is of rand 1, the Theorem is nothing but Lemma (2.9). We prove the theorem under the assumption that $S$ is of type III i) and $r = 2$. Proofs are similar in other case (in fact easier if $r = d(S)$). By Lemma (2.6), $\pi^*E$ is the direct sum of line bundles which are algebraically equivalent to 0, where $\pi: A \to S$ is the natural projection. We put $S' = S/(\bar{g}_5^2)$. Then we have a commutative diagram:



where $p$ and $p'$ are the natural projections. We put $\pi^*E = L_1 \oplus L_2$. Since $\bar{g}_5^*\pi^*E \simeq \pi^*E$, $\bar{g}_5^*L_1 \oplus \bar{g}_5^*L_2 \simeq L_1 \oplus L_2$. It follows $\bar{g}_5^*L_1 \simeq L_1$ or $\bar{g}_5^*L_1 \simeq L_2$. We shall show that we never have the first case $\bar{g}_5^*L_1 \simeq L_1$ hence $\bar{g}_5^*L_2 \simeq L_2$. If it were so, by the descent theory of Grothendieck there would be line bundles $L_1'$ $L_2'$ over $S$ such that $\pi^*L_1' \simeq L_1$ and $\pi^*L_2' \simeq L_2$. Hence $\pi^*E \simeq \pi^*(L_1' \oplus L_2')$. Taking $\pi_*$, $E \oplus E \otimes K \oplus E \otimes K^{\otimes 2} \oplus E \otimes K^{\otimes 3} \simeq (L_1' \oplus L_2') \oplus (L_1' \oplus L_2') \otimes K \oplus (L_1' \oplus L_2') \otimes K^{\otimes 2} \oplus (L_1' \oplus L_2') \otimes K^{\otimes 3}$. Hence $E$ would be the direct sum of line bundles. This contradicts to the $H$-stability of $E$. We conclude $L_2 \simeq \bar{g}_5^*L_1$ and $\bar{g}_5^{2*}L_1 \simeq L_1$ Hence $\pi^*E \simeq L_1 \oplus \bar{g}_5^*L_1$, with $\bar{g}_5^*L_1 \not\simeq L_1$ and $\bar{g}_5^{2*}L_1 \simeq L_1$. By Lemma (2.8), $\pi^*E$ is defined by the following unitary representation $\rho$ of the fundamental group $H$ of $A$:

$$\rho(g_i) = \begin{pmatrix} \rho_1(g_i) & 0 \\ 0 & \rho_1(g_5^{-1}g_ig_5) \end{pmatrix}, \qquad 1 \leq i \leq 4\,,$$

and $\rho_1(g_5^{-2}g_ig_5^2) = \rho_1(g_i)$ for any $1 \leq i \leq 4$ and $\rho_1(g_5^{-1}g_ig_5) \neq \rho_1(g_i)$ for some $i$. In view of Lemma (2.1.9), we conclude:

$$\rho(g_1) = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}, \qquad \rho(g_2) = \begin{pmatrix} \beta & 0 \\ 0 & \delta \end{pmatrix},$$
$$\rho(g_3) = \begin{pmatrix} \gamma & 0 \\ 0 & \gamma \end{pmatrix}, \qquad \rho(g_4) = \begin{pmatrix} \delta & 0 \\ 0 & \beta \end{pmatrix}$$

with

$$|\alpha| = |\gamma| = 1,\ \beta^2 = \delta^2 = 1 \quad \text{and} \quad \beta \neq \delta\,.$$

If we put $\rho(g_5) = \begin{pmatrix} 0 & \varepsilon \\ 1 & 0 \end{pmatrix}$ $\varepsilon^2 = \alpha, \rho$ is extended to a unitary representation of the fundamental group $G$ of $S$. Since there are two $\varepsilon$'s satisfying $\varepsilon^2 = \alpha$, it follows there are two non-isomorphic vector bundle $E_1, E_2$ over $S$ such that $\pi^*E \simeq \pi^*E_1 \simeq \pi^*E_2$. Hence $E$ should be one of them.

q.e.d.

COROLLARY (2.13). *Using the notation of Theorem (2.1.3), let $E$ be an $H$-stable vector bundle with numerically trivial Chern classes. Let $C_n$ be a generic hyperplane section of $nH$. Then the restriction $E_n$ of $E$ to $C_n$ is stable if $n$ is sufficiently big.*

*Proof.* By Theorem (2.12), $E$ is defined by an irreducible unitary representation $\rho$ of $G$. By Lefschetz theorem, the canonical homomorphism $i_*: \pi_1(C_n) \to \pi_1(S) = G$ is surjective if $n$ is sufficiently big (see Bott [2]). Hence the unitary representation $\rho \circ i_*$ is irreducible. Since $E_n$ is associated to this representation, by Seshadri [10], $E_n$ is stable.

COROLLARY (2.14). *Using the notation of Theorem* (2.1.3), *let $E$ be an $H$-stable vector bundle over $S$. Then the following are equivalent.*

(1) *$E$ is defined by an irreducible unitary representation of the fundamental group $G$.*
(2) *$E$ is defined by a unitary representation of the fundamental group $G$.*
(3) *$E$ is defined by a linear representation of the fundamental group $G$.*
(4) *$E$ has an integrable connection.*
(5) *$E$ has a connection.*
(6) *The Chern classes of $E$ are numerically trivial.*

*Proof.* The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6) follow from the general theory. (6) $\Rightarrow$ (1) follows from Theorem (2.12).

THEOREM (2.15). *Using the notation of Theorem* (2.1.3), *let $F_r(S)$ be the set of isomorphism classes of $H$-stable vector bundles of rank $r$ with numerically trivial Chern classes over $S$. Then $F_r(S)$ has a complex analytic structure. More precisely,*

*Case* I i). *$S$ is of type* I i).

*If* $r = 1$, $F_r(S) \simeq {}^*\mathrm{Pic}^0 S \coprod \mathrm{Pic}^0 S \coprod \mathrm{Pic}^0 S \coprod \mathrm{Pic}^0 S$
$\simeq \boldsymbol{C}/(1, \tau/2) \coprod \boldsymbol{C}/(1, \tau/2) \coprod \boldsymbol{C}/(1, \tau/2) \coprod \boldsymbol{C}/(1, \tau/2)$.

*If* $r = 2$, $F_r(S) \cong \boldsymbol{C}/(1, \tau) \times \boldsymbol{P}^1(\omega)$, $\boldsymbol{P}^1(\omega) = \boldsymbol{P}^1 - \{Q_1, Q_2, Q_3, Q_4\}$, *where $p: \boldsymbol{C}/(1, \omega) \to \boldsymbol{P}^1$ is a double covering of $\boldsymbol{P}^1$ defined by a divisor of degree* 2 *over $\boldsymbol{C}/(1, \omega)$ and $p$ is ramified at $Q_1, Q_2, Q_3$ and $Q_4$,*

*If* $r \neq 1, 2$, *$F_r(S)$ is empty.*

*Case* I ii). *$S$ is of type* I ii).

*If* $r = 1$, $F_r(S) \simeq \mathrm{Pic}^0 S \coprod \mathrm{Pic}^0 S \simeq \boldsymbol{C}/(1, \tau/2) \coprod \boldsymbol{C}/(1, \tau/2)$.

*If* $r = 2$, *$F_r(S)$ is an elliptic surface over $\Delta' = \boldsymbol{P}^1 - 2$ points i.e. $F_r(S)$ is a non-compact complex manifold of dimension* 2 *with surjective map $p: F_r(S) \to \Delta'$. Here $p$ is a proper morphism and $p^{-1}(x) \simeq \boldsymbol{C}/(1, \tau)$ for a general point $x \in \Delta'$.*

*If* $r \neq 1, 2$, *$F_r(S)$ is empty.*

---

* $\coprod$ denotes disjoint union.

*Case* II i). *S is of type* II i).

*If* $r=1$, $F_r(S) \simeq \mathrm{Pic}^0 S \coprod \mathrm{Pic}^0 S \coprod \mathrm{Pic}^0 S$
$\simeq \boldsymbol{C}/(1, \tau/3) \coprod \boldsymbol{C}/(1, \tau/3) \coprod \boldsymbol{C}/(1, \tau/3)$.

*If* $r=3$, $F_r = \boldsymbol{C}/(1, \tau) \times \Delta'$, *where* $\Delta'$ *is a non-singular and non-complete curve.*

*If* $r \neq 1, 3$, $F_r(S)$ *is empty.*

*Case* II ii). *S is of type* II ii).

*If* $r=1$, $F_r(S) \simeq \mathrm{Pic}^0 S \simeq \boldsymbol{C}/(1, \tau/3)$.

*If* $r=3$, $F_r(S)$ *is an elliptic surface with the general fibre* $\boldsymbol{C}/(1, \tau)$ *over a non-complete curve.*

*If* $r \neq 1, 3$, $F_r(S)$ *is empty.*

*Case* III i). *S is of type* III i).

*If* $r=1$, $F_r(S) \simeq \mathrm{Pic}^0 S \coprod \mathrm{Pic}^0 S \simeq \boldsymbol{C}/(1, \tau/4) \coprod \boldsymbol{C}/(1, \tau/4)$.

*If* $r=2$, $F_r(S) \simeq \mathrm{Pic}^0 S' \coprod \mathrm{Pic}^0 S' \simeq \boldsymbol{C}/(1, \tau/2) \coprod \boldsymbol{C}/(1, \tau/2)$, *where* $S' \simeq A/(\bar{g}_5^2)$.

*If* $r=4$, $F_r(S) \simeq \boldsymbol{C}/(1, \tau) \times \Delta''$, *where* $\Delta''$ *is a non-singular and non-complete curve.*

*If* $r \neq 1, 2, 4$, $F_r(S)$ *is empty.*

*Case* III ii). *S is of type* III ii).

*If* $r=1$, $F_r(S) \simeq \mathrm{Pic}^0 S \simeq \boldsymbol{C}/(1, \tau/4)$.

*If* $r=2$, $F_r(S) \simeq \mathrm{Pic}^0 S' \coprod \boldsymbol{C}/(1, \tau/2)$, *where* $S' \simeq A/(\bar{g}_5^2)$.

*If* $r=4$, $F_r(S)$ *is an elliptic surface with the general fibre* $\boldsymbol{C}/(1, \tau)$ *over a non-complete curve.*

*If* $r \neq 1, 2, 3, 4$ $F_r(S)$ *is empty.*

*Case* IV. *S is of type* IV.

*If* $r=1$, $F_r(S) \simeq \mathrm{Pic}^0 S \simeq \boldsymbol{C}/(1, \tau/6)$.

*If* $r=2$, $F_r(S) \simeq \mathrm{Pic}^0 S' \coprod \mathrm{Pic}^0 S' \simeq \boldsymbol{C}/(1, \tau/3) \coprod \boldsymbol{C}/(1, \tau/3)$, *where* $S' \simeq A/(\bar{g}_5^2)$.

*If* $r=3$, $F_r(S) \simeq \mathrm{Pic}^0 S'' \coprod \mathrm{Pic}^0 S'' \coprod \mathrm{Pic}^0 S''$
$\simeq \boldsymbol{C}/(1, \tau/2) \coprod \boldsymbol{C}(/1, \tau/2) \coprod \boldsymbol{C}/(1, \tau/2)$,
*where* $S'' \simeq A/(\bar{g}_5^3)$.

*If* $r=6$, $F_r(S)$ *is an elliptic surface with the general fibre* $\boldsymbol{C}/(1, \tau)$ *over a non-complete curve.*

*If* $r \neq 1, 2, 3, 6$, $F_r(S)$ *is empty.*

*Proof.* Case I i). If $r = 1$, a line bundle $\in F_r(S)$ is defined by a unitary representation, $\rho(g_1) = \alpha$, $\rho(g_2) = \pm 1$, $\rho(g_3) = \gamma$, $\rho(g_4) = \pm 1$, $\rho(g_5) = \varepsilon$ with $\varepsilon^2 = \alpha$. This shows $F_r(S) = \mathrm{Pic}^0 S \coprod \mathrm{Pic}^0 S \coprod \mathrm{Pic}^0 S \coprod \mathrm{Pic}^0 S$ and $\mathrm{Pic}^0 S \simeq \boldsymbol{C}/(1, \tau/2)$.

If $r = 2$, Theorem (2.11) shows that such a bundle $E$ is defined by an irreducible unitary representation

$$\rho(g_1) = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}, \qquad \rho(g_2) = \begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix},$$
$$\rho(g_3) = \begin{pmatrix} \gamma & 0 \\ 0 & \gamma \end{pmatrix}, \qquad \rho(g_4) = \begin{pmatrix} \delta & 0 \\ 0 & \delta^{-1} \end{pmatrix},$$
$$\rho(g_5) = \begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix} \qquad \text{with } \beta^2 \text{ or } \delta^2 \neq 1 .$$

Remark that this normalization of irreducible unitary representation is not unique since representation

$$\rho(g_1) = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}, \qquad \rho(g_5) = \begin{pmatrix} \beta^{-1} & 0 \\ 0 & \beta \end{pmatrix}, \qquad \rho(g_5) = \begin{pmatrix} \gamma & 0 \\ 0 & \gamma \end{pmatrix},$$
$$\rho(g_5) = \begin{pmatrix} \delta^{-1} & 0 \\ 0 & \delta \end{pmatrix}, \qquad \rho(g_5) = \begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix}$$

also defines $E$. These representations are equivalent. But this normalization shows that $E$ is the direct image of a line bundle $L$ of degree 0 over $A$. The condition $\beta^2$ or $\delta^2 \neq 1$ shows that if $L \simeq p_1^* L_1 \otimes p_2^* L_2$, $L_2^{\otimes 2} \not\simeq O_{C_2}$ where $p_i$ is the projection from $A = \boldsymbol{C}/(1, \omega) \times \boldsymbol{C}/(1, \tau) = C_1 \times C_2$ onto the $i$-th factor. Let $W$ be the set of fixed points on $\hat{A}$ with respect to the action of $\bar{g}_5^*$. Then $F_r(S) = \hat{A} - W(\bar{g}_5^*)$. Since the action of $\bar{g}_5^*$ of $\hat{A}$ is given by

$$\begin{aligned} C_1 \times C_2 &\longrightarrow C_1 \times C_2 \\ (x, y) &\longmapsto (x, -y) , \end{aligned}$$

the result follows.

By Theorem (2.11) and Theorem (2.12), or simply by Corollary (2.7), $F_r(S)$ is empty if $r \neq 1, 2$.

Case I ii). If $r = 1$, the proof is similar to the Case I i). Hence we omit it.

If $r = 2$, the proof is similar to the Case I i) but delicate. Hence we give it. Let $\hat{A}$ be the dual abelian variety of $A$. Then $\bar{g}_5$ induces an automorphism $\bar{g}_5^*$ on $\hat{A}$. Let $W$ be the set of fixed points on $\hat{A}$ with respect to the action of the cyclic group $(\bar{g}_5^*)$. Then $F_r(S) = \hat{A} - W/(\bar{g}_5^*)$ by Theorem (2.11). We determine the variety $\hat{A}/(\bar{g}_5^*)$. To do this, we write down the abelian variety $\hat{A}$, the automorphism $\bar{g}_5^*$ and the subvariety $W$ explicitly. $\hat{A}$ is given by the period matrix $\begin{pmatrix} 1 & 0 & \tau & \frac{1}{2} \\ 0 & 1 & 0 & \omega \end{pmatrix}$. To see this, let $B: \boldsymbol{C}^2 \times \boldsymbol{C}^2 \to \boldsymbol{C}$ be an $\boldsymbol{R}$-linear pairing $B((x_1, x_2, (y_1, y_2)) = (1/\operatorname{Im}\tau)x_1\bar{y}_1 + (1/\operatorname{Im}\omega)x_2\bar{y}_2$, $(x_1, x_2), y_1, y_2) \in \boldsymbol{C}^2$. Then it is easy to see that $B$ satisfies the following:

1° $B$ is complex linear on $(x_1, x_2)$ and complex anti-linear on $(y_1, y_2)$.
2° $B$ is non-degenerate.
3° If we put $\operatorname{Im} B = \beta$, $\beta$ is integral on $U_1 \times U_2$, where $U_1$ is the lattice defined by the column vectors of the period matrix $\begin{pmatrix} 1 & 0 & \tau & 0 \\ 0 & 1 & \frac{1}{2} & \omega \end{pmatrix}$ of $A$ and $U_2$ is the lattice defined by the column vectors of the period matrix $\begin{pmatrix} 1 & 0 & \tau & \frac{1}{2} \\ 0 & 1 & 0 & \omega \end{pmatrix}$.
4° $U_1$ and $U_2$ are dual lattices under $\beta$.

Hence by Mumford p. 86 [7], $\hat{A}$ is given by the period matrix $\begin{pmatrix} 1 & 0 & \tau & \frac{1}{2} \\ 0 & 1 & 0 & \omega \end{pmatrix}$. Let us now describe the automorphism $\bar{g}_5^*$. Since a translation $T_x$, $x \in A$ induces the identity on $\hat{A}$, we may assume $g_5$ is given by $g_5: (u, \zeta) \to (x_1, -x_2)$ using the notation above. By Mumford p. 85 [7], the line bundle corresponding to the point $(y_1, y_2) \in \hat{A}$ (or the universal covering space of $\hat{A}$) is given by the representation

$$\begin{array}{ccc} U_1 & \longrightarrow & \boldsymbol{C} \\ \cup & & \\ (u_1, u_2) & \longmapsto & \exp(-2\pi i\beta((u_1, u_2), (y_1, y_2))) \,. \end{array}$$

Hence the automorphism $(x_1, x_2) \mapsto (x_1, -x_2)$ of $A$ induces, the automorphism $(y_1, y_2) \mapsto (y_1, -y_2)$ on $\hat{A}$. We regard the abelian variety $\hat{A} \simeq \boldsymbol{C}/\{(1,0),(0,1),(\tau,0),(\frac{1}{2},\omega)\}$ as an elliptic surface over an elliptic curve $\boldsymbol{C}/(1,\omega)$ by considering the projection $p_2: \boldsymbol{C}^2 \to \boldsymbol{C}$. Then $\hat{A}$ has trivial functional and homological invariants. But $\hat{A}$ is not a basic elliptic surface. Let $\pi': \hat{A} \to \boldsymbol{C}/(1,\omega)$ be this fibration, then the action of the cyclic group $(\bar{g}_5^*)$ preserves this fibration i.e. the diagram

$$\begin{array}{ccc} \hat{A} & \xrightarrow{\bar{g}_5^*} & \hat{A} \\ \pi' \downarrow & & \downarrow \pi' \\ \boldsymbol{C}/(1, \omega) & \longrightarrow & \boldsymbol{C}/(1, \omega) \\ y & \longmapsto & -y \end{array}$$

The fixed points should be contained in the fibres of $[0], [1/2], [\omega/2]$ and $[\omega/2 + 1/2]$ where $[a]$ denotes the image of $a \in \boldsymbol{C}$ into $\boldsymbol{C}/(1, \omega)$. The points of the fibres over $[0]$ and $[1/2]$ are really fixed but the points over $[\omega/2]$ and $[\omega/2 + 1/2]$ are not fixed. The action of $\bar{g}_5^*$ on $\pi^{-1}([\omega/2])$ and $\pi^{-1}([\omega/2 + 1/2])$ is the addition by $[1/2]$ on the elliptic curve $\boldsymbol{C}/(1, \tau)$. Hence $W = \pi^{-1}([0]) \cup \pi^{-1}([1/2])$. Let us consider the quotient space $\hat{A}/(\bar{g}_5^*)$. It has a fibration $\hat{A}/(\bar{g}_5^*) \to \boldsymbol{P}^1 =$ the quotient space of the elliptic curve with respect to the action $y \mapsto -y$. Let $P_1, P_2, P_3$ and $P_4$ be images of $[0], [1/2], [\omega/2], [\omega/2 + 1/2]$ in $\boldsymbol{P}^1$.

$$\begin{array}{ccc} \hat{A} & \longrightarrow & \hat{A}/\bar{g}_5^* \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ \boldsymbol{C}/(1, \omega) & \longrightarrow & \boldsymbol{P}^1 \end{array}$$

It is not difficult to see that the fibration $\pi$ is not singular at $P_1$ and $P_2$ and is singular at $P_3$ and $P_4$. The homological and functional invariants of $\hat{A}/(\bar{g}_5^*)$ is trivial. The general fibre $\boldsymbol{C}/(1, \tau)$ is patched by the translation $T_{[1/2]}$ on $\boldsymbol{C}/(1, \tau)$ when we consider small circles around $P_3$ and $P_4$. We proved $F_r(S)$ is an elliptic surface with general fibre $\boldsymbol{C}/(1, \tau)$ over $\boldsymbol{P}^1 - 2$ points and has two singular fibres.

Case II i). $S$ is of type II i).
If $r = 1$, the proof is similar to the Case I i).
If $r = 3$, the proof is similar to the Case I i).

Case II ii). $S$ is of type II ii).
If $r = 1$, the proof is similar to the Case I i).
If $r = 2$, the proof is similar to the Case I ii) but much easier since the assertion in this case is weaker. Of course we can determine the surface explitly as in the Case I ii) but we do not use it in the sequel.

Crse III i). $S$ is of type III i).
If $r = 1$, the proof is similar to the Case I i), $r = 1$.
If $r = 2$, the proof is similar to the Case I i), $r = 2$ or $r = 1$.
If $r = 3$, the proof is similar to the Case I i), $r = 2$.

Case III ii). $S$ is of type III ii).
If $r = 1$, the proof is similar to the Case I i), $r = 1$.
If $r = 2$, the proof is similar to the Case I ii), $r = 2$.
If $r = 4$, the proof is similar to the Case I ii), $r = 2$.

Case IV. $S$ is of type IV.
If $r=1$, the proof is similar to the Case I i), $r=1$.
If $r=2$, the proof is similar to the Case III i), $r=2$.
If $r=3$, the proof is similar to the Case III i), $r=2$.
If $r=6$, the proof is similar to the Case I i), $r=2$.

Hence we omit them.

THEOREM (2.16). *$F_r(S)$ is the coarse moduli space for H-stable vector bundles of rank $r$ with numerically trivial Chern classes over $S$. More pricisely, let $T$ be a variety and if we are given a vector bundle $E$ over $T \times S$ such that the restriction $E_t$ of $E$ on $t \times S$ is H-stable of rank $r$ with numerically trivial Chern classes for any (closed) point $t \in T$, then there exists a morphism $\varphi: T \to F_r(S)$ such that $E_t$ is isomorphic to the vector bundle corresponding to $\varphi(t) \in F_r(S)$.*

*Proof.* Our proof depends on the following

LEMMA (2.17). *Using the notation above, for any closed point $t \in T$, there exist an étale neighbourhood $U_t$ of $t$ and a line bundle $L$ over $U_t \times S'$ such that $E_t$ is isomorphic to the restriction on $t \times S$ of the direct image of $L$, where $S'$ is a covering space of degree $r$ of $S$ uniquely determined by $r$.*

*Proof.* We prove the lemma under the assumption that $S$ is of type I i), $r=2$. The proofs in other cases are similar. Since the problem is local, we may assume $T$ affine. Let $K$ be the canonical bundle of $S$. We denote by $K$ the pull back $p_1^*K$ over $S \times T$. Then $A \times T = \widetilde{\operatorname{Spec}(O_{A\times T} \oplus K)}$. Let $F$ be a coherent sheaf on $A \times T$. Then $p_{2*}F \simeq \widetilde{H^0(A \times T, F)}$ since $T$ is affine. On the other hand, let $f: A \times T \to S \times T$ be the covering map. We have

$$\begin{aligned} H^0(A \times T, f^* \operatorname{End} E) &= H^0(S \times T, f_* f^* \operatorname{End} E) \\ &= \oplus_{i=0}^1 H^0(S \times T, \operatorname{Hom}(E, E \otimes K^{\otimes i})) \,. \end{aligned}$$

Since $\dim_C H^0(S, \operatorname{Hom}(E_{t'}, E_{t'} \otimes K^{\otimes i}) = 1$ for any point $t' \in T$, $p_{2*} \operatorname{Hom}(E, E \otimes K^{\otimes i})$ is locally free and of rank 1. Hence if we take a sufficiently small open neighbourhood $U_t$ of $t$, $p_{2*} \operatorname{Hom}(E, E \otimes K^{\otimes i}) \simeq O_{U_t}$. We may assume $T = U_t$. By the flat base change theorem,

$$p_{2*} \operatorname{Hom}(E, E \otimes K^{\otimes i}) \otimes k(t) \simeq \operatorname{Hom}(E_t, E_t \otimes K^{\otimes i}) \simeq \boldsymbol{C} \,.$$

Hence there exists an element $g$ of Hom $(E, E\otimes K)$ such that $g$ induces an isomorphism on the fibre of $t$. Hence there exists an open neighbourhood $U_t$ of $t$ such that $g$ induces a non-zero homomorphism $g_{t'}\in \mathrm{Hom}\,(E_{t'}, E_{t'}\otimes K)$ for any $t'\in U_t$. Since $E_{t'}$ and $E_{t'}\otimes K$ are $H$-stable and have the same numerical Chern classes, $g_{t'}$ is an isomorphism by Takemoto [12]. We may assume $T=U_t$. If we compose $g$ itself, we get $\underbrace{E\to E\otimes K\to E\otimes K^{\otimes 2}=E}_{g^2}$. $g^2$ is an isomorphism. Since $H^0(S\times T, \mathrm{Hom}\,(E,E))\simeq H^0(T, O_T)$, $g^2$ is nothing but the multiplication by an element $f\in H^0(T, O_T^*)$. If we take an étale neighbourhood $U_t$ of $t$ such that $\sqrt{f}$ is a regular function, $H^0(A\times U_t, \mathrm{End}\, t^*E)\simeq O_{U_t}[X]/X^2-1$. This shows $f^*E\simeq L_1\oplus L_2$. If we put $L=L_1$, then $L$ satisfies the condition. q.e.d.

*Proof of Theorem* (2.16). We give the proof under the assumption that S is of type I i), $r=2$. By the Krull-Remack theorem and Lemma (2.6), the restriction $L_{A'}$ of $L$ on the fibre $A\times t'$, $t'\in U_t$ is algebraically equivalent to 0. Hence there exists a morphism $\psi_{U_t}\colon U_t\to \hat{A}$ such that $L_{A'}$ is isomorphic to the line bundle corresponding to $\psi_{U'}(t')$ for any $t'\in U_t$. $\psi_{U'}$ does depend on the choice the étale neighbourhood $U_t$ and $L$. But the composition of the natural projection with $\psi_{U_t}\colon U_t\to\hat{A}\to \hat{A}/(\bar{g}_5^*)$ is independent of the choice of $U_t$ and $L$. Hence we get a morphism $T\to F_2(S)$ satisfying the condition of the theorem. The proofs in other cases are similar hence we omit them.

THEOREM (2.18). *If $r\geq 2$ and $r\,|\,d(S)$, $F_r(S)$ gives local moduli for $S$ i.e. the map $S\to F_r(S)$ locally separates the points of the local moduli space for $S$. If $S$ is neither of type* I i) *nor* I ii), *$F_1(S)$ gives local moduli for $S$.*

*Proof*. Assume that $S$ is of type I i), $r=2$. By Suwa [11], the hyperelliptic surfaces of the type I i) are completely and effectively parametrized by $\tau$ and $\omega$. Let $S, S'$ be hyperelliptic surfaces of type I i). Suppose that $F_2(S)$ and $F_2(S')$ are isomorphic. By Theorem (2.15), $F_2(S)\simeq \boldsymbol{C}/(1,\tau)\times \boldsymbol{P}^1-4$ points determined by $\omega, F_2(S')\simeq \boldsymbol{C}/(1,\tau')\times\boldsymbol{P}^1-4$ points determined by $\omega'$. Let $\varphi: F_2(S)\to F_2(S')$ be an isomorphism. Let $x\in \boldsymbol{P}^1-4$ points determined by $\omega$. Then the fibre $\boldsymbol{C}/(1,\tau)\times x$ should be mapped onto a fibre by $\varphi$ since $\boldsymbol{P}^1-4$ points is not complete. Hence we get a morphism $\varphi' : \boldsymbol{P}^1 - 4$ points determined by $\omega \to \boldsymbol{P}^2 - 4$ points determined by $\omega'$ which makes the following diagram commutative,

$$\begin{array}{ccc} F_2(S) & \xrightarrow{\quad\varphi\quad} & F_2(S') \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p'} \\ \boldsymbol{P} - {}^1 4 \text{ points determined by } \omega & \xrightarrow{\varphi'} & \boldsymbol{P}_1 - 4 \text{ points determined by } \omega' , \end{array}$$

where $p, p'$ are the projections. It follows that $\boldsymbol{C}/(1,\tau) \simeq \boldsymbol{C}/(1,\tau')$, $\boldsymbol{C}/(1,\omega) \simeq \boldsymbol{C}/(1,\omega')$, since the ramified covering $\boldsymbol{C}/(1,\omega)$ (resp. $\boldsymbol{C}/(1,\omega')$) is determined by 4 points where it ramifies. Hence $S \mapsto F_2(S)$ gives local moduli.

Assume now that $S$ is of type I ii), $r = 2$. Let $S$ and $S'$ be hyperelliptic surface of type I ii). By Theorem (2.15) $F_2(S)$ (resp. $F_2(S')$) is isomorphic to an elliptic surface over $\Delta$ (resp. $\Delta'$) which is $\boldsymbol{P}^1 - 2$ points. The elliptic surface $F_2(S)$ has 2 singular fibres. Let $\varphi$ be an isomorphism of elliptic surfaces $F_2(S)$ and $F_2(S')$. Then as in the Case I ii), $\varphi$ preserves the fibration hence it maps the singular fibres to the singular fibres. It follows that $\varphi$ induces an isomorphism of $\boldsymbol{P}^1 - 4$ points determined by $\omega$ and $\boldsymbol{P}^1 - 4$ points determined by $\omega'$. Therefore $\boldsymbol{C}/(1,\tau) \simeq \boldsymbol{C}/(1,\tau')$ and $\boldsymbol{C}/(1,\omega) \simeq \boldsymbol{C}/(1,\omega')$. Hence $F_2(S)$ gives local moduli for $S$.

The proof in other cases is similar and really is simpler since those surfaces are parametrized by one parameter $\tau$.

*Remark* (2.19). If $S$ is of type I i) or of I ii). $F_1(S)$ does not give local moduli. $F_1(S)$ is essentially the Picard scheme of $S$. Hence $F_r(S)$, $r \geq 2$ that can be regarded as a generalization the Picard scheme gives the finer result than the Picard scheme.

COROLLARY (2.20). *Any H-semi-stable vector bundle of rank* 2 *with numerically trivial Chern classes over S which is not H-stable can not be a limit of H-stable vector bundles of rank* 2 *with numerically trivial Chern classes if* $2 \neq d(S)$ (*See Definition* (2.21)). *There exists for any hyperelliptic surface S an H-semi-stable vector bundles of rank r over S which can not be a limit of H-stable vector bundles if* $r \geq 2$.

Before we give proof we need

DEFINITION (2.21). Let $E$ be an $H$-semi-stable with numerically trivial Chern classes. We say that $E$ is a limit of $H$-stable vector bundles with numerically trivial Chern classes if there exist an irreducible variety

$T$, a locally free sheaf $F$ over $S \times T$ and two points $t_0, t_1 \in T$ such that $F_{t_0} \simeq E$ and $F_{t_t}$ is $H$-stable. We remark, then $F_t$ is $H$-stable on an open dense set $T$ since $H$-stability is an open condition (see Maruyama [4]).

LEMMA (2.22). *Let $S$ denote a hyperelliptic surface as always. Let $E$ be an $H$-semi-stable vector bundle of rank 2 with numerically trivial Chern classes. Then the restriction $E_{|C}$ of $E$ on any non-singular curve $C$ on $S$ is semi-stable.*

*Proof.* Let $A$ be the abelian variety in Theorem (2.1.3). We use the notation of Definition (2.21). First of all we remark that $\pi^*E$ is homogeneous i.e. $\pi^*E \simeq T_x^*\pi^*E$ for any $x \in A$. In fact, if $E$ is $H$-stable, $\pi^*E$ is the direct sum of line bundles which are algebraically equivalent to 0 hence homogeneous. If $E$ is not $H$-stable but $H$-semi-stable, then $\pi^*E$ is not $H$-stable but $H$-semi-stable. It follows from Proposition (5.2) Takemoto [12] that $\pi^*E$ is homogeneous. Hence we conclude easily from Morimoto [6] that $\pi^*E_{|C'}$ is semi-stable for any non-singular curve $C'$ on $A$. If we consider the fibre product $\tilde{C} = C \times_S A$. $\tilde{C}$ is an unramified covering of degree $d(S)$ of $C$. Since $\pi^*E_{|C}$ is semi-stable, $E_{|C}$ is semi-stable. q.e.d.

*Proof of Corollary* (2.20). Assume $r \nmid d(S)$, then there is no $H$-stable vector bundle of rank $r$ with numerically trivial Chern classes over $S$. No semi-stable vector bundle can be a limit of $H$-stable vector bundles of rank $r$ with numerically trivial Chern classes over $S$. There does exist an $H$-semi-stable vector bundle of rank $r$ with numerically trivial Chern classes over $S$, for example $\overbrace{O_S \oplus \cdots \oplus O_S}^{r}$. Now we assume $r \mid d(S)$ and $r \neq d(S)$. Consider the $H$-semi-stable $E = \overbrace{O_S \oplus \cdots \oplus O_S}^{r}$. We show that $E$ is not a limit of $H$-stable bundles of rank $r$. Assume that $E$ is a limit of $H$-stable vector bundles of rank $r$ with numerically trivial Chern classes, we shall show this leads to a contradiction. We use the same notation as in Definition (2.21). If we take a generic hyperplane section $C$ of $S$, then, by the Lefschetz theorem, we get a family of semi-stable vector bundles $F_{t|C}$, $t \in U$ over $C$ where, $U$ is an open subset of $T$ containing the point $t_0$, since the stability is an open condition. Moreover we assume, by Theorem (2.12), that if $F_t$ is $H$-stable, then $F_{t|C}$ is stable. Let $M(C, r)$ denote the moduli space of the semi-stables vector bundles of rank $r$ and of degree 0 over $C$. Then we get a morphism $\varphi: U \rightarrow M(C, r)$ such that $F_t \simeq$ the vector bundle corresponding to $\varphi(t) \in M(C, r)$. $C$ is so determined that $F_t$ is $H$-stable if and only if $\varphi(t)$ is a stable point of $M(C, r)$. As in the proof of (2.16), we get a morphism $\psi: F_r(S) \rightarrow M(C, r)$ such that the restriction to $C$ of the vector bundle corresponding to a point $u \in F_r(S)$ is isomorphic to the vector bundle corresponding to the point $\psi(u) \in M(C, r)$. We may assume $\psi(u)$ is stable for any $u \in F_r(S)$. Under our hypothesis $r \mid d(S)$, $r \neq d(S)$, $F_r(S)$ is complete by Theorem (2.15). Hence $\psi(F_r(S))$ is a closed subset of $M(C, r)$. If we denote by $U^S$ the subset of the points $t$ of $U$ such that $F_t$ is $H$-stable, then $\varphi(U^S) \subset \psi(F_r(S))$. Hence $\varphi(U) \subset \varphi(U^S) \subset \psi(F_r(S))$. It follows that $E$ is $H$-stable. This is a contradiction. It should be remarked that we use only the hypothesis that the restriction $E_{|C}$ is semi-stable.

Assume now $r = d(S)$. We must show that there exist an $H$-semi-stable vector bundle of rank $r$ with numerically trivial Chern classes which is not limit of $H$-stable vector bundles. We show that the vector bundle $E = \overbrace{O_S \oplus \cdots \oplus O_S}^{r}$ can not be a limit of $H$-stable vector bundles. Assume that $E$ were a limit of $H$-stable vector bundles parameterized by $T$ as in Definition (2.21). Take a generic hyperplane section $C$ as above. We would get a morphism $\varphi: T \rightarrow M(C, R)$. On the other hand, under our hypothesis an $H$-stable vector bundle $E'$ of rank $r$, with trivial Chern classes is a direct image of a line bundle over $A$ which is algebraically equivalent to 0. Hence we have a family of $H$-semi-stable vector bundle over $S$ parametrized by $\hat{A}$ and this parametrization contains all the $H$-stable vector bundles of rank $r$ with numerically trivial Chern classes (cf. proof of Theorem 2.15). We get a morphism $\psi: \hat{A} \rightarrow M(C, r)$. It would follow as above, $\varphi(T) \subset \varphi(T^S) \subset \psi(\hat{A})$. Hence there would be a line bundle $L$ over $A$ such that $L$ is algebraically equivalent to 0 and such that $E_{|C}$ and the restriction to $C$ the direct image $\pi_* L$ over $S$ correspond to the same point of the moduli space $M(C, r)$. Hence $\pi_* L_{|C}$ would have a filtration $0 = G_0 \subset G_1 \cdots \subset G_r = \pi_* L_{|C}$ where $G_i$ is a sub-vector bundle of rank $i$ of $G_{i+1}$ with $G_{i+1}/G_i \simeq O_C$. This is impossible.

Let $E$ be an $H$-semi-stable vector bundle of rank 2 with numerically trivial Chern classes. We assume that $E$ is not $H$-stable. By Lemma (2.22) the restriction $E_{|C}$ is semi-stable. By the remark above, $E$ is not a limit if $H$-stable vector bundles. q.e.d.

## § 3. Final remarks

In view of § 1, it seems interesting to study the irreducible unitary representations of the fundamental group of an algebraic variety. But it very often happens that there is no irreducible unitary representation of degree $\geq 2$ of the fundamental group. Concerning this, we ask

PROBLEM 1. Under what hypothesis is the condition that a vector bundle is defined by an irreducible unitary representation open?

We ask

PROBLEM 1′. If $H^i(\tilde{V}, Q) = 0$ for $i \geq 1$, then is the condition above open? Here $\tilde{V}$ is the universal covering space of the variety.

Over a variety $V$ such that the condition above is open, the theory or irreducible unitary representation of the fundamental group seems interesting if it is not empty (see Proposition (1.3)).

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*Nagoya University*