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UNITS OF REAL QUADRATIC FIELDS

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1. Let D be a positive square-free integer. Throughout this note we shall use the following notations;

$d = d(D)$: the discriminant of $\boldsymbol{Q}(\sqrt{D})$,

$t_0,\ u_0$: the least positive solution of Pell's equation $t^2 - du^2 = 4$,

$\varepsilon_D = (t_0 + u_0\sqrt{d})/2$.

In this note we estimate ε_D. At first (in lemma) we prove that for $\boldsymbol{Q}(\sqrt{D})$ there exist integers ℓ, m and Δ (= square-free) such that D is one of three types

$$D = \Delta\Big(m^2\Delta \pm \frac{4}{2^\delta}\Big)\Big/\ell^2, \qquad (\delta = 0,\ 1 \text{ or } 2)$$

where $2\nmid m$, $2\nmid\Delta$ for $\delta = 0$ and $2\nmid\Delta$ for $\delta = 1$. Therefore we consider the above three types.

As for the estimate of ε_D Hua [1] proved

$$\log \varepsilon_D < \sqrt{d}\Big(\frac{1}{2}\log d + 1\Big). \tag{1}$$

Here we estimate ε_D in accordance with the above three types.

THEOREM. *We have*

$$\varepsilon_D < 2^\delta \ell^2 D, \tag{2}$$

where $D = \Delta(m^2\Delta + 4/2^\delta)/\ell^2$ *and* $\delta = 0,\ 1$ *or* 2. Δ *is a square-free integer* > 0, m *and* ℓ *are integers. In particular* $2\nmid m$, $2\nmid\Delta$ *for* $\delta = 0$ *and* $2\nmid\Delta$ *for* $\delta = 1$. *More precisely when* $\delta = 1$ *we have*

$$\varepsilon_D < \begin{cases} 2\ell^2 D & (\Delta = 1), \\ \ell^2 D & (\Delta \geqslant 2), \end{cases} \tag{3}$$

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and when $\delta = 2$ *we have*

$$\varepsilon_D < \begin{cases} 4\ell^2 D & (\Delta = 1), \\ 2\ell^2 D & (\Delta = 2, 3), \\ \ell^2 D & (\Delta \geqslant 4). \end{cases} \tag{4}$$

Hence if $m^2\Delta \pm 4/2^\delta$ *is square-free then, for* $D = \Delta(m^2\Delta \pm 4/2^\delta)$,

$$\varepsilon_D < 2^\delta D \tag{5}$$

holds, where $\delta = 0, 1$ *or* 2 *and* $2 \nmid \Delta$ *for* $\delta = 0, 1$.

2. Types of D and Proof of Theorem.

LEMMA. (A) (I) *If* $D \equiv 1 \pmod 4$ *then there exist* ℓ, m *and* Δ (*=square-free* > 0) *such that* D *is one of the following two forms*

$$D = \Delta(m^2\Delta + 4/2^\delta)/\ell^2,$$

where $\delta = 0$ *or* 2 *and* $2 \nmid m$, $2 \nmid \Delta$ *for* $\delta = 0$. *Then we have*

$$\varepsilon_D \leqslant \{(2^\delta m^2\Delta + 2) + 2^\delta \ell \, m\sqrt{D}\}/2.$$

(II) *If* $D \equiv 2, 3 \pmod 4$ *then there exist* ℓ, m *and* Δ (*= square-free* > 0) *such that* D *is one of the following two forms*

$$D = \Delta \, (m^2\Delta + 4/2^\delta)/\ell^2,$$

where $\delta = 1$ *or* 2 *and* $2 \nmid \Delta$ *for* $\delta = 1$. *Then we have*

$$\varepsilon_D \leqslant \{(2^\delta m^2\Delta + 2) + 2^\delta \ell m\sqrt{D}\}/2.$$

(B) *Let* $\Delta =$ *square-free* > 0 *and* $m > 0$ *then, for* $\boldsymbol{Q}(\sqrt{D}) = \boldsymbol{Q}(\sqrt{\Delta(m^2\Delta \pm 4/2^\delta)})$ ($m^2\Delta \pm 4/2^\delta$ *is not necessary square-free*),

$$\varepsilon_D \leqslant \frac{1}{2}\left\{2^\delta m^2\Delta \pm 2 + 2^\delta m\sqrt{\Delta(m^2\Delta \pm 4/2^\delta)}\right\} \tag{6}$$

holds, where $\delta = 0, 1$ *or* 2 *and* $2 \nmid \Delta$ *for* $\delta = 0, 1$.

Proof. (A) (I) Pell's equation

$$t^2 - du^2 = 4 \tag{7}$$

becomes $Du^2 = (t + 2)(t - 2)$, hence we have

$D = D_1 D_2$ such that $(D_1, D_2) = 1$, $D_1 \mid t + 2$, $D_2 \mid t - 2$.

If we write

$$t + 2 = m_1 D_1, \quad t - 2 = m_2 D_2. \tag{8}$$

then a relation

(9) $$m_1D_1 = m_2D_2 + 4$$

holds. From (7) we have

(10) $$u^2 = m_1m_2.$$

If m_1 and m_2 have a common divisor, from (9) it must be 1, 2 or 4. Let $(m_1, m_2) = 2^\delta$ ($\delta = 0$, 1 or 2), $m_1 = 2^\delta m_1'$ and $m_2 = 2^\delta m_2'$ then (10) becomes

(11) $$u^2 = (2^\delta)^2 m_1' m_2', \qquad (m_1', m_2') = 1.$$

Hence m_1' and m_2' are both square-numbers. Let $m_1' = \ell^2$, $m_2' = m^2$ and $D_2 = \varDelta$ (resp. $D_1 = \varDelta$), then, from (8) and (13), we have

$$\begin{cases} t = 2^\delta m^2\varDelta + 2 & (\text{resp. } t = 2^\delta \ell^2 \varDelta - 2) \\ u = 2^\delta \ell m & (\text{resp. } u = 2^\delta \ell m) \end{cases}$$
$$D_1 = (m^2\varDelta + 4/2^\delta)/\ell^2 \qquad (\text{resp. } D_2 = (\ell^2\varDelta - 4/2^\delta)/m^2).$$

But $\delta = 1$ does not happen. In fact if $D = \varDelta\,(m^2\varDelta + 2)/\ell^2$, we have

(12) $$\varDelta(m^2\varDelta + 2) \equiv \ell^2 \qquad (\text{mod } 4\ell^2).$$

Then (i) when $(m, 2) = 1$ eq.(12) becomes $1 + 2\varDelta \equiv \ell^2$ (mod 4). Hence ℓ = odd and $\varDelta \equiv 2$ (mod 4) and so

$$D = \varDelta\,(m^2\varDelta + 2)/\ell^2 \equiv 2(m^2\varDelta + 2)/\ell^2 \not\equiv 1 \ (\text{mod } 4).$$

On the other hand (ii) when $(m, 2) = 2$ let $m = 2m'$ then from (9) ℓ is even and this contradicts $(\ell, m) = 1$.

(II) Let $t = 2s$ then the Pell's equation becomes

(13) $$Du^2 = (s + 1)(s - 1).$$

Hence we have $D = D_1D_2$ such that $(D_1, D_2) = 1$, $D_1 | s + 1$ and $D_2 | s - 1$. If we write

(14) $$s + 1 = m_1D_1, \quad s - 1 = m_2D_2,$$

then, for m_1 and m_2, $m_1D_1 = m_2D_2 + 2$ holds. From (13) we have

(15) $$u^2 = m_1m_2.$$

Let $(m_1, m_2) = 2^\delta(\delta = 0$ or 1), $m_1 = 2^\delta m_1'$ and $m_2 = 2^\delta m_2'$, then m_1' and m_2' are both square numbers. Therefore let $m_1' = \ell^2$, $m_2' = m^2$ and $D_2 = \varDelta$ (resp. $D_1 = \varDelta$), then from (14) and (15) we have

$$\begin{cases} t = 2(2^\delta m^2\Delta + 1) & (\text{resp. } t = 2(2^\delta \ell^2\Delta - 1)) \\ u = 2^\delta \ell m & (\text{resp. } u = 2^\delta \ell m) \end{cases}$$

$$D_1 = (m^2\Delta + 2/2^\delta)/\ell^2 \qquad (\text{resp. } D_2 = (\ell^2\Delta - 2/2^\delta)/m^2.$$

(B) Since $2 \nmid \Delta$ for $\delta = 0$ and 1, the biggest square-factor ℓ^2 of $\Delta(m^2\Delta \pm 4/2^\delta)$ is the biggest square-factor of $m^2\Delta \pm 4/2^\delta$. As Pell's equation $t^2 - du^2 = 4$ of $Q(\sqrt{D}) = Q(\sqrt{\Delta(m^2\Delta \pm 4/2^\delta)})$ has a solution

$$\begin{cases} t = 2^\delta m^2\Delta \pm 2, \\ u = 2^\delta \ell m, \end{cases}$$

we have (6). q.e.d.

Remark 1. Let $\varepsilon = (t + u\sqrt{p})/2$ be the fundamental unit of the real quadratic fields $Q(\sqrt{p})$ ($p \equiv 1 \pmod 4$). Then for primes $p = m^2 \pm 4$ or $p = 4m^2 \pm 1$ we have

$$u \not\equiv 0 \pmod p.$$

In fact when $p = m^2 + 4$, from lemma (B), we have $u < \sqrt{p}$. When $p = m^2 - 4$ or $4m^2 \pm 1$, from lemma (B), we have $u < 4\sqrt{p}$. If $4\sqrt{p} \geqslant p$ i.e., $p = 5$ or 13 then

$$u = 1 \not\equiv 0 \pmod p$$

holds.

Remark 2. Applying the method of the proof of lemma we see the following. Let p and q be primes ($\neq 2$) and let D = square-free > 0, $D \equiv 1 \pmod 4$. Suppose that $Q\sqrt{D}$) has not a unit of norm -1. Then the necessary and sufficient conditions in order that $Q(\sqrt{D})$ has a unit $\varepsilon = (t + u\sqrt{D})/2$ of $u = pq$ is that D is one of the following four forms

$$D = m(mp^2 \pm 4)/q^2,$$

or

$$D = m(mp^2q^2 \pm 4),$$

where m is a square-free integer and $2 \nmid m$. The proof is easy.

Remark 3. There exist infinitely many fields $Q(\sqrt{D})$ ($D = \Delta(m^2\Delta \pm 4)$ = square-free). There also exist infinitely many fields $Q(\sqrt{D})$ ($D = \Delta\,(m^2\Delta \pm 2)$ = square-free or $D = \Delta(m^2\Delta \pm 1)$ = square-free). In fact from the prime number theorem of arithmetic progression, for $m(\neq 1)$ with $(m, 4) = 1$, there exist infinitely many primes p which satisfy

$$p \equiv 4 \pmod{m^2}.$$

Then for primes p and q which satisfy

$$\begin{cases} p = m^2 m_1^2 \Delta_1' + 4 > q = m^2 m_2^2 \Delta_2' + 4, \\ \Delta_1 = m_1^2 \Delta_1', \quad \Delta_2 = m_2^2 \Delta_2' \end{cases}$$

where Δ_1', Δ_2' are both square-free, if $p\Delta_1' = q\Delta_2'$ then

$$1 > \frac{\Delta_2'}{p} = \frac{\Delta_1'}{q}$$

holds. This is a contradiction. For $D = \Delta\,(m^2\Delta \pm 2)$ and $D = \Delta(m^2\Delta \pm 1)$, the proofs are also similar.

Proof of theorem; For $D = \Delta\ (m^2\Delta \pm 4/2^\delta)/\ell^2$, from lemma(B) we have

$$\begin{aligned} \varepsilon_D &\leqslant \{2^\delta m^2\Delta + 2 + 2^\delta m\sqrt{\Delta(m^2\Delta + 4/2^\delta)}\,\}/2 \\ &= \frac{2^\delta \ell^2}{2}\left\{\frac{1}{\ell^2}\left(m^2\Delta + \frac{2}{2^\delta}\right) + \frac{m}{\ell}\sqrt{\Delta\left(m^2\Delta + \frac{4}{2^\delta}\right)\Big/\ell^2}\right\} \qquad (16) \\ &< \frac{2^\delta \ell^2}{2}\left(D + \sqrt{D}\sqrt{D}\right) = 2^\delta \ell^2 D. \end{aligned}$$

Inequalities (3) and (4) are evidence by (16).

Reference

[1] L.K. Hua, On the least solution of Pell's equation, Bull. Amer. Math. Soc. **48** (1942) 731–735.

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