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NOTE ON THE INFINITE DIMENSIONAL LAPLACIAN OPERATOR

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To Professor Katuzi Ono on the occasion of his 60th birthday.

§0. Introduction.

The infinite dimensional Laplacian operator can be discussed in connection with the infinite dimensional rotation group ([1]). Our interest centers entirely on observing how each one-parameter subgroup of the infinite dimensional rotation group contributes to the determination of the Laplacian operator.

We shall start with the measure of white noise. Let E be a nuclear space of C^∞-functions which is dense in $L^2(R^1)$ and satisfies the relation

(1) $$E \subset L^2(R^1) \subset E^*,$$

where E^* stands for the dual space of E. Given a (characteristic) functional $C(\xi) = \exp\left(-\frac{1}{2}\|\xi\|^2\right)$, $\|\xi\|$ being the $L^2(R^1)$-norm of $\xi \in E$, we can form a probability measure μ on E^* such that

(2) $$C(\xi) = \int_{E^*} \exp[i\langle x, \xi\rangle] d\mu(x),$$

where $\langle x, \xi\rangle$, $x \in E^*$, $\xi \in E$, is the continuous bilinear form which links E and E^*. We call μ the measure of *white noise*.

By *the infinite dimensional rotation group*, we mean the group $O(E)$ which consists of all the linear transformations g on E satisfying the following two conditions:

i) Each g is an isomorphism of E,

ii) $C(g\xi) = C(\xi)$ for every $\xi \in E$.

For each one-parameter subgroup $\{g_t\}$ of $O(E)$ we are given a unitary group $\{U_t\}$ in the following manner:

(3) $$U_t\varphi(x) = \varphi(g_t^* x), \quad \varphi \in L^2(E^*, \mu),$$

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where g_t^* is the conjugate of g_t. With $\{U_t\}$ we can associate a generator X:

$$\frac{d}{dt} U_t \Big|_{t=0} = X. \tag{4}$$

We shall be interested in an operator Δ acting on $L^2(E^*, \mu)$ which enjoys the following properties:

(5)
i) Δ is a quadratic form of the X's,
ii) commutes with each X,
iii) annihilates constants,
iv) negative definite.

(cf. [2, Chapt. X]). It will be shown that such an operator Δ exists and is determined uniquely up to constant factor. Indeed, our Δ coincides with *the infinite dimensional Laplacian operator* given by Umemura [1].

In §2 we shall see that finite dimensional rotations play a dominant role in the determination of Δ giving attention to the property (5) ii). However, to determine Δ completely we shall need quite different requirements arising from (5) iii) and iv). In fact, we shall make use of the feature of the *support* of μ (§3).

Our method may not be the shortest way to obtain the explicit form of Δ, however the discussion in this note seems to be helpful to carry on the harmonic analysis on the Hilbert space $L^2(E^*, \mu)$.

§1. Preliminaries.

Let $\{\xi_n, n \geq 1\}$ be a complete orthonormal system (*c. o. n. s.*) in $L^2(R^1)$ such that each ξ_n belongs to E, and let μ be the measure of white noise. A tame function based on $\{\xi_n\}$ is a function on (E^*, μ) expressed in the form $f(\langle x, \xi_1\rangle, \cdots, \langle x, \xi_p\rangle)$ by a function f on R^p for some $p > 0$.

For a strongly continuous one-parameter subgroup $\{g_t, t \text{ real}\}$ we define the generator A:

$$A = \frac{d}{dt} g_t \Big|_{t=0}. \tag{6}$$

The unitary group $\{U_t\}$ and its generator X are given by (3) and (4). We now introduce the operator $\frac{\partial}{\partial \xi_j}$: If $\varphi(x) = f(\langle x, \xi_1\rangle, \langle x, \xi_2\rangle, \cdots)$, then

$\frac{\partial}{\partial \xi_j} \varphi$ is given by $\left(\frac{\partial}{\partial \xi_1} \varphi\right)(x) = \frac{\partial}{\partial t_j} f(t_1, t_2, \cdots)|_{t_j=<x, \xi_j>}$. By a formal computation we have the following assertion.

PROPOSITION 1. *Suppose that $A\xi_n \in E$ for every n. Then, for a tame function $\varphi(x)$ based on $\{\xi_n\}$, the generator X of the unitary group $\{U_t\}$ is expressed in the form*

$$(X\varphi)(x) = \sum_j \langle x, A\xi_j \rangle \left(\frac{\partial}{\partial \xi_j} \varphi\right)(x). \tag{7}$$

To avoid notational complication, we sometimes use the notations φ_j, φ_{jk}, $\cdots$ to ednote $\frac{\partial}{\partial \xi_j} \varphi$, $\frac{\partial^2}{\partial \xi_j \partial \xi_k} \varphi$, $\cdots$.

We now come to a consideration of a quadratic form of the X's of the form (7). Let X and Y be generators of unitary groups corresponding to one-parameter groups $\{g_t\}$ and $\{h_t\}$ with generators A and B, respectively. Suppose that $A\xi_j \in E$ and $B\xi_j \in E$ for every j. Set

$$A\xi_j = \sum_p \lambda_{jp} \xi_p \text{ and } B\xi_k = \sum_q \nu_{kq} \xi_q.$$

Then we have a formal expression

$$(XY)\varphi(x) = \sum_{kj} \alpha^{jk}(x) \varphi_{jk}(x) + \sum_j \beta^j(x) \varphi_j(x) \tag{8}$$

for a tame function φ, where

$$\alpha^{jk}(x) = \sum_{pq} \lambda_{jp} \nu_{kq} \langle x, \xi_p \rangle \langle x, \xi_q \rangle$$

and

$$\beta^j(x) = \sum_{kq} \lambda_{jk} \nu_{kq} \langle x, \xi_q \rangle.$$

Thus a quadratic form $\varDelta$ of the X's may be thought of as an operator expressed formally in the form

$$\varDelta = \sum_{jk} a^{jk}(x) \frac{\partial^2}{\partial \xi_j \partial \xi_k} + \sum_j b^j(x) \frac{\partial}{\partial \xi_j}. \tag{9}$$

Noting the expressions of α^{jk} and β^j in (8), a^{jk} and b^j in (9) must be the limits of quadratic forms and linear forms of the $\langle x, \xi_n \rangle$, $n \geq 1$, respectively.

Now our problem can be stated as follows:

Starting out with the expression (9), determine the coefficients $a^{jk}(x)$ and

$b^j(x)$ so that Δ satisfies all the conditions i) $\sim$ iv) in (5).
It is quite reasonable to assume that

(10) all the $a^{jk}(x)$ and $b^j(x)$ belong to the domains of $\frac{\partial}{\partial\xi_p}$ and $\frac{\partial^2}{\partial\xi_p\partial\xi_q}$, $p,\ q\geq 1$,

and that

$$a^{jk}(x)=a^{kj}(x),\ \ j,\ k\geq 1. \tag{11}$$

§2. Commutativity with finite dimensional rotations.

In this section we shall find a necessary condition which is imposed upon the coefficients of Δ given by (9) by the requirement that Δ be commutative with finite dimensional rotations.

If $g\in O(E)$ acts in such a way that $g\xi=\xi$ for every ξ orthogonal to some finite dimensional subspace of E, then g is called *a finite dimensional orthogonal transformation.* The collection of such g's forms a subgroup of $O(E)$. We can also define *a finite dimensional rotation* in a similar manner.

An arbitrary finite dimensional rotation g can be expressed as the product of two dimensional rotations via the Euler angles. Thus, in order that Δ be commutative with finite dimensional orthogonal transformations Δ must commute with two dimensional rotations. To be somewhat more specific let us take a two dimensional subspace spanned by ξ_p and ξ_q, and let g_t be the rotation through the angle t in the plane $\{\xi_p,\xi_q\}$. With this choice of g_t we are given a unitary group $\{U_t\}$ and its generator X_{pq} represented in the form

$$X_{pq}=\langle x,\xi_p\rangle\frac{\partial}{\partial\xi_q}-\langle x,\xi_q\rangle\frac{\partial}{\partial\xi_p}. \tag{12}$$

As in §1, let $\{\xi_n\}$ be a *c. o. n. s.* in $L^2(R^1)$ such that $\xi_n\in E$ for every n.

PROPOSITION 2. *Suppose that the operator Δ given by* (9) *commutes with X_{pq} for every pair* (p,q). *Then we have*

$$a^{jk}(x)=c\langle x,\xi_j\rangle\langle x,\xi_k\rangle+\delta_{j,k}d,\ \ j,k=1,2,\cdots, \tag{13}$$

$$b^j(x)=b\langle x,\xi_j\rangle,\ \ \ j=1,2,\ \cdots, \tag{14}$$

where b, c and d are constants.

Proof. The proof of (14) is quite easy. In fact, with a particular choice of φ: $\varphi(x)=\langle x, \xi_q\rangle$, the equation

$$X_{pq}\Delta\varphi = \Delta X_{pq}\varphi \tag{15}$$

implies that

$$b^p(x) = \langle x, \xi_p\rangle\, b_q^q(x) - \langle x, \xi_q\rangle\, b_p^q(x).$$

Noting that $b^p(x)$ belongs to the span of the $\langle x, \xi_n\rangle$'s, we see that b_q^q is a constant independent of q and that $b_p^q=0$ for $p\neq q$. Thus (14) is proved.

We proceed to the proof of (13). By using (14), the equation (15) for general φ can be expressed in the form

$$\begin{aligned}2\Big(\sum_k a^{pk}(x)\varphi_{qk}(x) - \sum_k a^{qk}(x)\varphi_{pk}(x)\Big)\\ = \sum_{j,k} a_q^{jk}(x)\langle x, \xi_p\rangle\, \varphi_{jk}(x) - \sum_{j,k} a_p^{jk}(x)\langle x, \xi_q\rangle\, \varphi_{jk}(x).\end{aligned} \tag{16}$$

Set $\varphi(x)=\langle x, \xi_p\rangle\langle x, \xi_q\rangle$, then we have

$$a^{pp}(x) - a^{qq}(x) = X_{pq}a^{pq}(x). \tag{17}$$

If both j and k are different from p and q, then we have

$$X_{pq}a^{jk}(x) = 0; \tag{18}$$

and for $k\neq q$ we have

$$X_{jq}a^{jk}(x) = -a^{qk}(x). \tag{19}$$

Since $a^{jk}(x)$ is quadratic in $\langle x, \xi_n\rangle$'s, direct computations of the relation (18) for all possible pairs (p, q) enable us to obtain the expression

$$a^{jk}(x) = a^{jk}(\langle x, \xi_j\rangle^2 + \langle x, \xi_k\rangle^2) + c^{jk}\langle x, \xi_j\rangle\langle x, \xi_k\rangle + d^{jk}.$$

For $j\neq k$ the relation (19) requires that $a^{jk}=0$. We may set $a^{jj}=0$. Finally, the equation (17) leads us to obtain $d^{pp}=d^{qq}$ and $c^{pp}=c^{qq}=c^{pq}$. Further, using (19) again, we see that $d^{jk}=0$ for $j\neq k$. Thus the equation (13) is proved.

So far we have just used the relation (15) to obtain the following formal expression:

$$\Delta = c\sum_{j,k}\langle x, \xi_j\rangle\langle x, \xi_k\rangle \frac{\partial^2}{\partial\xi_j\partial\xi_k} + d\sum_j \frac{\partial^2}{\partial\xi_j^2} + b\sum_j \langle x, \xi_j\rangle \frac{\partial}{\partial\xi_j}. \tag{9'}$$

§3. Conclusion.

By a *c. o. n. s.* $\{\xi_n; n \geq 1\}$ in $L^2(R^1)$ we are given a sequence $\{\langle x, \xi_n \rangle; n \geq 1\}$ of mutually independent standard Gaussian random variables. The strong law of large numbers shows that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \langle x, \xi_n \rangle^2 = 1 \text{ for almost all } x \in E^*, \tag{20}$$

and that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \langle x, \xi_n \rangle^4 = 3 \text{ for almost all } x \in E^*. \tag{21}$$

Now we can use the property (5) iii) which must be satisfied by Δ given by (9′). From (20) and (21) the relations $\Delta 1 = 0$ and $\Delta 3 = 0$ imply the following equations:

$$c + d + b = 0, \text{ and } 3c + d + b = 0,$$

that is, $c = 0$ and $b = -d$.

The negative difiniteness (5) iv) requires that for $\varphi(x) = \langle x, \xi_1 \rangle$

$$\int (\Delta\varphi(x))\varphi(x) d\mu(x) = b \int \langle x, \xi_1 \rangle^2 d\mu(x) = b \leq 0$$

must hold. To avoid trivial operator, the constant b should be strictly negative: $b < 0$.

Summing up the above discussions, we have

THEOREM. *If the operator* Δ *of the form* (9) *satisfies the conditions* (5) i) ~ iv), *then*

$$\Delta = d \sum_j \left(\frac{\partial^2}{\partial \xi_j^2} - \langle x, \xi_j \rangle \frac{\partial}{\partial \xi_j} \right) \tag{9''}$$

with a positive constant d.

The operator given by (9″) is exactly the same as the infinite dimensional Laplacian operator given by Umemura in [1]. In fact, the Δ given by (9″) acts on $L^2(E^*, \mu)$ and its domain is rich enough including all the so-called Fourier-Hermite polynomials. It is interesting to note that the properties (20) and (21), that is the feature of so to speak the support of μ, contribute in final determination of the infinite dimensional Laplacian operator.

References

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