SARIO'S POTENTIALS AND ANALYTIC MAPPINGS*

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To my teacher Professor KIYOSHI NOSHIRO on the occasion of his 60th birthday

In order to extend Nevanlinna's first and second fundamental theorems to arbitrary analytic mappings between Riemann surfaces, Sario [8, 9] introduced a kernel function on an arbitrary Riemann surface generalizing the elliptic kernel on the Riemann sphere. Because of the importance of the potential theoretic method in the value distribution theory, we discussed potentials of Sario's kernel in [4]. In that paper the validty of Frostman's maximum principle for Sario's potentials was left unsettled. The main object of this paper is to resolve this question (Theorem 1). As a consequence the fundamental theorem of the potential theory is obtained in its complete form for Sario's potentials (Theorem 2).

We apply these results to analytic mappings f of an arbitrary parabolic or regular hyperbolic Riemann surface R into an arbitrary Riemann surface S and show that among Nevanlinna's fundamental functions the relation $N(r, a) = T(r) + O(\sqrt{T(r)} \log T(r))$ holds for every a in S except those of a set of capacity zero (Theorem 4). This includes the known covering property of the Riemannian image of R under f over S. Here capacity means logarithmic one, considered locally.

Sario's kernel

1. We shall first review the definition of Sario's kernel [8, 9]. On an arbitrary Riemann surface S take arbitrary but then fixed points ζ_j and disjoint parametric disks D_j with centers ζ_j $(j=0, 1)$. Let $t_0(\zeta) = t(\zeta, \zeta_0, \zeta_1)$ be a harmonic function on $S - \{\zeta_0, \zeta_1\}$ such that $t_0(\zeta) + 2 \log |\zeta - \zeta_0|$ and $t_0(\zeta) - 2 \log |\zeta - \zeta_1|$ are harmonic in D_0 and D_1 respectively. Moreover, we require that $t_0 = L_1 t_0$

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in a neighborhood of the ideal boundary β of S, with L_1 a principal operator (Ahlfors-Sario [1]); this means that t_0 is, in a sense, a constant on β ([5]). Such a function t_0 is unique up to an additive constant, and for definiteness we assume that $t_0(\zeta)+2\log|\zeta-\zeta_0|\to 0$ as $\zeta\to\zeta_0$ in D_0. The functions $s_0(\zeta)=\log(1+e^{t_0(\zeta)})$ and $s_0(\zeta)+2\log|\zeta-\zeta_0|$ are finitely continuous on $S-\{\zeta_0\}$ and D_0 respectively.

For an arbitrary point a in $S-\{\zeta_0\}$ we construct the function $t(\zeta, a)=t(\zeta, a, \zeta_0)$ in the same manner as $t(\zeta, \zeta_0, \zeta_1)$, but we choose the normalization, which is essential this time, $t(\zeta, a)-2\log|\zeta-\zeta_0|\to s_0(a)$ as $\zeta\to\zeta_0$ in D_0.

Finally we put $s(\zeta, a)=s_0(\zeta)+t(\zeta, a)$ for $a\neq\zeta_0$ and $s(\zeta, \zeta_0)=s_0(\zeta)$, i.e. $t(\zeta, \zeta_0)=0$. The function $s(\zeta, a)$ thus defined on $S\times S$ is called *Sario's kernel* on S.

2. Assume that S is *hyperbolic*, i.e. that the *Green's kernel* $g(\zeta, a)$ exists on S. In this case, by the unicity of t_0 on S, we conclude that

$$t_0(\zeta)=2g(\zeta, \zeta_0)-2g(\zeta, \zeta_1)+c, \tag{1}$$

where the constant c is given by $c=2g(\zeta_0, \zeta_1)-h(\zeta_0)$ with $t_0(\zeta)\,|\,D_0=-2\log|\zeta-\zeta_0|+h(\zeta)$, and

$$s_0(\zeta)=\log(e^{-2g(\zeta,\zeta_0)}+e^{-2g(\zeta,\zeta_1)+c})+2g(\zeta, \zeta_0). \tag{2}$$

Similarly we obtain

$$t(\zeta, a)=2g(\zeta, a)-2g(\zeta, \zeta_0)+s_0(a)-c. \tag{3}$$

Therefore Sario's kernel is expressed in terms of Green's kernel as follows:

$$s(\zeta, a)=2g(\zeta, a)+u(\zeta)+u(a)+k, \tag{4}$$

where $k=h(\zeta_0)-2\log(1+e^c)$, a constant, and

$$u(\zeta)=\log(e^{-2g(\zeta,\zeta_0)}+e^{-2g(\zeta,\zeta_1)+c})-\log(1+e^c). \tag{5}$$

Frostman's maximum principle

3. Let S be an arbitrary Riemann surface and $s(\zeta, a)$ the Sario's kernel on S. Take a positive regular Borel measure μ on S with compact support S_μ. *Sario's potential* s^μ is, by definition,

$$s^\mu(\zeta)=\int s(\zeta, a)d\mu(a),$$

It is lower semicontinuous and bounded from below on S, and finitely continuous and subharmonic in $S-S_\mu$. For fundamentals of Sario's potentials see [4; nos. 7-9]. We shall now prove *Frostman's maximum principle*:

THEOREM 1. *If* $s^\mu \leq M$ *on* S_μ, *then* $s^\mu \leq M$ *on* S.

The proof for parabolic or compact S is found in [4; no. 8] and thus we give the proof only for *hyperbolic* S. The proof will be given in 4-5.

4. We may assume that $\mu(S)=1$. By (4), what we have to show is that the validity of

$$2\,g^\mu(\zeta)+u(\zeta)\leq M' \tag{6}$$

on S_μ implies its validity on all of S, where $g^\mu(\zeta)=\int g(\zeta,\ a)\,d\mu(a)$ and

$$M'=M-k-\int u(a)\,d\mu(a). \tag{7}$$

We now assume that (6) holds on S_μ, and first prove that $M'\geq 0$. Take the unit Borel measure ν on S such that $S_\nu\subset S_\mu$ and $\int g d\nu d\nu = V_g(S_\mu)=\inf_\theta\int g d\theta d\theta$, where θ's are unit Borel measures with $S_\theta\subset S_\mu$. It is well-known that such a ν exists, and that $g^\nu\leq V_g(S_\mu)$ on S and $g^\nu=V_g(S_\mu)$ on S_μ except for a set of capacity zero (Constantinescu-Cornea [2]). Integration of both sides of $g^\nu=V_g(S_\mu)$ with respect to μ and Fubini's theorem give $\int g^\mu d\nu = V_g(S_\mu)$. Therefore on integrating both sides of (6) we conclude

$$2\,V_g(S_\mu)+\int u(\zeta)\,d\nu(\zeta)\leq M'. \tag{8}$$

Here $\int u(\zeta)\,d\nu(\zeta) = -2\,g^\nu(\zeta_0)+\int\varphi(\zeta)\,d\nu(\zeta)$ with $\varphi(\zeta)=\psi(2\,g(\zeta,\ \zeta_0)-2\,g(\zeta,\ \zeta_1))$ and $\psi(\xi)=\log(1+e^{\xi+c})-\log(1+e^c)$.

Since $\dfrac{d^2}{d\xi^2}\psi(\xi)\geq 0$, ψ is a convex function. Therefore $\psi\left(\int\xi d\nu(\zeta)\right)\leq\int\psi(\xi)d\nu(\zeta)$ for any ν-integrable function $\xi=\xi(\zeta)$ and in particular for $\xi(\zeta)=2\,g(\zeta,\ \zeta_0)-2\,g(\zeta,\ \zeta_1)$. Hence

$$\log\frac{1+e^{2g^\nu(\zeta_0)-2g^\nu(\zeta_1)+c}}{1+e^c}\leq\int\varphi(\zeta)d\nu(\zeta)$$

and in turn

$$\log\frac{e^{-2g^\nu(\zeta_0)}+e^{-2g^\nu(\zeta_1)-c}}{1+e^c}\leq\int u(\zeta)\,d\nu(\zeta).$$

Since $g^{\nu}(\zeta_0)$ and $g^{\nu}(\zeta_1)$ are dominated by $V_g(S_\mu)$, we conclude that $\int u(\zeta)\,d\nu(\zeta) \geq -2\,V_g(S_\mu)$. This with (8) implies that $M' \geq 0$, as desired.

5. Consider the family $\mathfrak{F}$ of sequences $\{\zeta_n\} \subset S$ converging to the ideal boundary β of S, and the family $\mathfrak{F}^+ \subset \mathfrak{F}$ consisting of $\{\zeta_n\}$ with $\liminf_n g(\zeta_n, a) > 0$ for some and hence for all $a \in S$. There exists a positive superharmonic function v on S such that $\lim_n v(\zeta_n) = \infty$ for any $\{\zeta_n\} \in \mathfrak{F}^+$ (Constantinescu-Cornea [2; p. 27]). Consider

$$w(\zeta) = M' - (g^{\mu}(\zeta) + u(\zeta)) + \varepsilon v(\zeta)$$

for arbitrarily fixed $\varepsilon > 0$. This is clearly a superharmonic function on $S - S_\mu$. By (6) on S_μ and the local maximum principle for g^{μ}, $\liminf w(\zeta) \geq 0$ for $\zeta \in S - S_\mu$ converging to any point in ∂S_μ. We also have $\liminf_n w(\zeta_n) \geq 0$ for every $\{\zeta_n\} \in \mathfrak{F}$. In fact, if $\liminf_n w(\zeta_n) < 0$ for some $\{\zeta_n\} \in \mathfrak{F}$, then we can find a subsequence $\{\zeta'_n\}$ such that $\lim_n w(\zeta'_n) < 0$ and $\lim_n g(\zeta'_n, a) = 0$ for every $a \in S$. Then clearly $\lim_n g^{\mu}(\zeta'_n) = \lim_n u(\zeta'_n) = 0$ and we would obtain

$$0 > \lim_n w(\zeta'_n) \geq M' \geq 0.$$

Thus $w(\zeta) \geq 0$ on $S - S_\mu$ and hence on S for every $\varepsilon > 0$. On letting $\varepsilon \to 0$, we conclude (6) on S. This completes the proof of Theorem 1.

Capacity

6. We define a set function $V(K)$ first for compact sets $K \subset S$ by

$$V(K) = \inf_\mu \int s(\zeta, a)\,d\mu(\zeta)\,d\mu(a),$$

where μ runs over all unit Borel measures with $S_\mu \subset K$. It is known that there exists a unit Borel measure μ with $S_\mu \subset K$ and $V(K) = \int s\,d\mu\,d\mu$. This is called the *capacitary measure* of K with respect to s [4; no. 9]. For general sets X, we set $V(X) = \sup_K V(K)$, where K runs over all compact sets $K \subset X$.

A set $X \subset S$ with $V(X) = \infty$ is said to have (inner) *s-capacity zero.* Observe that for such a set X, $\mu(X) = 0$ for any measure μ with $\int s\,d\mu\,d\mu < \infty$. It is known that *$X \subset S$ has s-capacity zero if and only if X has capacity zero*, i.e. $X \cap D$ has (inner) logarithmic capacity zero for every parametric disk D of S ([4; no. 9]). The quantity $e^{-V(X)}$ is called the s-capacity of X. Sometimes it is more convenient to call $1/V(X)$ the s-capacity of X (see no. 7 below). However, since we are only interested in determining whether or not a given set has capacity zero, the definition of the s-capacity is immaterial.

By the weak form of the fundamental theorem ([4; no. 9]) and Theorem 1, we obtain the following complete form of *the fundamental theorem of potential theory*:

THEOREM 2. *Let K be a compact set of S with positive capacity and μ its capacitary measure. Then $s^{\mu} \leq V(K)$ on S and $s^{\mu} = V(K)$ on K except for a set of capacity zero.*

7. Since Sario's potentials enjoy the continuity principle, the local maximum principle ([4; nos. 6-9]), Frostman's maximum principle, and the fundamental theorem, most of the properties of elliptic potentials can be generalized to Sario's potentials. As an example and also for later use we shall prove

THEOREM 3. *Let X_n $(n = 1, 2, \ldots)$ be Borel sets in S and $X = \bigcup_{n=1}^{\infty} X_n$ be contained in an open set U such that $\inf_{U \times U} s > 0$. Then*

$$\frac{1}{V(X)} \leq \sum_{1}^{\infty} \frac{1}{V(X_n)}. \tag{9}$$

We may assume that $V(X) < \infty$, and X and X_n are compact. Let μ and μ_n be capacitary measures for X and X_n respectively. Then by Theorem 2

$$V(X) = \int_X s(\zeta, a)\, d\mu(a) \geq \int_{X_n} s(\zeta, a)\, d\mu(a) \tag{10}$$

for $\zeta \in X_n \subset X$ except for a set of capacity zero. Since $\int s^{\mu_n}(a)\, d\mu_n(a) = V(X_n)$, integration of both sides of (10) with respect to μ_n and Fubini's theorem give

$$V(X) \geq \int_{X_n} s^{\mu_n}(a)\, d\mu(a) = \int_{X_n} V(X_n)\, d\mu(a) = V(X_n)\, \mu(X_n),$$

with the convention $0 \cdot \infty = 0$. Therefore

$$\sum_{1}^{\infty} \frac{1}{V(X_n)} \geq \frac{1}{V(X)} \sum_{1}^{\infty} \mu(X_n) \geq \frac{1}{V(X)} \mu\Big(\bigcup_{1}^{\infty} X_n\Big) = \frac{1}{V(X)}.$$

Analytic mappings

8. Next we apply results obtained thus far to analytic mappings between Riemann surfaces. As the domain surface R, take an R_p-*surface* (Noshiro-Sario

[7]), i.e. an open Riemann surface R on which there exists a function $p(z) = p(z, z_0)$ such that p is harmonic on $R - z_0$, $p(z) - \log|z - z_0|$ is harmonic in a fixed parametric disk about z_0, and $\lim p(z) = k_R$ (constant) as $z \in R$ tends to the ideal boundary of R. Every parabolic Riemann surface R is an R_p-surface, and only in this case $k_R = \infty$ ([3]). A hyperbolic Riemann surface R is an R_p-surface if it is regular, i.e. if $\lim g(z, z_0) = 0$ as z tends to the ideal boundary of R. In this case, $k_R < \infty$ and $p(z) = k_R - g(z, z_0)$.

Set $r = r(z) = e^{p(z)}$ and $d\theta = d\theta(z) = {}^*dp(z)$. Then $re^{i\theta}$ can be used as local parameters on R except for a set of isolated points of R. Observe that $\rho_R = \sup r = \infty$ for every parabolic surface R and that $\rho_R < \infty$ for all regular hyperbolic surfaces R. The former surfaces generalize the z-plane and the latter generalize the disks. Denote $\beta_r = \{z \in R \mid r(z) = r\}$ and $R_r = \{z \in R \mid r(z) < r\}$ $(0 < r < \rho_R)$.

9. Consider an arbitrary non-constant analytic mapping f of an R_p-surface R into an arbitrary Riemann surface S. For $a \in S$, denote by $n(r, a)$ the number of a-points of f in R_r, counted with multiplicities. The *proximity function* for f is defined by

$$m(r, a) = \frac{1}{4\pi}\int_{\beta_r} s(f(z), a)\, d\theta(z)$$

and the *counting function* by

$$N(r, a) = \int_0^r \frac{n(r, a) - n(0+, a)}{r}\, dr + n(0+, a) \log r + c(a),$$

where $c(a)$ is a constant determined by the condition $\lim_{r\to 0}(m(r, a) + N(r, a)) = 0$. Finally, the *characteristic function* is given by

$$T(r) = \int_0^r \frac{S(r)}{r}\, dr,$$

where $S(r) = \frac{1}{4\pi}\int_{R_r} d\omega(f(z))$ with $d\omega(\zeta) = \Delta_\zeta s_0(\zeta)\, dS_\zeta$, dS_ζ being the local Euclidean area element on S. We then have the first fundamental theorem for analytic mappings (Sario [8, 9, 10]): for every $a \in S$ and $0 < r < \rho_R$

$$T(r) = m(r, a) + N(r, a). \tag{11}$$

We now state

THEOREM 4. *Let f be an analytic mapping of an R_p-surface R into an arbitrary surface S. Then*

(12) $$N(r, a) = T(r) + O(\sqrt{T(r)} \log T(r))$$

for every $a \in S$ *except for a Borel set of capacity zero.*

The relation (12) is meaningful only for f with

(13) $$\lim_{r \to \rho_R} T(r) = \infty,$$

otherwise (12) is trivial. The proof of this theorem will be given in 10, where we always assume (13). Needless to say, (13) is automatic for parabolic R.

10. Observe that there exists a number $0 < r_0 < \rho_R$ such that $T(r) - 2\sqrt{T(r)} \log T(r)$ is increasing for $r_0 < r < \rho_R$. We choose a sequence $r_0 < r_1 < \cdots < r_\nu \to \rho_R$ such that

(14) $$T(r_\nu) = T(r_{\nu-1}) + 2\sqrt{T(r_\nu)} \log T(r_\nu).$$

Let U be a parametric disk on S with $\inf_{U \times U} s > 0$. We call such a U *an admissible disk* on S. Fix an admissible U and consider compact sets $E_\nu \subset \overline{U}$ consisting of all points $a \in \overline{U}$ such that

(15) $$N(r_\nu, a) \leq T(r_\nu) - 2\sqrt{T(r_\nu)} \log T(r_\nu).$$

We first show that

(16) $$2\sqrt{T(r_\nu)} \log T(r_\nu) \leq V(E_\nu).$$

For the proof we may assume $V(E_\nu) < \infty$. Let μ be the capacitary measure of E_ν. On integrating both sides of (11) with respect to μ and by using Fubini's theorem, we obtain $T(r_\nu) \leq V(E_\nu) + \int_{E_\nu} N(r, a)\, d\mu(a)$. This with (15) implies (16).

Next set $E'_\nu = \bigcup_{j=\nu}^{\infty} E_j$. By Theorem 3, we conclude that

(17) $$\begin{aligned} \frac{1}{V(E'_\nu)} &\leq \sum_{j=\nu}^{\infty} \frac{1}{V(E_j)} \leq \sum_{j=\nu}^{\infty} \frac{1}{2\sqrt{T(r_j)} \log T(r_j)} \\ &\leq \frac{1}{4} \sum_{i=\nu}^{\infty} \int_{r_{j-1}}^{r_j} \frac{dT(r)}{T(r)(\log T(r))^2} \leq \frac{1}{2 \log T(r_\nu)} \end{aligned}$$

for sufficiently large ν. Next set $E_U = \bigcap_{j=1}^{\infty} E'_j$. Since $E'_j \supset E'_{j+1} \supset E_U$, by (17) we conclude that

(18) $$V(E_U) = \infty.$$

If $a \in U - E_U$, then there exists a ν such that $a \notin E_j$ for every $j \geq \nu$. Thus (15) implies that

$$N(r_j, a) \geq T(r_j) - 2\sqrt{T(r_j)} \log T(r_j) \tag{19}$$

for all $j \geq \nu$. If $r_j \leq r \leq r_{j+1}$, then by (14) and (19) it is easily seen that

$$N(r, a) \geq T(r) - 4\sqrt{T(r)} \log T(r). \tag{20}$$

Thus by (19) and (20), we deduce that

$$T(r) + O(1) \geq N(r, a) \geq T(r) - 4\sqrt{T(r)} \log T(r)$$

for $r_\nu \leq r < \rho_R$, since $m(r, a) \geq O(1)$ along with $s \geq O(1)$ (Sario [9]). Thus we conclude the validity of (12) for $a \in U - E_U$.

Cover S by a countable locally finite open covering $\{U\}$ of admissible disks U. By virtue of (17) we may conclude that $E = \bigcup_U E_U$ is of capacity zero, and (12) is valid for $a \in S - E$. The proof is herewith complete.

11. The geometric meaning of Theorem 4 is as follows (cf. Noshiro [6]). First assume that R is hyperbolic. If f is not of bounded characteristic, then (12) implies that $N(r, a) \to \infty$ and a fortiori $n(r, a) \to \infty$ $(r \to \rho_R)$ for every $a \in S$ except for a set of capacity zero. Thus the Riemannian image of R under f covers S infinitely often except for a set of capacity zero. This result is also true even for irregular hyperbolic R ([4]).

Next assume that R is parabolic. It is easy to see that $n(r, a) \to \infty$ if and only if $N(r, a)/\log r \to \infty$ $(r \to \infty)$. Therefore, by (12), if

$$\lim_{r \to \infty} \frac{T(r)}{\log r} = \infty, \tag{21}$$

then the Riemannian image of R under f covers S infinitely often except for a set of capacity zero, and vice versa. The condition (21) is also characterized by the Weierstrass property of f at the ideal boundary β of R, i.e. the global cluster set of f at β is total.

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