ON SOME PROPERTIES OF LOCALLY COMPACT GROUPS WITH NO SMALL SUBGROUP

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1. Let G be a locally compact group. Under a neighbourhood U we mean a symmetric (i.e. $U=U^{-1}$) neighbourhood of the identity e, with the compact closure $\overline{U}$. If there exists a neighbourhood U containing no subgroup other than the identity group, we say that G has *no small subgroup*. Now G has been called to have *property* (S) if

(S) for every $x \neq e$ in a sufficiently small neighbourhood U there exists an integer n so that $x^{2^n} \notin U$.[1]

If G has property (S), G is obviously with no small subgroup. Conversely we have

THEOREM 1. *A locally compact group has property* (S) *if it has no small subgroup.*

Proof. Let G be a locally compact group and V a neighbourhood with closure having no subgroup other than the identity group. Let W be a neighbourhood such as $W^2 \subset V$.

Suppose that the theorem is not true. Then there exist sequences $\{U_n\}$ and $\{x_n\}$ of neighbourhoods and elements such that

$$\ldots \supset U_n \supset U_{n+1} \supset \ldots,$$
$$\cap\, U_n = e,$$
$$U_n \ni x_n^{2^m}, \quad x_n \neq e, \quad m = 0, 1, 2, \ldots.$$

Because $\overline{V}$ has no non-trivial subgroup there exists j_n such that

$$x_n \in W, \ \ldots, \ x_n^{j_n-1} \in W, \ x_n^{j_n} \notin W,$$

for every n. Then the inequality $2^{m_n-1} < j_n \leqq 2^{m_n}$ determines a unique integer m_n. It is to be remarked that if $1 \leqq s_n \leqq 2^{m_n}$, then $x_n^{s_n} \in W^2 \subset V$. In particular $x_n^{j_n}$ is contained in V. Hence we can choose a subsequence $\{x_{n'}\}$ of $\{x_n\}$ such that $\lim x_{n'}^{j_{n'}}$ exists. Then the fact that $x_{n'}^{j_{n'}} \notin W$ implies that

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[1] See Kuranishi [4].

$$a = \lim x_{n'}^{j_{n'}} \notin W, \text{ whence } a \neq e.$$

Now let H be the totality of elements of the form $\lim x_{n'}^{s_{n'}}$, $1 \leqq s_{n'} \leqq 2^{m_{n'}}$. Then H is clearly contained in $\overline{V}$. Suppose

$$b = \lim x_{n'}^{s_{n'}} \quad \text{and} \quad c = \lim x_{n'}^{t_{n'}}$$

be in H. Let $u_{n'}$ be such that

$$s_{n'} - t_{n'} \equiv u_{n'} \,(\text{mod. } 2^{m_{n'}}),$$
$$1 \leqq u_{n'} \leqq 2^{m_{n'}}.$$

Then $\lim x_{n'}^{2^{m_{n'}}} = e$ implies

$$bc^{-1} = \lim x_{n'}^{s_{n'} - t_{n'}} = \lim x_{n'}^{u_{n'}} \notin H,$$

whence H is a subgroup in $\overline{V}$. Hence the non-triviality of H contradicts the hypothesis. Thus our theorem is proved.

Remark 1. A more general formulation of THEOREM 1 will appear in a forthcoming paper by the second author.[2]

Remark 2. The following proposition is a direct consequence of our THEOREM 1: *Let G be a locally euclidean group having no small subgroup. Then there exists a unique one-parameter subgroup through every point sufficiently near the identity.*[3]

2. The purpose of the present section is to prove the following

THEOREM 2. *Let G be a locally compact group with no small subgroup, and let L be a closed invariant subgroup of G. If L is a Lie group, then the factor group G/L has no small subgroup.*

In order to prove this we use the following lemmas.

LEMMA 1.[4] *Let a locally·compact group G have property* (S), *and Z the center of G. Then G/Z has no small subgroup.*

LEMMA 2. *Let G be a locally compact group, and N a closed invariant subgroup. If both G/N and N have no small subgroup, then the same is true for G.*

Proof. Let f be the natural homomorphic mapping from G onto G/N, and

[2] Yamabe [6].

[3] See Kuranishi [3].

[4] See Kuranishi [4].

let V be a neighbourhood of G/N having no non-trivial subgroup. Then there exists an neighbourhood W of G such that $f(W) \subset V$ and that $W \cap N$ contains no non-trivial subgroup. It is clear that the only subgroup in W is the identity group.

Proof of the theorem. First we consider some special cases.

i) *Let L be discrete.* In this case G and G/L are locally isomorphic. Hence the assertion is obvious.

ii) *Let L be a connected semi-simple Lie group with the center e.* Let $A(L)$ be the group of all continuous isomorphisms of L, and $I(L)$ the subgroup composed of inner automorphisms. It is well-known that $A(L)$ is a linear Lie group, and $I(L)$ coincides with the identity component of $A(L)$.

Now let g be an element of G. Putting

$$\delta(g)\, l = g^{-1} l g \text{ for } l \in L,$$

we obtain a continuous homomorphism δ of G into $A(L)$. Denote by C the kernel of the homomorphism: $C = \{c;\ lc = cl, \text{ for } l \in L\}$. Next let $\tilde{\delta}(g)$ be the coset of $A(L)$ mod. $I(L)$ containing $\delta(g)$. Then $\tilde{\delta}$ gives a continuous homomorphism of G into $A(L)/I(L)$. Let N be the kenrel of $\tilde{\delta}$.

Because $A(L)/I(L)$ is discrete, N is an open subgroup in G. Now every element of N induces an inner automorphism of L. Hence $N = CL$. On the other hand as the center of L is e, $C \cap L = e$, whence $N = C \times L$. Thus the isomorphism $N/L \cong C$ and the openness of N imply our assertion.

iii) *Let L be a connected commutative Lie group.* Denote by N the centralizer of L: $N = \{g;\ lg = gl, \text{ for } l \notin L\}$. By a similar argument as above C/N is a Lie group.

Now let Z be the center of N. Then by LEMMA 1 N/Z has no small subgroup. Now, because Z has no small subgroup and is commutative, Z is a Lie group. Hence Z/L is a Lie group. Thus by using LEMMA 2 twice we have the desired proposition.

iv) *General case.* Let L_1 be the identity component of L, and L_2 the largest solvable invariant subgroup of L_1. And let L_3 be the identity component of L_2. Denote by L_4 the topological commutator subgroup of L_3, L_5 the topological commutator subgroup of L_4, and so on. Then we get a sequence

$$L_0 = L \supset L_1 \supset L_2 \supset \ldots \supset L_n \supset L_{n+1} = e$$

of characteristic subgroups of L such that every L_i/L_{i+1} is either discrete, connected commutative, or connected semi-simple with no center. Considering G/L_1, G/L_2, ..., in order, we get the result in virtue of above i), ii) and iii). Q.E.D.

COROLLARY.[5] *A locally compact solvable (in the finite sense) group is a Lie group if it has no small subgroup.*

Proof. Let G, G^1, . . . , $G^{(m-1)}$, $G^{(m)}=e$, be the series of the topological commutator subgroups of G. We shall prove by the method of mathematical induction on m. Because $G^{(m-1)}$ has no small subgroup and is commutative, it is a Lie group. Now by THEOREM 2, $G/G^{(m-1)}$ has no small subgroup. Therefore by the assumption of induction $G/G^{(m-1)}$ is a Lie group. Hence the assertion follows from the extension theorem of Lie groups.[6]

3. *Applications of* THEOREM 2.

THEOREM 3. *Let G be a locally compact group with no small subgroup, and H a closed subgroup. If H is a maximal connected Lie group in G, then the identity component of the normalizer $n(H)$ coincides with H.*

LEMMA 3.[7] *Let G be a locally compact group having no small subgroup. If G is not 0-dimensional, then G contains a non-trivial commutative Lie group.*

Proof of the theorem. Let $n(H)^*$ be the identity component of $n(H)$. By THEOREM 2 $n(H)^*/H$ has no small subgroup. Hence if $n(H)^*$ does not coincide with H, then there exists a connected Lie group A in $n(H)^*/H$. Then the complete inverse image of A in the natural homomorphism $n(H)^* \sim n(H)^*/H$ is a connected Lie group in virtue of the extension theorem. This contradicts the fact that H is a maximal connected Lie group.

LEMMA 4. *Let G be a locally compact group with no small subgroup, and H_1 a closed local subgroup. If H_1 is a local Lie group, then the closure H of the subgroup generated by H_1 is a Lie group.*

Proof. We have proved that H is an (L)-group in the sense of K. Iwasawa[8] for general G.[9] On the other hand H has no small subgroup. Hence H is a Lie group.

Using THEOREM 3 and LEMMA 4 we have readily

THEOREM 4. *Let G be a locally compact group with no small subgroup, and H_1 a closed local subgroup. If H_1 is a maximal local Lie group, then the identity component of the normalizer of H_1 coincides with H_1 locally.*

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[5] The corollary has been proved by C. Chevally, A. Melcev, and K. Iwasawa separately. See e.g. Iwasawa [2]. The authors do not know whether their methods can be applied for non-connected case. (The authors have had no access to the former two, and Iwasawa proved the corollary only in connected case.)

[6] See Kuranishi [3] and Iwasawa [2].

[7] See Montgomery [5].

[8] See Iwasawa [2].

[9] See Gotô [1].

BIBLIOGRAPHY

[1] M. Gotô: *On local Lie groups in a locally compact group*, forthcoming shortly.

[2] K. Iwasawa: *On some types of topological groups*, Ann. of Math., Vol. **50** (1949).

[3] M. Kuranishi: *On euclidean local groups satisfying certain conditions*, Proc. Amer. Math. Soc., Vol. **1** (1950).

[4] M. Kuranishi: *On conditions of differentiability of locally compact groups*, Nagoya Math. Journ., Vol. **1** (1950).

[5] D. Montgomery: *Connected one dimensional groups*, Ann. of Math., Vol. **49** (1948).

[6] H. Yamabe: *On the structure of locally compact groups*, Osaka Math. Journ., Vol. **3** (1951).

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